LECTURE 16

Linear coupling

Two coupled harmonic oscillators

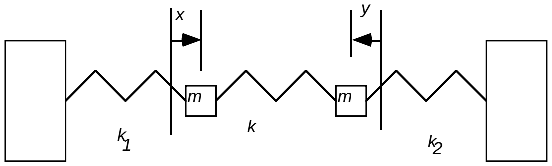
Equations of linear coupling

Difference resonances

Sum resonances

Two coupled harmonic oscillators

The motion of a particle in an accelerator may exhibit coupling between the two transverse planes. Such motion is very similar to the motion of two coupled simple harmonic oscillators. The position of one oscillator is analogous to the particle's x-motion, while that of the other oscillator is analogous to the y-motion.



The equations of motion for the masses are

$$m\ddot{x} = -k_1 x - k(x+y)$$

$$m\ddot{y} = -k_2 y - k(x+y)$$

This can be written in matrix form as

$$\ddot{\vec{z}} + \mathbf{M}\vec{z} = 0, \quad \vec{z} = \begin{pmatrix} x \\ y \end{pmatrix},$$

$$\mathbf{M} = \frac{1}{m}\begin{pmatrix} k_1 + k & k \\ k & k_2 + k \end{pmatrix} = \begin{pmatrix} \omega_1^2 & q^2 \\ q^2 & \omega_2^2 \end{pmatrix}$$

The standard technique for a solution is to find the *normal modes* of the motion. The normal modes $\vec{\zeta}$ are linear combinations of *x* and *y*, given by the transformation matrix **S:**

$$\vec{z} = \mathbf{S}\vec{\zeta}$$

The normal modes are uncoupled, so that

$$\ddot{\vec{\zeta}} + \mathbf{S}^{-1}\mathbf{M}\mathbf{S}\vec{\zeta} = 0$$

in which the matrix $\mathbf{S}^{-1}\mathbf{M}\mathbf{S} = \mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ is diagonal. The quantities $\sqrt{\lambda_1}$ and $\sqrt{\lambda_2}$ are the normal mode frequencies.

To find the normal modes and frequencies, we want to find the matrix **S** which makes $\mathbf{\Lambda} = \mathbf{S}^{-1}\mathbf{M}\mathbf{S}$ diagonal.

This problem is equivalent to that of finding the eigenvalues and eigenvectors of the matrix **M**. If the two eigenvectors are $\vec{e}_1$ and $\vec{e}_2$, and eigenvalues are λ_1 and λ_2, the eigenvector equation is $\mathbf{M}\vec{e}_k = \lambda_k \vec{e}_k$.

Then

$$(\mathbf{M}-\mathbf{I}\lambda_k)\vec{e}_k = 0$$

For these linear homogeneous equations to have a solution, the determinant $|\mathbf{M}-\mathbf{I}\lambda_k| = 0$: this is called the *secular equation.* It provides a set of equations for the eigenvalues λ_k. Given these, the solutions to $(\mathbf{M}-\mathbf{I}\lambda_k)\vec{e}_k = 0$ can be found, which yield the eigenvectors. If we construct a matrix $S_{ik} = (e_i)_k$ (that is, with the kth column of **S** equal to the kth eigenvector), then the eigenvector equation becomes

$$\mathbf{M}\vec{e}_k = \lambda_k \vec{e}_k \Rightarrow \mathbf{MS} = \mathbf{S\Lambda} \Rightarrow \mathbf{S}^{-1}\mathbf{MS} = \mathbf{\Lambda}$$

which shows that, if we find the eigenvalues and eigenvectors of **M,** we will know the normal mode frequencies, and the transformation from the (x,y) to the normal modes.

