LECTURE 17

Linear coupling (continued)

Coupling coefficients for distributions of skew quadrupoles and solenoids

Pretzel Orbits

Motivation and applications

Implications

Linear coupling (continued)

Coupling coefficients for distributions of skew quadrupoles and solenoids

The previous discussion focused on a single skew quadrupole, for simplicity. Actual machines typically have a distribution of skew quadrupoles, and also may include solenoids. The axial solenoid field couples to the slope of the trajectory; the end fields couple to the trajectory itself: (c.f., Lecture 3, p 10:)

$$x'' = \left(\frac{T'}{2}y + Ty'\right); \quad y'' = -\left(\frac{T'}{2}x + Tx'\right)$$

$$T = \frac{B_s}{B_0\rho}, \;\; T' = \frac{1}{B_0\rho}\frac{\partial B_s}{\partial s}$$

Let's see how to calculate the coupling coefficient for an arbitrary distribution of skew quadrupole and solenoid strength around the ring.

We'll call the location at which we want to evaluate the trajectories s=0. At some other point in the ring, s', let the skew quadrupole strength be $\tilde{k}(s')$, and the solenoid strength $T(s')$. For the moment, we assume that this is the only point of coupling in the ring. At the end of the discussion, we'll integrate over the whole ring to get the result for a distribution of strengths.

The incremental kick delivered to a trajectory at this point by these fields, which extend a distance $\Delta s'$, is

$$\frac{\Delta x}{\Delta s'} = \frac{1}{2}Ty \qquad \frac{\Delta x'}{\Delta s'} = y\tilde{k} + \frac{1}{2}Ty'$$

$$\frac{\Delta y}{\Delta s'} = -\frac{1}{2}Tx \qquad \frac{\Delta y'}{\Delta s'} = x\tilde{k} - \frac{1}{2}Tx'$$

In Floquet coordinates, we have

$$\frac{\Delta\dot{\xi}_x - \alpha_x Q_x \Delta\xi_x}{\Delta s' Q_x\sqrt{\beta_x}} = y\tilde{k} + \frac{1}{2}Ty' = \tilde{k}\xi_y\sqrt{\beta_y} + \frac{1}{2}T\frac{\dot{\xi}_y - \alpha_y Q_y \xi_y}{Q_y\sqrt{\beta_y}}$$

$$\frac{\Delta\xi_x}{\Delta s'} = \frac{1}{2}T\xi_y\sqrt{\frac{\beta_y}{\beta_x}}$$

This gives

$$\Delta\dot{\xi}_x = \kappa_{1x} Q_x \xi_y + \kappa_{2x} \frac{Q_x}{Q_y} \dot{\xi}_y, \quad \Delta\xi_x = \kappa_{3x}\xi_y$$

$$\kappa_{1x} = \left(\sqrt{\beta_x \beta_y}\tilde{k} - \frac{\alpha_y T}{2}\sqrt{\frac{\beta_x}{\beta_y}} + \frac{\alpha_x T}{2}\sqrt{\frac{\beta_y}{\beta_x}} \right)\Delta s'$$

$$\kappa_{2x} = \frac{T}{2}\sqrt{\frac{\beta_x}{\beta_y}}\Delta s' \quad \kappa_{3x} = \frac{T}{2}\sqrt{\frac{\beta_y}{\beta_x}}\Delta s'$$

in which everything is evaluated at the point s'.

There are similar equations for *y*, in which *y* and *x* are interchanged, and *T*->-*T*.

