LECTURE 20

Collective effects in multi-particle Beams

Tune shifts and spreads:

Transverse space charge: direct and indirect

Beam-beam interaction

Collective Effects in multi-particle Beams

To this point in these lectures, we have not considered the effects of the electromagnetic fields generated by the beam itself. The effects of these fields are called *collective effects*, because they depend on the field of a collection of charged particles (the beam). For intense beams, these collective effects can be very important: the fields of the beam can be comparable to or larger than the magnetic guide fields or the rf accelerating fields.

The fields of the beam can cause static effects (such as tune shifts, lattice function distortions, resonance excitation) just like any perturbing field in the machine. Since the collective fields follow the motion of the beam, and can also affect it, they can also have dynamical effects, leading to damping or growth (instability) of

beam motion. We'll start the study of collective effects with the simplest topic, the static effects of the collective fields.

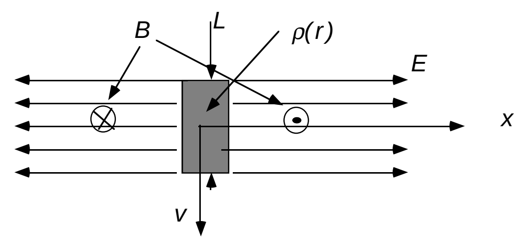
Transverse space charge.

This is the simplest collective effect: the beam constitutes a charge and current distribution, and the fields generated by this distribution will act on the trajectories of the individual constituents of the beam. We can understand what happens by inserting the beam's fields into the trajectory equations that govern the motion of the individual particles in the beam.

Direct space charge effect

Consider a highly relativistic bunch of length L, with N particles of charge e, which has a round Gaussian charge distribution in the

transverse direction $\rho(r) = \dfrac{Ne}{2\pi\sigma^2}\exp\left(-\dfrac{r^2}{2\sigma^2}\right)$. Since the fields are "flattened" to be perpendicular to the direction of motion, the picture is



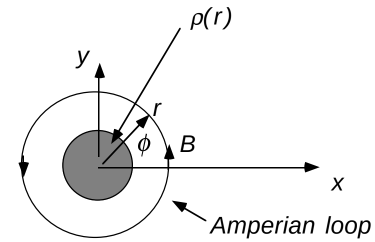
To find the electric field a distance r from the axis of the bunch, we surround the bunch with a Gaussian surface and apply Gauss' Law:

$$\oint \vec{E} \bullet d\vec{a} = E(2\pi rL) = \frac{Q_{enclosed}}{\varepsilon_0} \frac{Ne}{2\pi\varepsilon_0\sigma^2}\int_0^r rdr\exp\left(-\frac{r^2}{2\sigma^2}\right) \Rightarrow$$

$$\vec{E} = \frac{Ne}{2\pi\varepsilon_0 rL}\left(1-\exp\left(-\frac{r^2}{2\sigma^2}\right)\right)\hat{r}$$

This field is directed radially outward.

Similarly, Ampere’s Law will give the magnetic field



$$\oint \vec{B} \bullet d\vec{l} = B(2\pi r) = \mu_0 I_{enclosed} = \mu_0 \frac{Nev}{2\pi\sigma^2}\int_0^r rdr\exp\left(-\frac{r^2}{2\sigma^2}\right)$$

$$\Rightarrow \quad \vec{B} = \mu_0 \frac{Nev}{2\pi rL}\left(1-\exp\left(-\frac{r^2}{2\sigma^2}\right)\right)\hat{\phi}$$

This field is directed in the $\hat{\phi}$ direction. Now consider a particle of charge e in the beam. The Lorentz force it feels from these fields is called the *space charge force*.

$$\vec{F} = e\left(\vec{E} + \vec{v}\times\vec{B}\right) = \frac{Ne^2}{2\pi\varepsilon_0 rL}\left(1-\exp\left(-\frac{r^2}{2\sigma^2}\right)\right)\left(\hat{r} + \hat{s}\times\hat{\phi}\frac{v^2}{c^2}\right)$$

$$= \hat{r}\frac{Ne^2}{2\pi\varepsilon_0 rL}\left(1-\exp\left(-\frac{r^2}{2\sigma^2}\right)\right)\left(1-\frac{v^2}{c^2}\right)$$

$$= \hat{r}\frac{Ne^2}{2\pi\varepsilon_0\gamma^2 rL}\left(1-\exp\left(-\frac{r^2}{2\sigma^2}\right)\right)$$

For large γ, the forces due to the electric and magnetic fields tend to cancel; the space charge forces goes like $1/\gamma^2$.

