LECTURE 2

Review of basic electrodynamics
Magnetic guide fields used in accelerators
Particle trajectory equations of motion in accelerators

Review of basic electrodynamics

Maxwell's equations: Electric field $\vec{E}(\vec{r},t)$, magnetic field $\vec{B}(\vec{r},t)$, charge density $\rho(\vec{r},t)$, current density $\vec{J}(\vec{r},t)$

$$\vec{\nabla}\bullet\vec{E}=\frac{\rho}{\varepsilon_0} \quad => \quad \oint_S \vec{E}\bullet d\vec{a}=\frac{Q_{enclosed}}{\varepsilon_0} \quad \text{Gauss' Law}$$

$$\vec{\nabla}\bullet\vec{B}=0$$

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \quad => \quad \oint_C \vec{E}\bullet d\vec{l}= -\oint_S \frac{\partial\vec{B}}{\partial t}\bullet d\vec{a} \quad \text{Faraday's Law}$$

$$\vec{\nabla}\times\vec{B}=\mu_0 J+\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} \quad =>$$

$$\oint_C \vec{B}\bullet d\vec{l}=\mu_0 I_{enclosed}+ \mu_0\varepsilon_0 \oint_S \frac{\partial\vec{E}}{\partial t}\bullet d\vec{a} \quad \text{Ampere's Law}$$

Electrodynamic potentials: $\vec{A}(\vec{r},t)$, $V(\vec{r},t)$

$$\vec{B}=\vec{\nabla}\times\vec{A} \qquad \vec{E}=-\vec{\nabla}V-\frac{\partial\vec{A}}{\partial t}$$

Conservation of charge:

$$\vec{\nabla}\bullet\vec{J}+\frac{\partial\rho}{\partial t}=0$$

Ohm's Law:

$$\vec{J}=\sigma\vec{E}$$

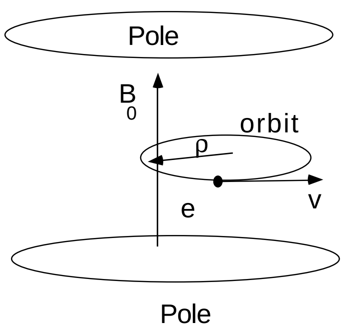
Lorentz Force Law: force on charge *e*

$$\vec{F}=e\left(\vec{E}+\vec{v}\times\vec{B}\right)$$

The Lorentz force generated by the accelerator's *guide field* determines the trajectory of particles in the accelerator. Before we discuss the trajectory equations, we should make some remarks about the guide field.

Magnetic guide fields used in accelerators

The simplest guide field for a circular accelerator is a uniform field.



The Lorentz force provides the centripetal acceleration

$$\frac{v^2}{\rho} = \frac{evB_0}{m}, \quad \frac{1}{\rho} = \frac{eB_0}{mv}$$

$$\frac{1}{\rho}\left[\mathrm{m}^{-1}\right] = 0.2998 \frac{B_0[\mathrm{T}]}{p[\mathrm{GeV/c}]}$$

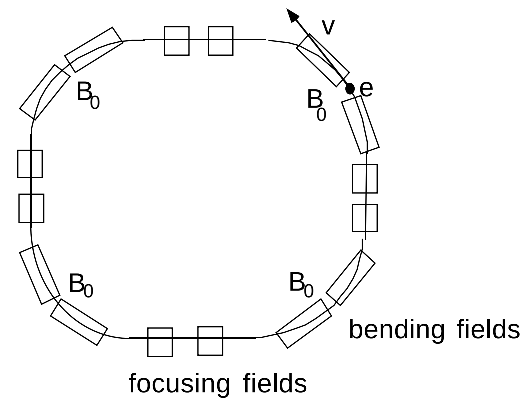
Using this relation, we sometimes measure momentum in units of T-m:

$$(B\rho)[\mathrm{T\text{-}m}] = \frac{p[\mathrm{GeV/c}]}{0.2998}$$

The $(B\rho)$ product is called the *magnetic rigidity*.