Now consider a trajectory which starts at s=0, with phase space coordinates

$$\xi_x(0) = r_x \cos\phi_x \quad \dot{\xi}_x(0) = -Q_x r_x \sin\phi_x$$
$$\xi_y(0) = r_y \cos\phi_y \quad \dot{\xi}_y(0) = -Q_y r_y \sin\phi_y$$

at that point. It travels to s', at which the betatron phase is $\Phi' = \Phi(s')$. The phase space coordinates there are

$$\xi_x(s') = r_x \cos(\phi_x + \Phi'_x) \quad \dot{\xi}_x(s') = -Q_x r_x \sin(\phi_x + \Phi'_x)$$
$$\xi_y(s') = r_y \cos(\phi_y + \Phi'_y) \quad \dot{\xi}_y(s') = -Q_y r_y \sin(\phi_y + \Phi'_y)$$

The changes in the Floquet coordinates at this point are then

$$\Delta\dot{\xi}_x = \kappa_{1x} Q_x r_y \cos(\phi_y + \Phi'_y) - \kappa_{2x} Q_x r_y \sin(\phi_y + \Phi'_y),$$
$$\Delta\xi_x = \kappa_{3x} r_y \cos(\phi_y + \Phi'_y)$$
$$\Delta\dot{\xi}_y = \kappa_{1y} Q_y r_x \cos(\phi_x + \Phi'_x) - \kappa_{2y} Q_y r_x \sin(\phi_x + \Phi'_x),$$
$$\Delta\xi_y = \kappa_{3y} r_x \cos(\phi_x + \Phi'_x)$$

We then continue from this point to *s*=*C*, where we started. The Floquet coordinates at *s*=*C* are given by

$$\begin{pmatrix} \xi_x \\ \dot{\xi}_x \end{pmatrix}_C = \begin{pmatrix} \cos(2\pi Q_x - \Phi'_x) & \dfrac{\sin(2\pi Q_x - \Phi'_x)}{Q_x} \\ -Q_x \sin(2\pi Q_x - \Phi'_x) & \cos(2\pi Q_x - \Phi'_x) \end{pmatrix} \begin{pmatrix} \xi_x(s') + \Delta\xi_x \\ \dot{\xi}_x(s') + \Delta\dot{\xi}_x \end{pmatrix}$$

with a similar equation for *y*.

We then calculate the changes in the phase-amplitude variables over the turn, using

$$\frac{dr_x^2}{dn} = \xi_x(C)^2 - \xi_x(0)^2 + \frac{\dot{\xi}_x(C)^2 - \dot{\xi}_x(0)^2}{Q_x^2}$$

$$\frac{d\phi_x}{dn} = \tan^{-1}\frac{Q_x \xi_x(C)}{\dot{\xi}_x(C)} - \tan^{-1}\frac{Q_x \xi_x(0)}{\dot{\xi}_x(0)}$$

In the following results, the parameters κ are assumed to be small, so only the linear terms are retained. The trigonometric functions have also been expanded, only terms driving the difference resonance have been retained, and the change of variables to the rotating coordinate system has been made. The resulting equations are

$$\frac{dr_x}{dn} = -\frac{r_y}{2}\left(\vartheta_{x1}\cos\left(\phi'_x - \phi'_y\right) + \vartheta_{x2}\sin\left(\phi'_x - \phi'_y\right)\right)$$

$$\frac{d\phi'_x}{dn} = \pi\delta Q - \frac{r_y}{2r_x}\left(\vartheta_{x2}\cos\left(\phi'_x - \phi_y\right) - \vartheta_{x1}\sin\left(\phi'_x - \phi'_y\right)\right)$$

$$\vartheta_{x1} = \kappa_{1x}\sin\left(\Phi'_x - \Phi'_y\right) - \left(\kappa_{2x} + \kappa_{3x}\right)\cos\left(\Phi'_x - \Phi'_y\right)$$

$$\vartheta_{x2} = \kappa_{1x}\cos\left(\Phi'_x - \Phi'_y\right) + \left(\kappa_{2x} + \kappa_{3x}\right)\sin\left(\Phi'_x - \Phi'_y\right)$$

with $\delta Q = Q_x - Q_y$. There are similar equations for y, obtained by interchanging x and y .