The effect of this force on the trajectory equations, from Lect 2, p 35, is

$$x'' = -\frac{e}{p}\left(B_y - \frac{E_x}{v}\right) = \frac{F_x}{vp} \qquad y'' = \frac{e}{p}\left(B_x + \frac{E_y}{v}\right) = \frac{F_y}{vp}$$

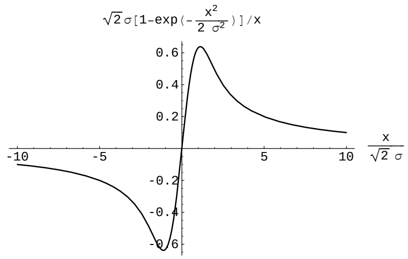
where, from the result above,

$$F_x(x,y) = \frac{Ne^2}{2\pi\varepsilon_0\gamma^2 L}\frac{x}{\left(x^2+y^2\right)}\left(1-\exp\left(-\frac{\left(x^2+y^2\right)}{2\sigma^2}\right)\right)$$

$$F_y(x,y) = \frac{Ne^2}{2\pi\varepsilon_0\gamma^2 L}\frac{y}{\left(x^2+y^2\right)}\left(1-\exp\left(-\frac{\left(x^2+y^2\right)}{2\sigma^2}\right)\right)$$

The force is nonlinear in x and y, and introduces coupling.

Plot of the space charge force $F_x(x,0)$ vs. x



For small x, the force is linear in x, but it reaches a maximum at about x=2σ, and then falls off slowly.

For small x and y, we have

$$F_x(x,0) \approx x\frac{Ne^2}{4\pi\sigma^2\varepsilon_0\gamma^2 L} \qquad F_y(0,y) \approx y\frac{Ne^2}{4\pi\sigma^2\varepsilon_0\gamma^2 L}$$

so the trajectory equations become

$$x'' = \frac{Ne^2}{4\pi\sigma^2\varepsilon_0 vp\gamma^2 L}x \qquad y'' = \frac{Ne^2}{4\pi\sigma^2\varepsilon_0 vp\gamma^2 L}y$$

The space charge force is equivalent to a quadrupole error, with strength

$$k = -\frac{Ne^2}{4\pi\sigma^2\varepsilon_0 vp\gamma^2 L} = -\frac{Ne^2}{4\pi\sigma^2\varepsilon_0\beta^2c^2m_0\gamma^3 L} = -\frac{Nr_0}{\beta^2\gamma^3\sigma^2 L}$$

where $r_0 = \dfrac{e^2}{4\pi\varepsilon_0 m_0 c^2}$

Such a quadrupole error produces a tune shift

$$\Delta Q_x = \frac{1}{4\pi}\oint\beta_x(s)k\,ds = -\frac{Nr_0}{4\pi\beta^2\gamma^3 L}\oint\frac{\beta_x(s)}{\sigma^2}ds$$

$$= -\frac{Nr_0}{4\pi\beta\gamma^2 L\varepsilon_n}\oint ds = -\frac{Nr_0 R}{2\beta\gamma^2 L\varepsilon_n}$$

where $R = \dfrac{\oint ds}{2\pi}$ is the mean radius of the machine, and

$\varepsilon_n = \beta\gamma\varepsilon = \beta\gamma\dfrac{\sigma^2}{\beta_x}$ is the normalized rms emittance. The negative sign indicates the space charge force is defocusing. There is an equal tune shift in the y direction.