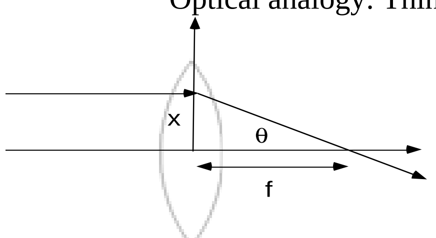
More than a simple uniform bending field is required. Focusing fields are required to insure stability to small displacements from the orbit.

In a synchrotron, the guide field (including bending and focusing fields) is achieved by a series of separate magnets.



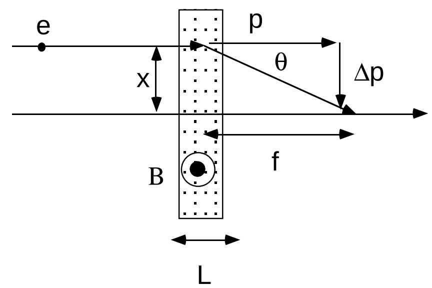
The focusing fields may be separate from the bending fields ("*separated function*" machine) or may be combined with the bending fields ("*combined function*" machine).
If the bending fields all have the same design value, the machine is called *isomagnetic*.

Magnetic focusing fields:
Optical analogy: Thin lens, focal length *f*



$$\theta = \frac{x}{f}$$

Magnetic lens:



$$\Delta p = F\Delta t = evB_y\left(\frac{L}{v}\right) = eB_yL$$

$$\theta \approx \frac{\Delta p}{p} = \frac{eB_yL}{p}$$

If $B_y = \frac{\partial B_y}{\partial x}x = B'x$, then $\theta \approx \frac{eB'xL}{p} = \frac{x}{f}$

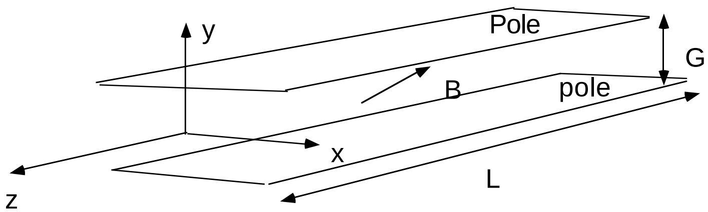
The field acts like a focusing lens of focal length

$$\frac{1}{f} \approx \frac{eB'L}{p} = \frac{B'L}{(B\rho)}$$

This linear dependence of field on position can be generated by a quadrupole magnet

General description of the guide fields in the region of the particle beam

For typical accelerator magnets, the magnet length is much larger than the transverse dimensions of the field gap
(L>>G)



The *idealized fields* extend only over the length L ("effective length") and are independent of z and transverse to the beam. Idealized fields ignore the fringe fields. (Note that this is does not

apply to solenoids, and is a poor approximation for wigglers or undulators).

In the region of the beam,

$$\vec{\nabla}\times\vec{B}=0 \text{ so } \vec{B}=-\vec{\nabla}\phi$$
$$\vec{\nabla}\bullet\vec{B}=0 \text{ so } \nabla^2\phi=0$$

The magnetic scalar potential is a solution to LaPlace's equation. For the idealized fields, the magnetic potential is only a function of *x* and *y*.
In this case, the general solution to LaPlace's equation in Cartesian coordinates

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 \text{ can be written as}$$

$$\phi(x,y)=\mathrm{Re}\sum_{m=0}^{\infty}C_m(x+iy)^m$$

where C_m is a (complex) constant determined by the boundary conditions.

The vector potential has only one component and it is

$$A_z(x,y)=\mathrm{Re}\sum_{m=0}^{\infty}iC_m(x+iy)^m$$

Then

$$B_x=-\frac{\partial\phi}{\partial x}=-\mathrm{Re}\sum_{m=1}^{\infty}mC_m(x+iy)^{m-1}$$

$$B_y=-\frac{\partial\phi}{\partial y}=-\mathrm{Re}\sum_{m=1}^{\infty}imC_m(x+iy)^{m-1}$$

These can be combined to give

$$B_y+iB_x=-\sum_{m=1}^{\infty}imC_m(x+iy)^{m-1}.$$

This is the general multipole expansion for the two-dimensional magnetic field in a current-free region.