At this point, we can generalize to a distribution of skew quadrupole and sextupole strength around the ring. The above equations give the contribution from an element of length $\Delta s'$ at s':

for a distribution of skew quadrupole and solenoid strength around the ring, we make the following replacements:

$$\vartheta_{x1} \to \int_0^C ds' \left(\sqrt{\beta_x\beta_y}\tilde{k} - \frac{\alpha_y T}{2}\sqrt{\frac{\beta_x}{\beta_y}} + \frac{\alpha_x T}{2}\sqrt{\frac{\beta_y}{\beta_x}}\right)\times$$

$$\sin\left(\Phi_x(s') - \Phi_y(s')\right) - \frac{T}{2}\left(\sqrt{\frac{\beta_x}{\beta_y}} + \sqrt{\frac{\beta_y}{\beta_x}}\right)\cos\left(\Phi_x(s') - \Phi_y(s')\right)$$

$$\vartheta_{x2} \to \int_0^C ds \left(\sqrt{\beta_x\beta_y}\tilde{k} - \frac{\alpha_y T}{2}\sqrt{\frac{\beta_x}{\beta_y}} + \frac{\alpha_x T}{2}\sqrt{\frac{\beta_y}{\beta_x}}\right)\times$$

$$\cos\left(\Phi_x(s') - \Phi_y(s')\right) + \frac{T}{2}\left(\sqrt{\frac{\beta_x}{\beta_y}} + \sqrt{\frac{\beta_y}{\beta_x}}\right)\sin\left(\Phi_x(s') - \Phi_y(s')\right)$$

As before, we now introduce the complex quantities

$$w_x = r_x\exp\left(i\phi'_x\right) \quad w_y = r_y\exp\left(i\phi'_y\right)$$

The result is the pair of complex equations

$$\frac{dw_x}{dn} = i\left(\delta Q\pi w_x + \frac{i\vartheta_{x1} - \vartheta_{x2}}{2}w_y\right)$$

$$\frac{dw_y}{dn} = -i\left(\delta Q\pi w_y - \frac{i\vartheta_{y1} - \vartheta_{y2}}{2}w_x\right)$$

The equivalent matrix equation is

$$\frac{d\vec{w}}{dn} + i\pi\delta Q\mathbf{M}\vec{w} = 0, \qquad \mathbf{M} = \begin{pmatrix} -1 & \dfrac{\vartheta_{x2} - i\vartheta_{x1}}{2\delta Q\pi} \\ \dfrac{\vartheta_{y2} - i\vartheta_{y1}}{2\delta Q\pi} & 1 \end{pmatrix}$$

Substituting from above, we can write the matrix as

$$\mathbf{M} = \begin{pmatrix} -1 & \dfrac{\Theta}{2\delta Q\pi} \\ \dfrac{\Theta^*}{2\delta Q\pi} & 1 \end{pmatrix}$$

in which

$$\Theta = \int_0^C ds\exp\left(-i\left(\Phi_x - \Phi_y\right)\right)\begin{pmatrix} \sqrt{\beta_x\beta_y}\tilde{k} + \dfrac{T}{2}\left[\alpha_x\sqrt{\dfrac{\beta_y}{\beta_x}} - \alpha_y\sqrt{\dfrac{\beta_x}{\beta_y}}\right] + \\ i\dfrac{T}{2}\left[\sqrt{\dfrac{\beta_y}{\beta_x}} + \sqrt{\dfrac{\beta_x}{\beta_y}}\right] \end{pmatrix}$$

The eigenvalues are

$$\lambda_1 = -\sqrt{1+\varepsilon^2} \qquad \lambda_2 = \sqrt{1+\varepsilon^2}$$

in which

$$\varepsilon = \frac{|\Theta|}{2\pi\delta Q}$$

The minimum tune split, on the difference resonance, is

$$(Q_2 - Q_1)_{\min} = \frac{|\Theta|}{2\pi}$$

Correction of coupling.