Let's carry this out for the two masses above. Solving the secular equation gives

$$\lambda_1 = \frac{1}{2}\left[\omega_2^2 + \omega_1^2 - \sqrt{4q^4 + \left(\omega_1^2 - \omega_2^2\right)}\right]$$

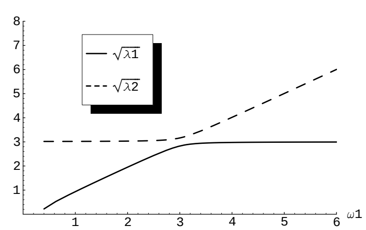
$$\lambda_2 = \frac{1}{2}\left[\omega_2^2 + \omega_1^2 + \sqrt{4q^4 + \left(\omega_1^2 - \omega_2^2\right)}\right]$$

The eigenvectors determine the matrix **S**. In terms of this matrix, the conversion from normal modes to (x,y) is

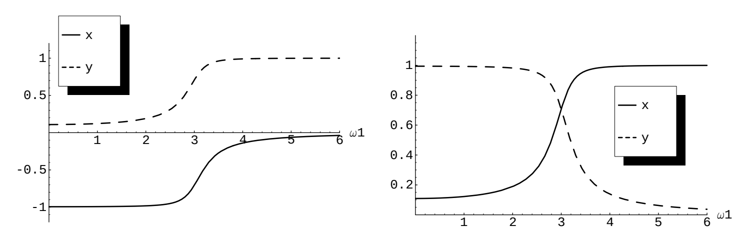
$$\zeta_1 = A_1 \exp\left(i\sqrt{\lambda_1}t + \phi_1\right) = S_{11}^{-1}x + S_{12}^{-1}y$$

$$\zeta_2 = A_2 \exp\left(i\sqrt{\lambda_2}t + \phi_2\right) = S_{21}^{-1}x + S_{22}^{-1}y$$

The following figures show a numerical example, for ω_2 = 3 Hz, q=1 Hz. The normal mode frequencies, and the coupling matrix elements, are plotted as a function of ω_1 (in Hz).



For $\omega_1 < \omega_2$, $\sqrt{\lambda_1} \approx \omega_1$ and $\sqrt{\lambda_2} \approx \omega_2$. The situation is reversed for $\omega_1 > \omega_2$. Note that the normal mode frequencies are never equal, even for $\omega_1 = \omega_2$. Coupling: left, mode 1; right, mode 2



For $\omega_1 < \omega_2$, $\zeta_1 \approx -x$, $\zeta_2 \approx y$**;** For $\omega_1 > \omega_2$, $\zeta_1 \approx y$, $\zeta_2 \approx x$**.** For $\omega_1 = \omega_2$, $\zeta_1 \approx \frac{1}{\sqrt{2}}(y-x)$, $\zeta_2 \approx \frac{1}{\sqrt{2}}(y+x)$**.**

Equations of linear coupling

With this introduction to coupled oscillators, let's see how the coupled trajectory equations of motion can be understood in terms of coupled oscillators. For simplicity, we're only going to consider the coupling resulting from a single skew quadrupole, located at a position s_0 in the ring. This simplifies the math, while maintaining the essential physical features of coupling.

From Lecture 3, p 9: the coupling produced by a skew quadrupole is given by

$$x'' = y\tilde{k}; \qquad y'' = x\tilde{k}$$

in which $\tilde{k} = \dfrac{\tilde{B}'}{B_0\rho}$ is the skew quadrupole strength. For a thin lens of length L_s, these equations become

$$\Delta x' = y\tilde{k}L_s = \frac{y}{\tilde{f}} \qquad \Delta y' = x\tilde{k}L_s = \frac{x}{\tilde{f}}$$

in which $\tilde{f} = \dfrac{1}{\tilde{k}L_s}$ is the skew quad focal length. In Floquet coordinates, using $\Delta x' = \dfrac{\Delta\dot{\xi}_x}{Q_x\sqrt{\beta_x(s_0)}}$, $\Delta y' = \dfrac{\Delta\dot{\xi}_y}{Q_y\sqrt{\beta_y(s_0)}}$, gives

$$\Delta\dot{\xi}_x = Q_x\kappa\xi_y \qquad \Delta\dot{\xi}_y = Q_y\kappa\xi_x$$

in which $\kappa = \dfrac{\sqrt{\beta_x(s_0)\beta_y(s_0)}}{\tilde{f}}$ is a measure of the coupling.