Since

$$-imC_m = \frac{1}{(m-1)!}\left[\left.\frac{\partial^{m-1}B_y}{\partial x^{m-1}}\right|_{x=y=0} + i\left.\frac{\partial^{m-1}B_x}{\partial x^{m-1}}\right|_{x=y=0}\right]$$

we can write

$$B_y + iB_x = \sum_{m=0}^{\infty}\frac{1}{m!}(B^{(m)} + i\tilde{B}^{(m)})(x+iy)^m$$

where

$$B^{(0)} = B_0; \quad \tilde{B}^{(0)} = \tilde{B}_0;$$

$$B^{(1)} = B' = \left.\frac{\partial B_y}{\partial x}\right|_{x=y=0}; \quad \tilde{B}^{(1)} = \tilde{B}' = \left.\frac{\partial \tilde{B}_x}{\partial x}\right|_{x=y=0}$$

$$B^{(2)} = B'' = \left.\frac{\partial^2 B_y}{\partial x^2}\right|_{x=y=0}; \quad \tilde{B}^{(2)} = \tilde{B}'' = \left.\frac{\partial^2 B_x}{\partial x^2}\right|_{x=y=0}$$

$$B^{(m)} = \left.\frac{\partial^m B_y}{\partial x^m}\right|_{x=y=0}; \quad \tilde{B}^{(m)} = \left.\frac{\partial^m B_x}{\partial x^m}\right|_{x=y=0}$$

Vector potential in this language:

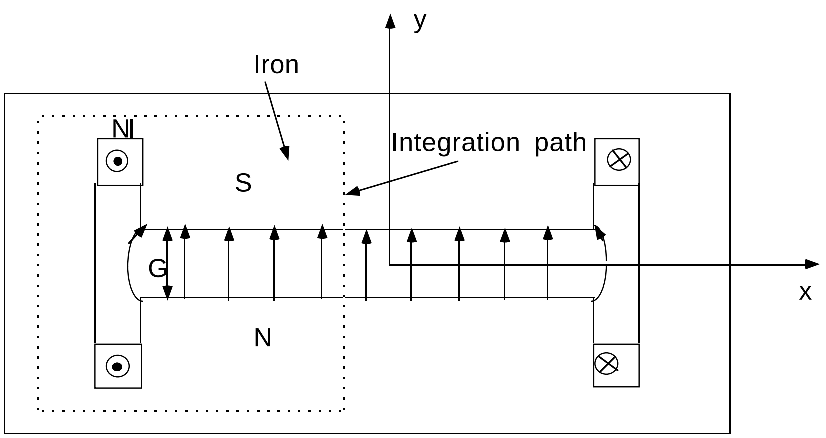
$$A_z(x,y) = -\mathrm{Re}\sum_{m=1}^{\infty}\frac{1}{m!}(B^{(m-1)} + i\tilde{B}^{(m-1)})(x+iy)^m$$

Let us write out the first few terms:

$$B_y = B_0 + B'x - \tilde{B}'y + \frac{B''}{2}(x^2 - y^2) - \tilde{B}''xy + \ldots$$

$$B_x = \tilde{B}_0 + \tilde{B}'x + B'y + \frac{\tilde{B}''}{2}(x^2 - y^2) + B''xy + \ldots$$

The terms without the twiddle are called “normal” terms; the terms with the twiddle are called “skew” terms. The 0 coefficients are pure dipole terms, the linear are quadrupole terms, the quadratic are sextupole terms.



Pure dipole: NI turns/pole

Evaluate $\oint \vec{H} \bullet d\vec{l} = I_{enclosed}$ around the integration path shown.