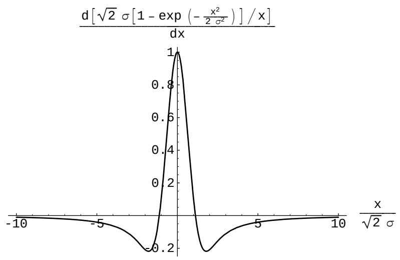
Example: an unbunched proton beam, containing $6\text{x}10^{12}$ particles, with a normalized rms emittance of 2 mm-mrad, is injected into the Fermilab Booster at 400 MeV. This beam is not relativistic, but the

treatment given above is valid for non-relativistic beams also, if they are not bunched. In this case, $L = 2\pi R$, and N represents the total number of particles in the machine:

$$\Delta Q_x = -\frac{Nr_0}{4\pi\beta\gamma^2\varepsilon_n} = -\frac{6\times10^{12}\times1.53\times10^{-18}}{4\pi\times0.713\times1.426^2\times2\times10^{-6}} = -0.252$$

Because of the nonlinear form of the space charge force (for a Gaussian beam), the tune shift will depend on the amplitude of the particle's oscillation. The tune shift will be proportional to the local gradient of the force.

Plot of the gradient of $F_x(x,0)$ vs. x



The largest tune shift occurs for particles at small x; particles at large *x* see a smaller tune shift. This results in a tune *spread* in the beam, roughly equal to the small amplitude tune shift. This tune spread can cause problems if some parts of the beam are shifted onto resonances. Amazingly enough, it has been found that beams can survive (for a short time: the acceleration cycle in a rapidly cycling machine) with space charge tune spreads as large as 0.4.

The space charge effect just discussed is called the *direct* space charge effect. The tune shifts it produces are called *incoherent* tune shifts, because they are shifts of the tune of individual particles in the beam. The oscillation frequency of the beam as a whole (a *coherent* oscillation) is not affected by the direct space charge effect.

There is another effect of space charge, called the *indirect* space charge effect, which can cause both incoherent tune spreads and coherent tune shifts.

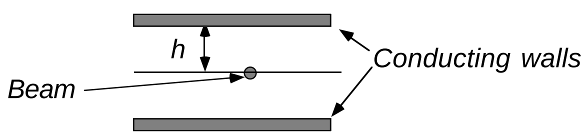
Indirect space charge effect

This effect is due to the fact that the beam is traveling inside a vacuum chamber with conducting walls, and generally also inside the poles of a magnet. The fields of the beam will produce induced charges in the vacuum chamber and induced magnetization in the

magnet poles. The induced charge and magnetization produces fields that can act on the beam. The effect of these is called the indirect space charge effect.

The induced charges and magnetization can be found by requiring solutions of Maxwell's equations to satisfy the boundary conditions: $E_{\|} = 0$ at a conducting surface, and $B_{\|} = 0$ at an magnetic surface.

For the case of parallel conducting walls:



The additional electric and magnetic fields in the region of the beam can be obtained through the use of image charges to find

solutions that satisfy the boundary conditions. The result, for small *x* and *y*, is

$$F_x(x,0) = x\frac{e^2}{2\pi\varepsilon_0}\left(\frac{N}{\gamma^2 L}\left[\frac{1}{2\sigma^2} - \frac{\pi^2}{24h^2}\right] - \frac{N_{tot}}{2\pi R}\frac{\pi^2}{24h^2}\right)$$

$$F_y(0,y) = y\frac{e^2}{2\pi\varepsilon_0}\left(\frac{N}{\gamma^2 L}\left[\frac{1}{2\sigma^2} + \frac{\pi^2}{24h^2}\right] + \frac{N_{tot}}{2\pi R}\frac{\pi^2}{24h^2}\right)$$

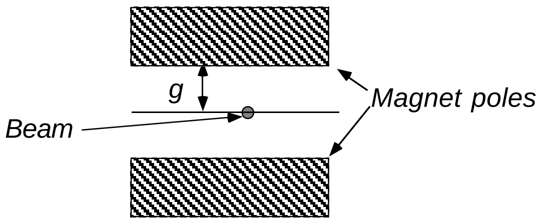
The first two terms in brackets are due to the ac image charges and currents, and have the suppression factor $1/\gamma^2$ due to electric-magnetic field cancellation. The last term is due to the dc current, proportional to the average line density $\frac{N_{tot}}{2\pi R}$, with N_{tot} the total number of particles in the machine. Static magnetic fields from the dc beam current can penetrate the conducting walls. There are no induced dc currents, and so this term has no $1/\gamma^2$ suppression factor. It will dominate at high energies.