At s_0, we can write the Floquet coordinates in phase-amplitude form as

$$\xi_x = r_x\cos\phi_x \qquad \xi_y = r_y\cos\phi_y$$

$$\dot{\xi}_x = -Q_x r_x\sin\phi_x \quad \dot{\xi}_y = -Q_y r_y\sin\phi_y$$

So we have for the changes in the Floquet coordinates

$$\Delta\dot{\xi}_x = Q_x\kappa r_y\cos\phi_y \qquad \Delta\dot{\xi}_y = Q_y\kappa r_x\cos\phi_x$$

We want to get equations entirely in terms of the phase-amplitude variables. Since

$$r^2 = \xi^2 + \left(\frac{\dot{\xi}}{Q}\right)^2, \text{ we have } \quad \Delta r^2 = \frac{2\dot{\xi}\Delta\dot{\xi}}{Q^2} = -\frac{2r\sin\phi\Delta\dot{\xi}}{Q}, \text{ so}$$

$$\Delta r_x^2 = -2r_x r_y\kappa\sin\phi_x\cos\phi_y$$

$$\Delta r_y^2 = -2r_x r_y\kappa\sin\phi_y\cos\phi_x$$

For the changes in the phase, we have

$$\tan\phi = -\frac{\dot{\xi}}{Q\xi} \Rightarrow \Delta\phi = -\cos^2\phi\frac{\Delta\dot{\xi}}{Q\xi} = -\cos\phi\frac{\Delta\dot{\xi}}{Qr}, \text{ so}$$

$$\Delta\phi_x = -\frac{r_y}{r_x}\kappa\cos\phi_x\cos\phi_y$$

$$\Delta\phi_y = -\frac{r_x}{r_y}\kappa\cos\phi_x\cos\phi_y$$

Now we've assumed only one coupling element in the ring. For the motion in all the rest of the ring, r_x and r_y do not change, and ϕ_x and

ϕ_y advance by $2\pi Q_x$ and $2\pi Q_y$ respectively. So, we can write differential equations for the changes of the phase and amplitude per turn (n=turn number)

$$\frac{dr_x^2}{dn} = -2 r_x r_y \kappa \sin\phi_x \cos\phi_y$$

$$\frac{dr_y^2}{dn} = -2 r_x r_y \kappa \sin\phi_y \cos\phi_x$$

$$\frac{d\phi_x}{dn} = 2\pi Q_x - \frac{r_y}{r_x}\kappa \cos\phi_x \cos\phi_y$$

$$\frac{d\phi_y}{dn} = 2\pi Q_y - \frac{r_x}{r_y}\kappa \cos\phi_x \cos\phi_y$$

Difference resonances

We now expand the trigonometric functions to identify the resonant coupling terms:

$$\sin\phi_x \cos\phi_y = \frac{1}{2}\left[\sin\left(\phi_x - \phi_y\right) + \sin\left(\phi_x + \phi_y\right)\right]$$

If $Q_x \approx Q_y$, then the first term is slowly varying and can drive resonant coupling. This condition is referred to as the *difference resonance.* In this case, we can neglect the second term, which will be rapidly varying, and the coupled equations of motion become

$$\frac{dr_x}{dn} = -\frac{r_y}{2}\kappa \sin\left(\phi_x - \phi_y\right)$$

$$\frac{dr_y}{dn} = \frac{r_x}{2}\kappa \sin\left(\phi_x - \phi_y\right)$$

$$\frac{d\phi_x}{dn} = 2\pi Q_x - \frac{r_y}{2r_x}\kappa \cos\left(\phi_x - \phi_y\right)$$

$$\frac{d\phi_y}{dn} = 2\pi Q_y - \frac{r_x}{2r_y}\kappa \cos\left(\phi_x - \phi_y\right)$$