For a difference resonance corresponding to $Q_x - Q_y = m + \delta Q$, we can approximate

$$\Phi_x(s') - \Phi_y(s') \approx \left(Q_x\theta - Q_y\theta\right) = (m + \delta Q)\theta$$

in which $\theta = \frac{2\pi s}{C}$ is the azimuthal angle. Then, for small δQ, the coupling coefficients become

$$\Theta \cong \int_0^C ds \exp\left(-im\frac{2\pi s}{C}\right)\left(\begin{array}{l}\sqrt{\beta_x\beta_y}\tilde{k} + \frac{T}{2}\left[\alpha_x\sqrt{\frac{\beta_y}{\beta_x}} - \alpha_y\sqrt{\frac{\beta_x}{\beta_y}}\right] + \\ i\frac{T}{2}\left[\sqrt{\frac{\beta_y}{\beta_x}} + \sqrt{\frac{\beta_x}{\beta_y}}\right]\end{array}\right)$$

The coefficients which drive the $Q_x - Q_y = m$ difference resonance are the mth Fourier components of the coupling strength.

To correct a general set of coupling errors, at least two correctors are needed, to cancel the two Fourier harmonics (real and

imaginary parts of Θ). If the coupling errors and the lattice functions have superperiodicity N, this will suppress Fourier harmonics which do not satisfy $m = jN$, for integral j.

Pretzel Orbits

Motivation and applications

The term "pretzel orbits" refers to the deliberate introduction of closed orbit distortions, through the use of electric fields, in order to provide orbit separation at undesired collision points in multiple bunch particle-antiparticle colliders.

Pretzel orbits were invented and first developed at CESR. They are now in use here, and also in LEP at CERN, and in the Tevatron at Fermilab, to allow multiple bunch operation and higher luminosity.

Why do more bunches give higher luminosity?

Recall, Lecture 1, p 38, luminosity formula:

$$\mathsf{L} = f_c \frac{N_b^2}{4\pi\sigma^2}$$

Here N_b=number of particles per colliding bunch, and f_c=collision frequency. If there are B bunches per species, then $f_c = fB$, where f is the revolution frequency, and so

$$\mathsf{L} = f\frac{BN_b^2}{4\pi\sigma^2}$$

If there is some limit on N_b (e.g, the beam-beam limit, which is proportional to N_b), then more bunches will give more luminosity.

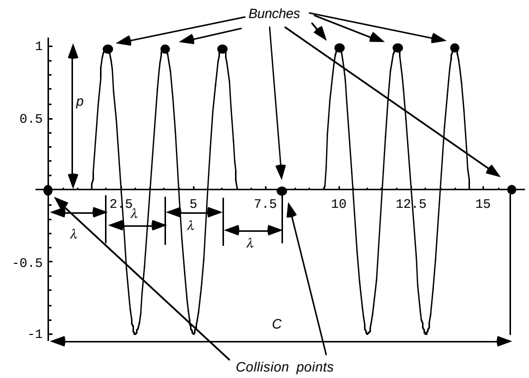
If, however, I can make N_b as big as I want, but have a fixed total number of particles $N = BN_b$, then I can write

$$\mathcal{L} = f\,\frac{1}{B}\frac{\left(BN_b\right)^2}{4\pi\sigma^2} = \frac{f}{B}\frac{N^2}{4\pi\sigma^2}$$

and I want to make B as small as I can (i.e., 1) to maximize luminosity.

The typical situation in particle-antiparticle colliders is operation at the beam-beam limit, and we want to have as many bunches as possible. However, B bunches have $2B$ collision points, while typically there are only one or two detectors. At each collision point, we suffer from the beam-beam interaction, so we want to minimize the number of collision points. Thus, we want to separate the bunches everywhere in the machine, so they do not collide,

except at the collision points where we have detectors. This is the purpose of "pretzel orbits".



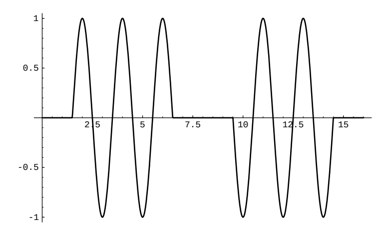
The figure above illustrates a possible ideal realization of the basic idea, providing two collision points with 8 bunches. Two closed orbit distortions are generated, of wavelength λ and amplitude *p*.