The associated incoherent tune shifts are

$$\Delta Q_x = -\frac{r_0 R}{\beta}\left(\frac{N}{\gamma^2 L}\left[\frac{1}{2\varepsilon_n} - \frac{\pi^2\langle\beta_x\rangle}{24\beta\gamma h^2}\right] - \frac{N_{tot}}{2\pi R}\frac{\pi^2\langle\beta_x\rangle}{24\beta\gamma h^2}\right)$$

$$\Delta Q_y = -\frac{r_0 R}{\beta}\left(\frac{N}{\gamma^2 L}\left[\frac{1}{2\varepsilon_n} + \frac{\pi^2\langle\beta_y\rangle}{24\beta\gamma h^2}\right] + \frac{N_{tot}}{2\pi R}\frac{\pi^2\langle\beta_y\rangle}{24\beta\gamma h^2}\right)$$

For the permeable magnet poles:



The additional magnetic fields in the region of the beam can be obtained through the use of image currents. Only the dc component of the beam current contributes. The incoherent tune shifts, including both the direct and indirect fields, are

$$\Delta Q_x = -\frac{r_0 R}{\beta}\left(\frac{N}{\gamma^2 L}\left[\frac{1}{2\varepsilon_n} - \frac{\pi^2\langle\beta_x\rangle}{24\beta\gamma h^2}\right] - \frac{N_{tot}\pi\langle\beta_x\rangle}{24\beta\gamma R}\left(\frac{1}{2h^2} + \frac{\beta^2}{g^2}\right)\right)$$

$$\Delta Q_y = -\frac{r_0 R}{\beta}\left(\frac{N}{\gamma^2 L}\left[\frac{1}{2\varepsilon_n} + \frac{\pi^2\langle\beta_y\rangle}{24\beta\gamma h^2}\right] + \frac{N_{tot}\pi\langle\beta_y\rangle}{24\beta\gamma R}\left(\frac{1}{2h^2} + \frac{\beta^2}{g^2}\right)\right)$$

Example: Indirect space charge tune shift in CESR, due to magnet poles (dominant term). For 45 bunches with $1.5\text{x}10^{11}$ per bunch, $N_{tot}=6.75\ 10^{12}$; $\langle\beta_x\rangle=20$ m, $g=3$ cm=>

$$\Delta Q_x = \frac{N_{tot}\pi\langle\beta_x\rangle r_0}{24\gamma g^2} = 0.0054$$

Coherent tune shift

For a beam oscillating *as a whole* in the vertical direction between horizontal plates or poles, the image charges and currents oscillate also, and produce a force on the beam that is proportional to the average *y* position of the beam

$$F_{y,coh}(0,\bar{y}) = \bar{y}\,\frac{N_{tot}}{2\pi R}\frac{e^2}{\pi\varepsilon_0}\frac{\pi^2}{16}\left(\frac{1}{h^2} + \frac{\beta^2}{g^2}\right)$$

resulting in a *coherent* tune shift

$$\Delta Q_{y,coh} = -\frac{r_0 N_{tot}\pi\langle\beta_y\rangle}{16\beta^2\gamma}\left(\frac{1}{h^2} + \frac{\beta^2}{g^2}\right)$$

This tune shift affects the beam as a whole (like a standard quadrupole error). If the beam is given a kick and the oscillation frequency measured with a spectrum analyzer, this tune shift can be measured.