Combining the first two equations gives

$$\frac{1}{r_y}\frac{dr_x}{dn} = -\frac{1}{r_x}\frac{dr_y}{dn} \Rightarrow \frac{dr_x^2}{dn} + \frac{dr_y^2}{dn} = 0$$

$$r_x^2 + r_y^2 = a_x^2 + a_y^2 = \text{constant}$$

Although there can be exchange of motion from one plane to another, the motion is bounded and thus stable. To see what the motion looks like, we must solve both sets of equations. To do this, we make a change of variables to

$$\phi_x' = \phi_x - \pi\left(Q_x + Q_y\right)n$$

$$\phi_y' = \phi_y - \pi\left(Q_x + Q_y\right)n$$

This is a change of variables to a rotating coordinate system, rotating with a frequency corresponding to the tune $\frac{Q_x+Q_y}{2}$. The equations of motion become

$$\frac{dr_x}{dn}=-\frac{r_y}{2}\kappa\sin\left(\phi_x'-\phi_y'\right)$$
$$\frac{dr_y}{dn}=\frac{r_x}{2}\kappa\sin\left(\phi_x'-\phi_y'\right)$$
$$\frac{d\phi_x'}{dn}=\pi\delta Q-\frac{r_y}{2r_x}\kappa\cos\left(\phi_x'-\phi_y'\right)$$
$$\frac{d\phi_y'}{dn}=-\pi\delta Q-\frac{r_x}{2r_y}\kappa\cos\left(\phi_x'-\phi_y'\right)$$

in which $\delta Q=Q_x-Q_y$. These can be reduced to the form of two complex linear coupled first order equations with the substitution

$$w_x=r_x\exp\left(i\phi_x'\right)\quad w_y=r_y\exp\left(i\phi_y'\right)$$

The result is the pair of complex equations

$$\frac{dw_x}{dn}=i\left(\delta Q\pi w_x-\frac{\kappa}{2}w_y\right)$$
$$\frac{dw_y}{dn}=-i\left(\delta Q\pi w_y+\frac{\kappa}{2}w_x\right)$$

We can now solve these equations just as we did for the coupled harmonic oscillators earlier in the lecture. Written as a matrix equation, the pair of equations above is

$$\frac{d\vec{w}}{dn}+i\pi\delta Q\mathbf{M}\vec{w}=0,\qquad \mathbf{M}=\begin{pmatrix}-1 & \varepsilon\\ \varepsilon & 1\end{pmatrix}$$

in which $\varepsilon=\frac{\kappa}{2\pi\delta Q}$

The normal modes of the motion will be given by $\vec{\zeta}$, where

$$\vec{w}=\mathbf{S}\vec{\zeta}$$

The equation of motion of the normal modes is

$$\frac{d\vec{\zeta}}{dn}+i\pi\delta Q\mathbf{S}^{-1}\mathbf{M}\mathbf{S}\vec{\zeta}=\frac{d\vec{\zeta}}{dn}+i\pi\delta Q\mathbf{\Lambda}\vec{\zeta}=0$$

in which the matrix $\mathbf{S}^{-1}\mathbf{M}\mathbf{S}=\mathbf{\Lambda}=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$ is diagonal.

The solutions are
$$\zeta_1(n)=\zeta_{10}\exp\left(i\pi\delta Q\lambda_1 n\right)$$
$$\zeta_2(n)=\zeta_{20}\exp\left(i\pi\delta Q\lambda_2 n\right)$$

The normal modes frequencies and the normal modes are the eigenvalues and the eigenvectors of the matrix **M**. Since the matrix is pretty simple, the eigenvalues and eigenvectors are relatively simple also. The eigenvalues are

$$\lambda_1=-\sqrt{1+\varepsilon^2}\quad \lambda_2=\sqrt{1+\varepsilon^2}$$

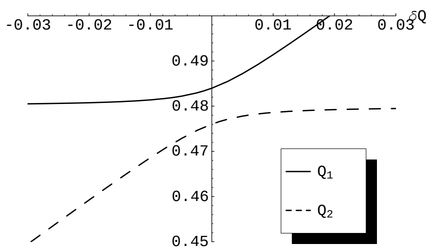
so the normal mode solutions are

$$\zeta_1(n)=\zeta_{10}\exp\left(-i\pi\delta Q\sqrt{1+\varepsilon^2}n\right)$$
$$\zeta_2(n)=\zeta_{20}\exp\left(i\pi\delta Q\sqrt{1+\varepsilon^2}n\right)$$