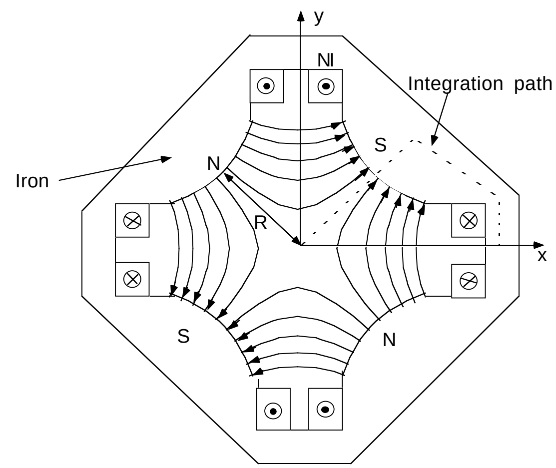
For infinite permeability iron $\vec{H} = \frac{\vec{B}}{\mu} \rightarrow 0$ inside the iron, so in the gap

$$H = \frac{2NI}{G} \Rightarrow B = \mu_0 H = \mu_0 \frac{2NI}{G}$$

$$B[\mathrm{T}] = 2.52 \frac{NI[\mathrm{kA-turns}]}{G[\mathrm{mm}]}$$

This dipole bends a positive particle moving in the z-direction to the left.

If the dipole is rotated clockwise by 90^{o} about the z-axis, it becomes a pure *skew dipole.*



Pure quadrupole, NI turns/pole

Evaluate $\oint \vec{H} \bullet d\vec{l} = I_{enclosed}$ around the integration path shown.

For infinite permeability iron $\vec{H} = \frac{\vec{B}}{\mu} \rightarrow 0$ inside the iron, so in the gap

$$\oint \vec{H} \bullet d\vec{l} = \frac{1}{\mu_0} \int_0^R B' r dr = \frac{B' R^2}{2\mu_0} = NI \Rightarrow B' = \mu_0 \frac{2NI}{R^2}$$

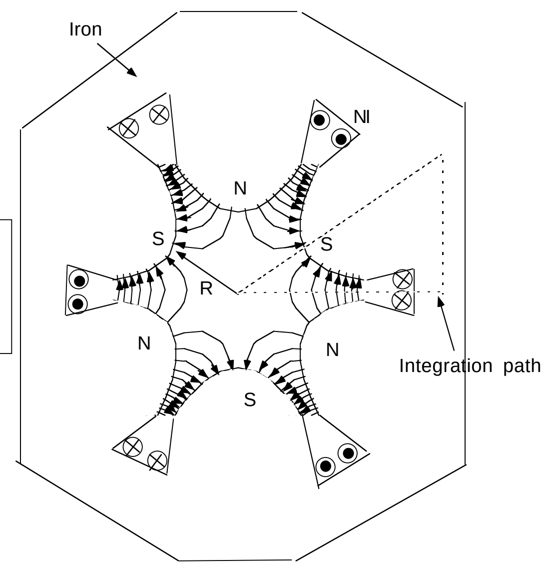
$$B'[\frac{\mathrm{T}}{\mathrm{m}}] = 2.51 \frac{NI[\mathrm{A-turns}]}{R[\mathrm{mm}]^2}$$

Quadrupole focal length $f \approx \frac{p}{eB'L} = \frac{(B\rho)}{B'L}$

Note that (for a positive particle moving in the z-direction), this quadrupole is focusing in x, but defocusing in y.

If the quadrupole is rotated clockwise by 45^{o} about the z-axis, it becomes a pure *skew quadrupole.*

Pure sextupole NI turns/pole



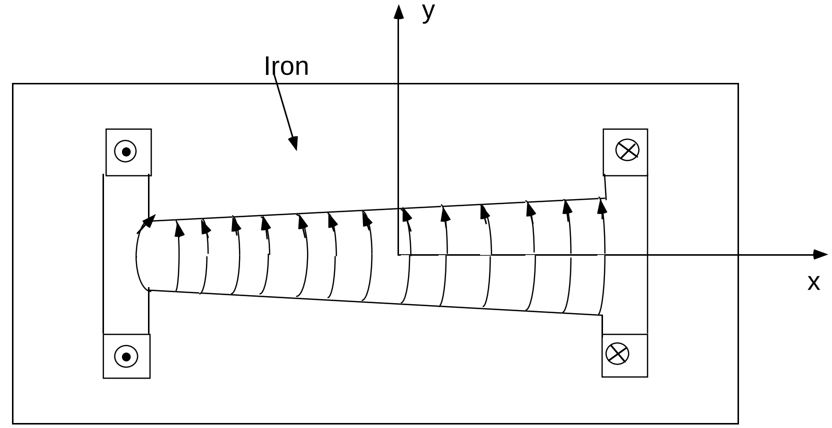
Evaluate $\oint \vec{H} \bullet d\vec{l} = I_{enclosed}$ around the integration path shown.