Beam-beam interaction

We have discussed the long range beam-beam interaction in Lecture 19. We'll now consider the "head-on" interaction, when a particle in one beam passes through the charge distribution of the opposing beam. This problem is essentially identical to the one that we have just examined: the fields from the bunch are those obtained above on p. 5 and 6; the force experienced by the particle in the opposing beam, which has a charge $-e$ and a velocity in the $-\hat{s}$ direction, is

$$\vec{F} = -e\left(\vec{E} + \vec{v} \times \vec{B}\right) = -\frac{Ne^2}{2\pi\varepsilon_0 rL}\left(1 - \exp\left(-\frac{r^2}{2\sigma^2}\right)\right)\left(\hat{r} - \hat{s} \times \hat{\phi}\frac{v^2}{c^2}\right)$$

$$= -\hat{r}\frac{Ne^2}{2\pi\varepsilon_0 rL}\left(1 - \exp\left(-\frac{r^2}{2\sigma^2}\right)\right)\left(1 + \frac{v^2}{c^2}\right) \approx -\hat{r}\frac{Ne^2}{\pi\varepsilon_0 rL}\left(1 - \exp\left(-\frac{r^2}{2\sigma^2}\right)\right)$$

The electric and magnetic forces **add** in this case, and are almost equal for a relativistic particle (we will take β=1). The trajectory equations, $x'' = \frac{F_x}{vp} \qquad y'' = \frac{F_y}{vp}$, can be integrated over the effective length of the fields Δs, giving $\Delta x' = \frac{F_x}{vp}\Delta s \qquad \Delta y' = \frac{F_y}{vp}\Delta s$ as the

angular kicks. As explained in Lecture 17, p 30, the effective length of the fields is $\Delta s = \frac{L}{2}$: so the angular kicks are

$$\Delta x' = -\frac{2Nr_0}{\gamma}\frac{x}{\left(x^2 + y^2\right)}\left(1 - \exp\left(-\frac{\left(x^2 + y^2\right)}{2\sigma^2}\right)\right)$$

$$\Delta y' = -\frac{2Nr_0}{\gamma}\frac{y}{\left(x^2 + y^2\right)}\left(1 - \exp\left(-\frac{\left(x^2 + y^2\right)}{2\sigma^2}\right)\right)$$

As in the case of space charge, the kicks are nonlinear in *x* and *y*, and introduce coupling. Linearizing for small x and y gives

$$\Delta x' = -\frac{Nr_0}{\gamma}\frac{x}{\sigma^2} \qquad \Delta y' = -\frac{Nr_0}{\gamma}\frac{y}{\sigma^2}$$

This corresponds to an effective focal length

$$\frac{1}{f_x} = -\frac{\Delta x'}{x} = \frac{Nr_0}{\sigma^2\gamma}$$

resulting in a tune shift

$$\Delta Q_x = \frac{1}{4\pi}\frac{\beta_x^*}{f_x} = \frac{Nr_0\beta_x^*}{4\pi\sigma^2\gamma} = \xi_x.$$

where β_x^* is the value of β_x at the collision point, and ξ_x is called the *tune shift parameter*.

This expression can be written in terms of the rms normalized emittance $\varepsilon_{n,x} = \varepsilon_x\gamma = \gamma\frac{\sigma^2}{\beta_x^*}$ as

$$\Delta Q_x = \frac{N r_0}{4\pi \varepsilon_{n,x}}. \text{ Similarly, } \Delta Q_y = \frac{N r_0}{4\pi \varepsilon_{n,y}}.$$

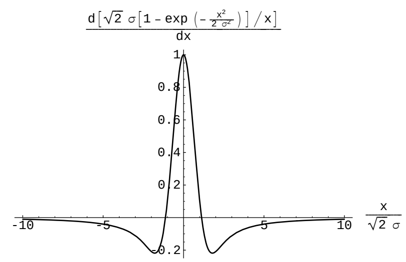
For <u>round beams</u>, the beam-beam tune shifts are independent of the beam size and lattice functions at the collision point, and depend only on the number of particles per bunch and the normalized emittance.

Example: the Tevatron collider operates with about $2\text{x}10^{11}$ protons per bunch with a normalized emittance of about 2.5 mm-mrad. The beam-beam tune shift per collision experienced by the antiprotons is

$$\Delta Q_x = \frac{N r_0}{4\pi \varepsilon_{n,x}} = \frac{2\times 10^{11}\times 1.53\times 10^{-18}}{4\pi\times 2.5\times 10^{-6}} \approx 0.01$$

Just like the case for space charge, because of the nonlinear form of the beam-beam force (for a Gaussian beam), the tune shift will depend on the amplitude of the particle's oscillation. The tune shift will be proportional to the local gradient of the force.