The bunch spacing is equal to λ. The bunches are arranged as shown, so that while two are at the collision points, the others are at the pretzel antinodes. The orbit distortion is generated using electric fields (typically electrostatic separators), so that the oppositely charged, counter-rotating bunches follow an orbit with the opposite sign. The bunches passing at the pretzel antinodes are separated by a separation 2*p*, while those at the collision points collide.

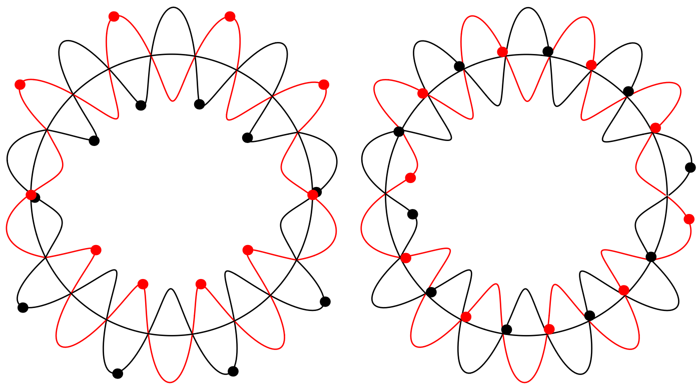
The scheme accommodates $B = C/\lambda$ bunches, where λ is the betatron wavelength. Since $Q \approx C/\lambda$, the value of the tune sets the maximum number of bunches.

This limitation has been overcome at CESR and LEP by using *trains* of bunches, with a spacing much smaller than λ. The trains must be short enough to fit in the region of pretzel antinode. A small *crossing angle* is introduced in the straight sections to prevent undesired collisions for bunches in a single train.

The pretzel shown above is symmetric about each collision point. An antisymmetric pretzel is also possible, and in fact desireable:

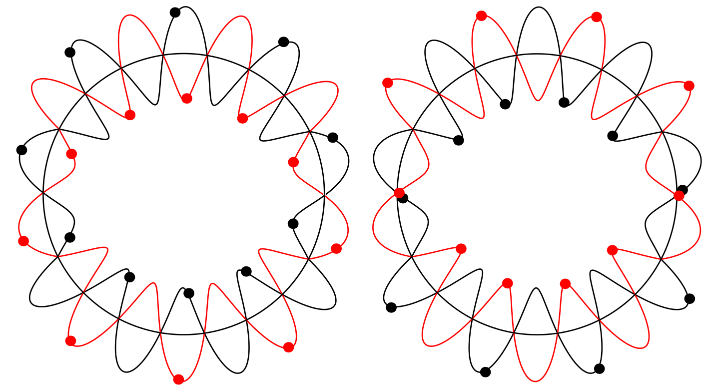


The following figures illustrate the orbit separation scheme. (Animations of these figures are available in the animations folder).



Left: collisions at two points, other bunches at pretzel antinodes

Right: after collision, most bunches at pretzel nodes.



Left: All bunches near pretzel antinodes

Right: two collisions, other bunches at pretzel antinodes.

Implications:

There are a number of issues associated with pretzel orbit operation.

- Long-range beam-beam collisions. The long-range collisions cause closed orbit errors, tune shifts, beta function distortion, and resonance excitation. The need to limit these effects sets the size of the pretzel amplitude p, upon which all other effects depend.
- Aperture. The deformed orbits, plus betatron oscillations around them, must fit into the good field region of the magnet apertures.

- Pretzel closure. If the orbit deformation “leaks” into the collision regions, the colliding bunches may fail to collide head-on, or even miss each other.
- Dispersion. The deformed closed orbit generates dispersion; this will be vertical dispersion if the pretzel is vertical, and will contribute to quantum excitation of the vertical emittance in an electron machine.
- Path length changes. The path length on the deformed orbit will change. This can result in an energy difference between the colliding beams.
- Sextupole effects:

If the pretzel is horizontal: The closed orbit deformation in the sextupoles causes horizontal dipole errors, which will modify the closed orbit. It also causes quadrupole errors in both planes, which in turn result in tune shifts, beta function distortion, and second order resonance enhancement.