These solutions have been found in the rotating coordinate system. Therefore, we have to add back $\pi\left(Q_x + Q_y\right)n$ to get the motion in the usual phase space. The tunes of the normal modes are then

$$Q_{1,2} = Q_x + \frac{\delta Q}{2} \pm \frac{1}{4\pi}\sqrt{\kappa^2 + 4\pi^2 \delta Q^2}$$

Example: $Q_x = 0.48$, κ=0.05



The tune split between the two normal modes is

$$Q_1 - Q_2 = \frac{1}{2\pi}\sqrt{\kappa^2 + (2\pi\delta Q)^2}$$

The minimum tune split, on the difference resonance at δQ=0, is

$$\left(Q_2 - Q_1\right)_{\min} = \frac{\kappa}{2\pi} = \frac{\sqrt{\beta_x \beta_y}}{2\pi\tilde{f}}$$

Example: 1 mrad rotation of CESR permanent magnet quadrupole: focal length 0.8 m: 1 mrad roll=> $\tilde{f} = 0.8/0.002 = 400$ m. β_x = 10 m, β_y = 84 m. Then

$$\Delta Q_{\min} = \frac{\sqrt{84 \times 10}}{2\pi \times 400} = 0.011 .$$

Normal mode orientations:

$\vec{\zeta} = \mathbf{S}^{-1}\vec{w}$ implies that

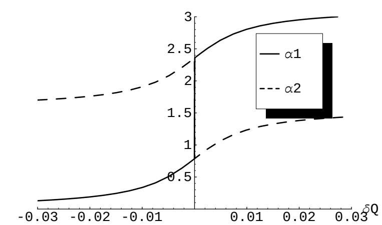
$$\zeta_1 = S_{11}^{-1} w_x + S_{12}^{-1} w_y = w_x \cos\alpha_1 + w_y \sin\alpha_1$$

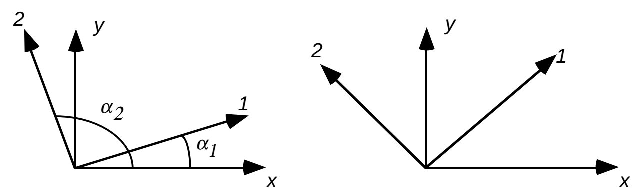
$$\zeta_2 = S_{21}^{-1} w_x + S_{22}^{-1} w_y = w_x \cos\alpha_2 + w_y \sin\alpha_2$$

These angles are given by from the eigenvectors by

$$\tan\alpha_1 = \frac{1 - \sqrt{1+\varepsilon^2}}{\varepsilon} \quad \tan\alpha_2 = \frac{1 + \sqrt{1+\varepsilon^2}}{\varepsilon}$$

The following is a plot of the angles, for κ=0.05, as a function of δQ:





Left: δQ<0. Right: on resonance (for $\delta Q \le 0$).

Emittance exchange:

The amplitude of the motion in the *x* and *y* planes oscillates:

The transformation from *w* to ζ is given by $\vec{w} = \mathbf{S}\vec{\zeta}$. In terms of the angles α_1 and α_2 defined above,

$$\mathbf{S} = \operatorname{sgn}\varepsilon \begin{pmatrix} -\sin\alpha_2 & \sin\alpha_1 \\ \cos\alpha_2 & -\cos\alpha_1 \end{pmatrix}$$

$$r_x^2 = |w_x|^2 = |-\sin\alpha_2\zeta_1 + \sin\alpha_1\zeta_2|^2$$

$$r_y^2 = |w_y|^2 = |\cos\alpha_2\zeta_1 - \cos\alpha_1\zeta_2|^2$$

Using $\zeta_i(n) = \zeta_{i0}\exp\left(\mp i\pi\delta Q\sqrt{1+\varepsilon^2}n\right)$, (i=1,2), and simplifying to the case of real ζ_{i0}, we have

$$r_x^2 = \zeta_{20}^2\sin^2\alpha_1 + \zeta_{10}^2\sin^2\alpha_2$$
$$-2\zeta_{10}\zeta_{20}\sin\alpha_1\sin\alpha_2\cos\left(2\pi\delta Q\sqrt{1+\varepsilon^2}n\right)$$
$$r_y^2 = \zeta_{20}^2\cos^2\alpha_1 + \zeta_{10}^2\cos^2\alpha_2$$
$$-2\zeta_{10}\zeta_{20}\cos\alpha_1\cos\alpha_2\cos\left(2\pi\delta Q\sqrt{1+\varepsilon^2}n\right)$$

So that the amplitude squared is modulated with a period of

$n_\kappa = \dfrac{1}{\delta Q\sqrt{1+\varepsilon^2}} \approx \dfrac{2\pi}{\kappa}$ near the resonance. This corresponds to about 100 turns in our previous example.