For infinite permeability iron $\vec{H} = \frac{\vec{B}}{\mu} \to 0$ inside the iron, so in the gap

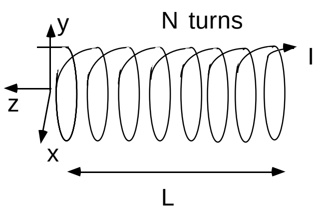
$$\oint \vec{H} \bullet d\vec{l} = \frac{1}{\mu_0} \int_0^R \frac{B''}{2} r^2 dr = \frac{B'' R^3}{6\mu_0} = NI \Rightarrow B = \mu_0 \frac{6NI}{R^3}$$

$$B''[\frac{\mathrm{T}}{\mathrm{m}}] = 7540 \frac{NI[\mathrm{A-turns}]}{R[\mathrm{mm}]^3}$$

If the sextupole is rotated clockwise by 30° about the z-axis, it becomes a pure *skew sextupole*.



Gradient magnet (for combined function machines): provides both a bending field and a gradient (due to changing gap dimension).



Solenoid: produces an axial field

$$B_z = \mu_0 \frac{NI}{L}$$

$$B_z[\mathrm{T}] = 1.26 \times 10^{-3} \frac{NI[\mathrm{kA-turns}]}{L[\mathrm{m}]}$$

The transverse fringe fields at the solenoid ends are crucial to the solenoid focusing action, so in this case the idealized fields must include end fields. Maxwell's equations allow the end fields to have the form $B_x = -\frac{1}{2}\frac{\partial B_z}{\partial z} x \qquad B_y = -\frac{1}{2}\frac{\partial B_z}{\partial z} y$

Solenoid focal length:

$$f = \frac{2}{L}\left(\frac{p}{eB_z}\right)^2 = \frac{2}{L}\left(\frac{(B\rho)}{B_z}\right)^2$$

Particle trajectory equations of motion in accelerators

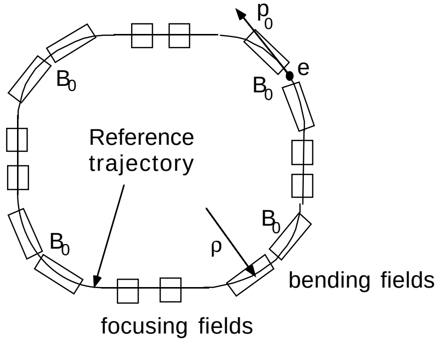
Consider a particle of charge e, rest mass m_0, momentum $\vec{p} = m_0 \gamma \vec{v}$ $(\gamma^2 = \frac{1}{1-\left(\frac{v}{c}\right)^2})$, moving under the action of the Lorentz force:

The trajectory equation of motion is given by Newton's Law:

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}(m_0 \gamma \vec{v}) = \vec{F} = e\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

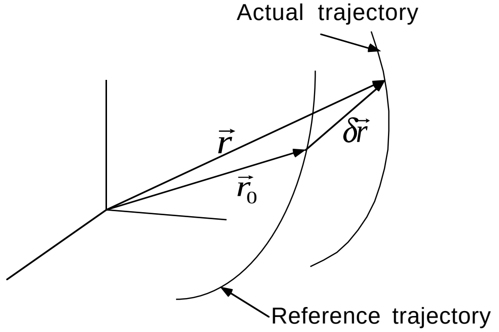
Solution is this equation is the trajectory $\vec{r}(t)$

Write $\vec{r}(t)$ in terms of the "reference trajectory":



The "reference trajectory" $\vec{r}_0(t)$ is the trajectory of a "reference particle" of momentum $\vec{p}_0$ that passes through the center of symmetry of all the *idealized* magnetic guide fields. The reference trajectory is an *idealization*.