If the pretzel is vertical: The closed orbit deformation in the sextupoles causes horizontal dipole errors, and skew quadrupole errors in both planes, which increases the coupling.

- Particle-antiparticle energy differences: If the pretzel is present in the rf cavities, and the rf field varies with position, there may be energy differences between the two beams.

- Nonlinear resonances from field errors. The large amplitude excursions of the beams may allow them to enter nonlinear field regions, increasing the sensitivity to resonances.

- Injection. During the damped betatron oscillations which occur after injection, the separation between the bunches may be reduced, potentially leading to beam loss.

- Electrostatic separators. The requirements on these devices are challenging. In addition to having to provide high electric fields (typically > 100 kV/cm), for high current electron-positron machines, they must have low impedance. For proton-antiproton colliders, they must be very reliable, as sparks often cause loss of the stored beam.

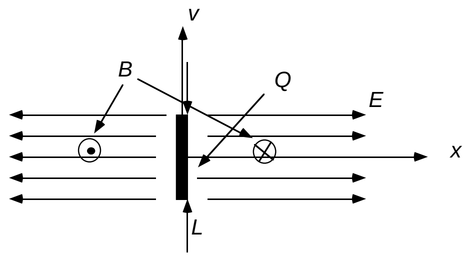
Machines that operate with flat beams must strictly limit the amount of vertical dispersion and coupling, in order to minimize the vertical emittance. Vertical pretzel closure errors at the collision point are also very damaging, because of the small

vertical beam size. Hence, electron colliders typically choose the pretzel to be in the horizontal plane.

Let's examine some of these effects quantitatively, for the case of horizontally separated orbits.

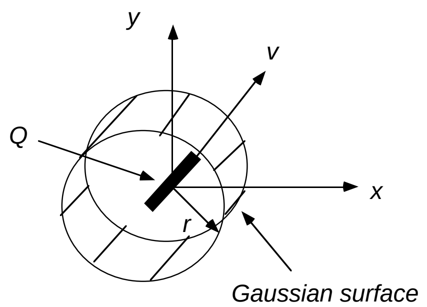
Long-range beam-beam collisions.

To estimate the effect of these collisions, we need to know the fields produced by a bunch. Imagine the bunch to have a length L along the direction of motion. We will be seeking the "long-range" fields, at a distance from the bunch large compared to its transverse size. So, we imagine the bunch to have a very small transverse size.



The bunch is taken to be composed of ultra-relativistic point charges, which have "flattened" fields that are directed perpendicular to the direction of motion (see figure above).

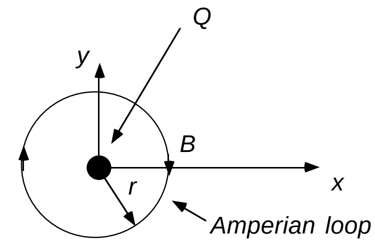
To find the electric field at a point a distance r from the bunch, we surround the bunch with a Gaussian surface as shown:



Applying Gauss' Law to find the field gives

$$\oint \vec{E} \bullet d\vec{a} = E(2\pi rL) = \frac{Q}{\varepsilon_0} \Rightarrow E = \frac{Q}{2\pi rL\varepsilon_0}$$

To find the magnetic field at *r*, use Ampere's Law

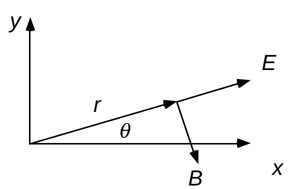


$$\oint \vec{B} \bullet d\vec{l} = B(2\pi r) = \mu_0 I = \mu_0 \frac{dQ}{dt}$$

$$\frac{dQ}{dt} = \frac{\Delta Q}{\Delta t} = \frac{Q}{\Delta s}v = \frac{Qv}{L} \Rightarrow \quad B = \mu_0 \frac{Qv}{2\pi rL}$$

Now consider a point charge -*e*, moving opposite to the bunch, at the point *r* . The effect of the long-range fields of the bunch on the trajectory of this particle is given by (see Lect 2, p. 35):

$$x'' = \frac{eB_y}{p} - \frac{eE_x}{vp} \quad y'' = -\frac{eB_x}{p} - \frac{eE_y}{vp}$$