Suppose that we start a betatron oscillation with amplitude in the x-plane only. Then $w_x = w_{x0},\quad w_y = 0$. From the equations on p. 23, the initial values of the normal modes are

$$\zeta_{10} = w_{x0}\cos\alpha_1 \qquad \zeta_{20} = w_{x0}\cos\alpha_2$$

Plugging these values into the equations for the emittances, and using the expressions given above for the angles in terms of ε, gives

$$r_x^2 = w_{x0}^2\,\frac{2+\varepsilon^2\left(1+\cos\left(2\pi\delta Q\sqrt{1+\varepsilon^2}n\right)\right)}{2\left(1+\varepsilon^2\right)}$$

$$r_y^2 = w_{x0}^2\,\frac{\varepsilon^2}{1+\varepsilon^2}\sin^2\left(\pi\delta Q\sqrt{1+\varepsilon^2}n\right)$$

Thus, x-amplitude appears as y-motion; the peak value of the y-emittance is

$$\frac{\varepsilon_y}{\varepsilon_x} = \frac{r_y^2}{w_{x0}^2} = \frac{\varepsilon^2}{1+\varepsilon^2} = \frac{\kappa^2}{\kappa^2 + (2\pi\delta Q)^2}$$

For example, with κ=0.05, and δQ=0.003, we have

$$\frac{\varepsilon_y}{\varepsilon_x} \approx 0.066$$

This would be unacceptably large for an electron-positron collider operating with flat beams.

Sum resonances

Returning to our trig expansion

$$\sin\phi_x \cos\phi_y = \frac{1}{2}\left[\sin\left(\phi_x - \phi_y\right) + \sin\left(\phi_x + \phi_y\right)\right]$$

If $Q_x \approx m - Q_y$, then the second term is slowly varying and can drive resonant coupling. This condition is referred to as the *sum resonance*. In this case, we can drop the first term, which will be rapidly varying, and the coupled equations of motion become

$$\frac{dr_x}{dn} = -\frac{r_y}{2}\kappa\sin\left(\phi_x + \phi_y\right)$$

$$\frac{dr_y}{dn} = -\frac{r_x}{2}\kappa\sin\left(\phi_x + \phi_y\right)$$

$$\frac{d\phi_x}{dn} = 2\pi Q_x - \frac{r_y}{2r_x}\kappa\cos\left(\phi_x + \phi_y\right)$$

$$\frac{d\phi_y}{dn} = 2\pi Q_y - \frac{r_x}{2r_y}\kappa\cos\left(\phi_x + \phi_y\right)$$

Combining the first two equations gives

$$\frac{1}{r_y}\frac{dr_x}{dn} = \frac{1}{r_x}\frac{dr_y}{dn} \Rightarrow \frac{dr_x^2}{dn} - \frac{dr_y^2}{dn} = 0$$

$$r_x^2 - r_y^2 = a_x^2 - a_y^2 = \text{constant}$$

In this case, the amplitude of the motion is unbounded, as both r_x and r_y may grow, provided their difference is bounded. Unstable motion can result from a sum resonance. A complete analysis shows that one of the eigenmodes will have an n dependence of the form

$$\exp\left(-i\pi n\sqrt{(\delta Q)^2 - \left(\frac{\kappa}{2\pi}\right)^2}\right),$$

with $\delta Q = m - \left(Q_x + Q_y\right)$. If $\delta Q < \frac{\kappa}{2\pi}$, this gives exponential growth, independent of the initial amplitude: all particles are unstable. This is the stopband width of the linear sum resonance.