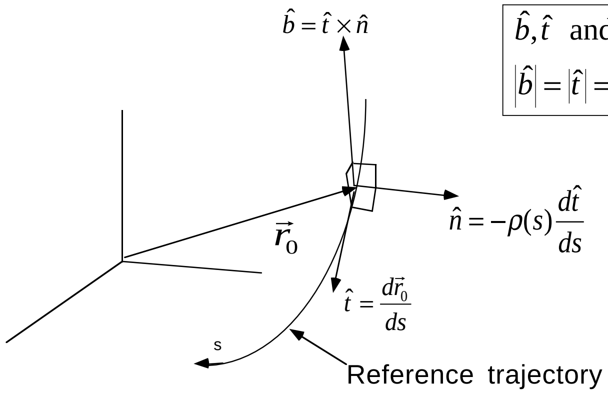
For a circular accelerator, this trajectory is closed (i.e., it repeats itself exactly on every revolution)



$$\vec{r}(t) = \vec{r}_0(t) + \delta\vec{r}(t)$$

We want to write the trajectory equation of motion in the "natural" coordinate system of the reference trajectory.

From differential geometry: "natural" coordinate system of a curve in space



$\hat{b}, \hat{t}$ and $\hat{n}$ are unit vectors

$|\hat{b}| = |\hat{t}| = |\hat{n}| = 1$

For any space curve $\vec{r}_0(s)$, where s is the arc length along the curve, the vector $\hat{t} = \frac{d\vec{r}_0}{ds}$ is a unit vector tangent to the curve. The vector $\frac{d\hat{t}}{ds}$ measures the rate at which the tangent vector changes, which is inversely proportional to $\rho(s)$, the radius of curvature: $\frac{d\hat{t}}{ds} = -\frac{\hat{n}}{\rho(s)}$, where $\hat{n}$ (called the principal normal to the curve) is a unit vector normal to $\hat{t}$. The vectors $\hat{t}$, $\hat{n}$, and $\hat{b} = \hat{t} \times \hat{n}$ form an orthogonal, right-handed coordinate system which moves (and may rotate) as we progress along the space curve.

The change in the unit vectors with s is given by the

Frenet-Serret relations:

$$\frac{d\hat{t}}{ds} = -\frac{\hat{n}}{\rho} \qquad \frac{d\hat{n}}{ds} = \tau\hat{b} + \frac{\hat{t}}{\rho} \qquad \frac{d\hat{b}}{ds} = -\tau\hat{n}$$

where $\tau(s)$ is called the torsion.

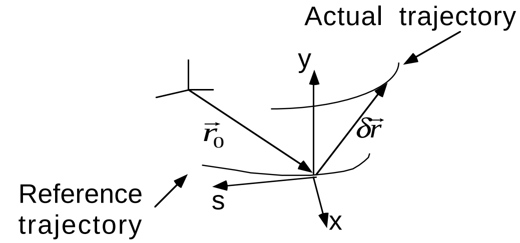
For most accelerators, the reference orbit lies in a plane: this corresponds to $\tau(s)$=0. For circular accelerators, since the orbit is closed,

$$\oint \frac{ds}{\rho(s)} = \oint \frac{d\theta}{ds} ds = 2\pi$$

Take the space curve to be the reference trajectory, and write the trajectory equations using the Frenet-Serret co-ordinate system:

$$\vec{r} = \vec{r}_0 + \delta\vec{r} = \vec{r}_0 + x\hat{n} + y\hat{b}$$

($\delta\vec{r}$ lies in x-y plane since coordinate system moves with the particle)



$$\dot{f} \equiv \frac{df}{dt} = \frac{ds}{dt}\frac{df}{ds} = \dot{s}\frac{df}{ds}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}_0}{dt} + \dot{x}\hat{n} + x\frac{d\hat{n}}{dt} + \dot{y}\hat{b} = \dot{s}\frac{d\vec{r}_0}{ds} + \dot{x}\hat{n} + x\dot{s}\frac{d\hat{n}}{ds} + \dot{y}\hat{b}$$

$$= \dot{s}\hat{t} + \dot{x}\hat{n} + \frac{x}{\rho}\dot{s}\hat{t} + \dot{y}\hat{b} = \dot{s}\hat{t}\left(1 + \frac{x}{\rho}\right) + \dot{x}\hat{n} + \dot{y}\hat{b}$$