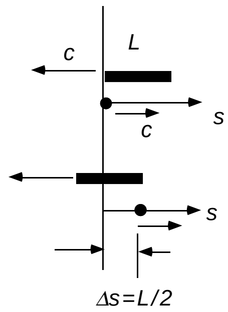
For small θ, we have

$$E_x \approx E \quad E_y \approx E\frac{y}{r} \quad B_x \approx \frac{y}{r}B \quad B_y \approx -B$$

So the total change in slopes of the trajectory produced by the fields of the bunch is

$$\Delta x' = \left(-\frac{eB}{p} - \frac{eE}{vp}\right)\Delta s = \frac{-eQ}{m_0\pi\varepsilon_0\gamma c^2 L}\frac{\Delta s}{r}$$

$$\Delta y' = \frac{y}{r}\left(-\frac{eB}{p} - \frac{eE}{vp}\right)\Delta s = \frac{-eQ}{m_0\pi\varepsilon_0\gamma c^2 L}\frac{y\Delta s}{r^2}$$



From the figure to the right, we see that $\Delta s = L/2$: the effective length of the fields seen by the particle is half the bunch length.

For a bunch with N_b particles of charge e, the angular kicks are

$$\Delta x' = -\frac{N_b e^2}{2\pi\gamma m_0 \varepsilon_0 c^2}\frac{1}{r} = -\frac{2N_b}{\gamma}\frac{r_0}{r} \qquad \Delta y' = -\frac{2N_b}{\gamma}\frac{y r_0}{r^2},$$

in which

$$r_0 = \frac{e^2}{4\pi\varepsilon_0 m_0 c^2}$$ =2.82x10^{-15} m is the classical electron radius.

On pretzel orbits, the beams are separated by a distance $2p$. Hence, we have $r = \sqrt{(2p+x)^2 + y^2}$, where x and y measure the betatron oscillations about the pretzel orbit. Thus

$$\Delta x' = -\frac{2N_b}{\gamma}\frac{r_0}{\sqrt{(2p+x)^2 + y^2}} \qquad \Delta y' = -\frac{2N_b}{\gamma}\frac{y r_0}{(2p+x)^2 + y^2}$$

Typically $(x, y) \ll p$, so we can expand

$$\frac{1}{\sqrt{(x+2p)^2 + y^2}} \cong \frac{1}{2p}\left(1 - \frac{x}{2p} + \ldots\right)$$

and to lowest order in (x,y) we have

$$\Delta x' = -\frac{2N_b r_0}{\gamma}\left(\frac{1}{2p} - \frac{x}{4p^2}\right) \qquad \Delta y' = -\frac{2N_b r_0}{\gamma}\frac{y}{4p^2}$$

The first term in parentheses in the x-equation corresponds to a dipole error. Since it is linear in p, the errors will have different signs for particles and antiparticles, resulting in differential orbit changes and pretzel closure errors. In principle, this can be corrected by adjusting the separators. The second term in x, and the only term in y, is a quadrupole error. The effective focal length is

$$\frac{1}{f_x} = -\frac{\Delta x'}{x} = -\frac{N_b}{\gamma}\frac{r_0}{2p^2}$$

defocusing for both types of particles, in x, and focusing in y. For B bunches, producing $2B$-1 long-range crossings, the tune shift due to the long-range crossings is

$$\Delta Q_{LR,x} = \frac{1}{4\pi}\sum_i \frac{\beta_{xi}}{f_{xi}} \approx -\frac{2B-1}{4\pi}\frac{N_b}{\gamma}\frac{\beta_x r_0}{2p^2}$$

in which β_x is a typical lattice function at the crossings. For a given tolerable tune shift, the required pretzel amplitude is

$$p = \sqrt{\frac{2B-1}{4\pi}\frac{N_b}{\gamma}\frac{\beta_x r_0}{2\Delta Q_{LR,x}}}$$

Example: We want ΔQ_{LR} to be small compared to a typical maximum head-on tune shift, which might be ΔQ_{HO} =0.05. Taking ΔQ_{LR} to be 0.005, for CESR parameters β=30 m, B=30, N_b=10^{11}, γ=10^4, we have p = 16 mm, which requires a full aperture of 32 mm plus room for betatron oscillations.