Plot of the gradient of $F_x(x,0)$ vs. x



The largest tune shift occurs for particles at small x; particles at large x see a smaller tune shift. This results in a tune *spread* in the beam, roughly equal to the small amplitude tune shift. This tune spread can cause problems if some parts of the beam are shifted onto resonances.

The quadrupole part of the beam-beam force also causes a distortion of the lattice functions. This is sometimes called the "dynamic beta effect".

Experimentally, it has been found that proton-antiproton colliders, which operate with round beams, can tolerate up to a total beam-beam tune shift of about 0.025.

Electron-positron colliders (perhaps because of radiation damping) can be operated with beam-beam tune shifts as large as 0.06.

For electron-positron colliders, radiation damping results in flat beam. For such flat beams, in which the charge distribution has a different size in the horizontal and vertical directions, the expression for the beam-beam tune shift is a little more complex.

$$\Delta Q_x = \xi_x = \frac{2 N r_0 \beta_x^*}{4\pi\sigma_x(\sigma_x+\sigma_y)\gamma} \approx \frac{N r_0 \beta_x^*}{2\pi\sigma_x^2\gamma}$$

$$\Delta Q_y = \xi_y = \frac{2 N r_0 \beta_y^*}{4\pi\sigma_y(\sigma_x+\sigma_y)\gamma} \approx \frac{N r_0 \beta_y^*}{2\pi\sigma_x\sigma_y\gamma}$$

If the machine is operated with $\frac{\beta_x^*}{\sigma_x} = \frac{\beta_y^*}{\sigma_y}$, then the tune shift will be the same in both planes and can be written in terms of the rms horizontal emittance ε_x as

$$\Delta Q \approx \frac{Nr_0}{2\pi\varepsilon_x \gamma}$$

Example: CESR, with 2x10[11] electrons per bunch with an rms horizontal emittance of about 0.2 mm-mrad. The beam-beam tune shift per collision experienced by the positrons and the electrons is

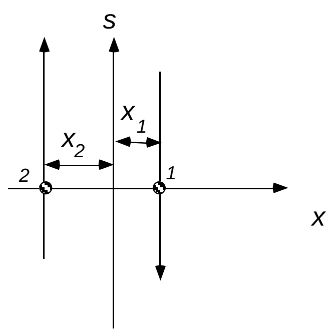
$$\Delta Q = \frac{Nr_0}{2\pi\gamma\varepsilon_x} = \frac{2\times10^{11}\times2.82\times10^{-15}}{2\pi\times2\times10^{-7}\times10^{4}} \approx 0.044$$

The focal length of the beam-beam "lens" in the *y*-plane is $f_y = \frac{\beta_y^*}{4\pi\Delta Q} = \frac{1.8\ \text{cm}}{4\pi\times0.04} = 3.26\ \text{cm}$

which can be compared with the focal length of the strongest quadrupole in CESR: about 80 cm.

Coherent beam-beam effect

The tune shifts discussed above are <u>incoherent</u> tune shifts, affecting individual particles in a bunch. There are also coherent beam-beam effects, which involve the interactions of one entire bunch with the opposing bunch. In the simplest model of the coherent collective interaction, we consider each bunch to be a rigid "macroparticle", interacting as a whole with the other bunch.



Let the displacement of the two bunches as they pass the collision point be x_1 and x_2. These displacements are assumed to be non-zero because of a coherent betatron oscillation of each bunch, driven by the electromagnetic forces exerted by one bunch on the other. The angular kick given to bunch 1 by bunch 2 can be obtained by

averaging the force seen by bunch 1 due to bunch 2 over the charge distribution of bunch 1. For equal size round beams, with x_1 and $x_2 \ll \sigma$, we find

$$\Delta x_1' = -2\pi \frac{\xi_x}{\beta_x^*}(x_1 - x_2) \qquad \Delta x_2' = 2\pi \frac{\xi_x}{\beta_x^*}(x_1 - x_2)$$