Newton's Law: $\frac{d\vec{p}}{dt} = \frac{d}{dt}(m_0\gamma\vec{v}) = \vec{F} = e\left(\vec{E} + \vec{v}\times\vec{B}\right)$

$$\frac{d}{dt}[m_0\gamma\vec{v}] = \frac{d}{dt}[m_0\gamma\dot{x}\hat{n}] + \frac{d}{dt}[m_0\gamma\dot{y}\hat{b}] + \frac{d}{dt}\left[m_0\gamma\dot{s}\left(1+\frac{x}{\rho}\right)\hat{t}\right]$$

$$= \hat{n}\left[\frac{d}{dt}[m_0\gamma\dot{x}] - \frac{m_0\gamma\dot{s}^2}{\rho}\left(1+\frac{x}{\rho}\right)\right] + \hat{b}\frac{d}{dt}[m_0\gamma\dot{y}] + \hat{t}\left[\frac{d}{dt}\left[m_0\gamma\dot{s}\left(1+\frac{x}{\rho}\right)\right] + \frac{m_0\gamma\dot{x}\dot{s}}{\rho}\right]$$

$$\vec{v}\times\vec{B} = \begin{vmatrix} \hat{n} & \hat{b} & \hat{t} \\ \dot{x} & \dot{y} & \dot{s}\left(1+\frac{x}{\rho}\right) \\ B_x & B_y & B_s \end{vmatrix}$$

$$= \hat{n}\left[\dot{y}B_s - \dot{s}\left(1+\frac{x}{\rho}\right)B_y\right] + \hat{b}\left[\dot{s}\left(1+\frac{x}{\rho}\right)B_x - \dot{x}B_s\right] + \hat{t}\left[\dot{x}B_y - \dot{y}B_x\right]$$

Three trajectory equations:

$$\frac{d}{dt}[m_0\gamma\dot{x}] = \frac{m_0\gamma\dot{s}^2}{\rho}\left(1+\frac{x}{\rho}\right) + \dot{y}eB_s - \dot{s}\left(1+\frac{x}{\rho}\right)eB_y + eE_x$$

$$\frac{d}{dt}[m_0\gamma\dot{y}] = \dot{s}\left(1+\frac{x}{\rho}\right)eB_x - \dot{x}eB_s + eE_y$$

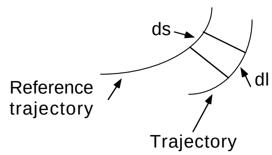
$$\frac{d}{dt}\left[m_0\gamma\dot{s}\left(1+\frac{x}{\rho}\right)\right] = -\frac{m_0\gamma\dot{x}\dot{s}}{\rho} + \dot{x}eB_y - \dot{y}eB_x + eE_s$$

Write derivatives in terms of arc length *s* along the reference curve as the independent variable:

(with $f' \equiv \frac{df}{ds}$)

$$\frac{df}{dt} = \frac{df}{ds}\dot{s} = f'\dot{s}$$

and introduce l=the path length of the particle along the orbit



$$\frac{dl}{dt} = v = \frac{dl}{ds}\dot{s} = l'\dot{s} \;\Rightarrow l' = \frac{v}{\dot{s}}$$

$$\frac{v}{l'}\frac{d}{ds}\left[m_0\gamma\frac{v}{l'}x'\right] = \frac{m_0\gamma}{\rho}\left(\frac{v}{l'}\right)^2\left(1+\frac{x}{\rho}\right) + y'\frac{v}{l'}eB_s - \frac{v}{l'}\left(1+\frac{x}{\rho}\right)eB_y + eE_x$$

$$\frac{v}{l'}\frac{d}{ds}\left[m_0\gamma\frac{v}{l'}y'\right] = \frac{v}{l'}\left(1+\frac{x}{\rho}\right)eB_x - x'\frac{v}{l'}eB_s + eE_y$$

$$\frac{v}{l'}\frac{d}{ds}\left[m_0\gamma\frac{v}{l'}\left(1+\frac{x}{\rho}\right)\right] = -\frac{m_0\gamma x'}{\rho}\left(\frac{v}{l'}\right)^2 + x'\frac{v}{l'}eB_y - y'\frac{v}{l'}eB_x + eE_s$$