In practice, it is not this tune shift itself which causes problems, but rather smaller, higher order nonlinear effects which are difficult to correct. Nevertheless, this simple estimate correctly sets the scale of the required pretzel separation.

Sextupole effects of horizontal pretzel orbits

The vertical field of a sextupole is $B_y = \frac{B''}{2}\left(x^2+y^2\right)$. Let the closed orbit deformation produced by the pretzel be $p(s)$. Then, on the pretzel, the sextupole field is

$$B_y = \frac{B''}{2}\left((x+p(s))^2+y^2\right) = \frac{B''}{2}\left(p(s)^2+2xp(s)+x^2+y^2\right)$$

in which (x,y) now refer to betatron oscillations about the pretzel orbit. We see that the effect of the sextupoles is to produce a dipole field error $\frac{B''p^2}{2}$, which is the same for both species. This error can be corrected with standard correction dipoles. There is also a tune shift due to the quadrupole error $\Delta k = \frac{B''p}{(B_0\rho)} = mp$, in which m is the sextupole strength. The total tune shift, integrated around the ring, is

$$\Delta Q_x = \frac{1}{4\pi}\int_0^C ds\, m(s)\beta_x(s)p(s).$$

The tune shift per unit pretzel amplitude is called the *tonality*.

This tune shift will have opposite signs for particles and antiparticles. If the ring has superperiodicity two, and the pretzel is antisymmetric about the symmetry point, ($p\left(s+\frac{C}{2}\right) = -p(s)$)then

$$\Delta Q_x = \frac{1}{4\pi}\left[\int_0^{C/2} ds\, m(s)\beta_x(s)p(s) + \int_{C/2}^{C} ds\, m(s)\beta_x(s)p(s)\right]$$

$$\Delta Q_x = \frac{1}{4\pi}\left[\int_0^{C/2} ds\, m(s)\beta_x(s)p(s) + \int_0^{C/2} ds\, m(s+C/2)\beta_x(s+C/2)p(s+C/2)\right]$$

$$= \frac{1}{4\pi}\left[\int_0^{C/2} ds\, m(s)\beta_x(s)p(s) - \int_0^{C/2} ds\, m(s)\beta_x(s)p(s)\right] = 0$$

The tonality is zero to lowest order. The quadrupole errors produce a lattice function distortion (from Lect 8, p 21)

$$\frac{\Delta\beta_x(s)}{\beta_{x0}(s)}$$
$$= -\frac{1}{2\sin 2\pi Q_{x0}} \int_0^C ds' m(s') p(s') \beta_{x0}(s') \cos\left[2\left(\left|\Phi_{x0}(s) - \Phi_{x0}(s')\right| - \pi Q_{x0}\right)\right]$$

For tunes near a half-integer, this perturbation is maximally antisymmetric about C/2. The tonality, calculated using the perturbed lattice functions, will thus be non-zero in next to lowest order in pretzel amplitude.

Path length changes.

In one of the homework problems, it was shown that a dipole error θ at a location where the dispersion is η produces a path length change $\Delta C = \eta\theta$. If the separators that produce the pretzel are located at dispersive points, then the path length change on the pretzel will be

$$\Delta C = \sum_i \eta(s_i)\theta(s_i)$$

where the sum is over all the pretzel kicks. This change is opposite for the two species of particles. Since the circumference is fixed by the rf wavelength and harmonic number, the path length change on the pretzel results in an energy change given by $\delta = \frac{\Delta C}{\alpha_C C}$. The two

species will then have different energies, which can be a problem if there is residual vertical dispersion at the interaction point.

To lowest order in the pretzel amplitude (i.e., neglecting the changes in η due to the pretzel itself) ΔC is zero for an antisymmetric pretzel in a superperiod 2 lattice