Divide by $\frac{v}{l'}$, use $p = m_0\gamma v$, and expand LHS:

$$\frac{p}{l'}x'' + x'\frac{d}{ds}\left[\frac{p}{l'}\right] = \frac{p}{\rho l'}\left(1+\frac{x}{\rho}\right) + y'eB_s - \left(1+\frac{x}{\rho}\right)eB_y + eE_x\frac{l'}{v}$$

$$\frac{p}{l'}y'' + y'\frac{d}{ds}\left[\frac{p}{l'}\right] = \left(1+\frac{x}{\rho}\right)eB_x - x'eB_s + eE_y\frac{l'}{v}$$

$$\left(1+\frac{x}{\rho}\right)\frac{d}{ds}\left[\frac{p}{l'}\right] + \frac{p}{l'}\frac{d}{ds}\left[1+\frac{x}{\rho}\right] = -\frac{px'}{\rho l'} + x'eB_y - y'eB_x + eE_s\frac{l'}{v}$$

Let the electric field be zero: the particle energy is then constant.
Divide by $\frac{p}{l'}$ and use $\frac{d}{ds}\left[\frac{1}{l'}\right] = -\frac{l''}{l'^2}$

$$x'' - x'\frac{l''}{l'} = \frac{1}{\rho}\left(1+\frac{x}{\rho}\right) + y'l'\frac{eB_s}{p} - \left(1+\frac{x}{\rho}\right)l'\frac{eB_y}{p}$$

$$y'' - y'\frac{l''}{l'} = l'\left(1+\frac{x}{\rho}\right)\frac{eB_x}{p} - x'l'\frac{eB_s}{p}$$

$$\left(1+\frac{x}{\rho}\right)\frac{l''}{l'} - \frac{d}{ds}\left[1+\frac{x}{\rho}\right] = \frac{x'}{\rho} - x'l'\frac{eB_y}{p} + y'l'\frac{eB_x}{p}$$

From the relation for the velocity,

$$v^2 = \dot{s}^2(1+Kx)^2 + \dot{x}^2 + \dot{y}^2 = \dot{s}^2\left[\left(1+\frac{x}{\rho}\right)^2 + x'^2 + y'^2\right]$$

Solve for $\dot{s}$

$$\dot{s} = \frac{v}{\sqrt{\left(1+\frac{x}{\rho}\right)^2 + x'^2 + y'^2}} \quad \text{and} \quad l' = \frac{v}{\dot{s}} = \sqrt{\left(1+\frac{x}{\rho}\right)^2 + x'^2 + y'^2}$$

Approximation:

"paraxial" motion=> the derivatives $x'^2,\ y'^2 \ll 1$ so

$l' \approx 1 + \frac{x}{\rho}$ and we neglect $x'\frac{l''}{l'}$ and $y'\frac{l''}{l'}$ terms in the trajectory equations

(These neglected terms are called "kinematic terms")

Typical values of

$$x(s) \approx x_{\max} \cos\left(\frac{s}{\beta}\right)$$

$$x' = \frac{dx}{ds} \approx \frac{x_{\max}}{\beta} \approx \frac{10\ \text{mm}}{10\ \text{m}} \approx 10^{-3}$$

Trajectory equations in the paraxial approximation:

$$x'' = \frac{1}{\rho}\left(1+\frac{x}{\rho}\right) - \left(1+\frac{x}{\rho}\right)^2 \frac{eB_y}{p} + y'\frac{eB_s}{p}\left(1+\frac{x}{\rho}\right)$$

$$y'' = \left(1+\frac{x}{\rho}\right)^2 \frac{eB_x}{p} - x'\frac{eB_s}{p}\left(1+\frac{x}{\rho}\right)$$

$$l' = \left(1+\frac{x}{\rho}\right)$$