This represents a pair of coupled equations, which describe the coupled motion of the two bunches. Anticipating the result for the form of the coupled motion, we define new variables

$$u_1 = \frac{1}{\sqrt{2}}(x_1 + x_2) \quad u_2 = \frac{1}{\sqrt{2}}(x_1 - x_2)$$

The changes in the slopes of these variables is

$$\Delta u_1' = 0 \qquad \Delta u_2' = -4\pi \frac{\xi_x}{\beta_x^*} u_2$$

The one-turn matrix for the two-bunch system can be written as $\mathbf{M}_{tot} = \mathbf{M}\mathbf{M}_c$:

$$\begin{pmatrix} u_1 \\ u_1' \\ u_2 \\ u_2' \end{pmatrix}\Bigg|_{C+IP} = \mathbf{M_{tot}} \begin{pmatrix} u_1 \\ u_1' \\ u_2 \\ u_2' \end{pmatrix}\Bigg|_{IP}$$

M represents the standard one-turn matrix at the interaction point (IP), expanded to describe the motion of the centroid of the two bunches:

$$\mathbf{M} = \begin{pmatrix} \cos 2\pi Q_x & \beta_x^* \sin 2\pi Q_x & 0 & 0 \\ -\frac{1}{\beta_x^*}\sin 2\pi Q_x & \cos 2\pi Q_x & 0 & 0 \\ 0 & 0 & \cos 2\pi Q_x & \beta_x^* \sin 2\pi Q_x \\ 0 & 0 & -\frac{1}{\beta_x^*}\sin 2\pi Q_x & \cos 2\pi Q_x \end{pmatrix}$$

(we assume $\alpha^*=0$). The coupling is represented by

$$\mathbf{M}_c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -4\pi\frac{\xi_x}{\beta_x^*} & 1 \end{pmatrix}$$

The product matrix $\mathbf{M}_{tot} = \mathbf{M}\mathbf{M}_c$ takes us once around the machine, including the coupling due to the beam-beam interaction. The result is

$$\mathbf{M} = \begin{pmatrix} \cos 2\pi Q_x & \beta_x \sin 2\pi Q_x & 0 & 0 \\ -\frac{1}{\beta_x}\sin 2\pi Q_x & \cos 2\pi Q_x & 0 & 0 \\ 0 & 0 & \cos 2\pi Q_x - 4\pi\xi_x \sin 2\pi Q_x & \beta_x \sin 2\pi Q_x \\ 0 & 0 & -\frac{1}{\beta_x}\sin 2\pi Q_x - 4\pi\frac{\xi_x}{\beta_x^*}\cos 2\pi Q_x & \cos 2\pi Q_x \end{pmatrix}$$

The absence of off-diagonal terms shows that the modes u_1 and u_2 are uncoupled. The submatrix associated with the u_1 variable has the unperturbed tune. This corresponds to a mode in which both bunches move together: this is called the σ mode.

The mode u_2, in which the displacements of the bunches are equal and opposite, is called the π mode. It has a tune given by

$$\cos 2\pi Q_\pi = \cos 2\pi Q_x - 2\pi\xi_x \sin 2\pi Q_x \quad .$$

For small values of the tune shift parameter ξ_x, we have

$$\cos 2\pi Q_\pi = \cos 2\pi\left(Q_x + \Delta Q_\pi\right) \approx \cos 2\pi Q_x - 2\pi\Delta Q_\pi \sin 2\pi Q_x$$

$$\text{so } \Delta Q_\pi \approx \xi_x$$

The frequency shift of the coherent π mode can be observed on a spectrum analyzer and is an approximate indicator of the beam-beam parameter.

Tune shift parameter and luminosity

It is very often the case that the beam-beam effect is the limit to the attainable peak luminosity in a particle-antiparticle collider. In this case, it is useful to cast the luminosity formula into a different form that explicitly includes the beam-beam parameter.

For round beams, we have for the luminosity

$$\mathsf{L} = f\,\frac{BN^2}{4\pi\sigma^2}$$

in which N is the number of particles per bunch, B the number of bunches. In terms of the tune shift parameter $\xi = \frac{N r_0 \beta^*}{4\pi\sigma^2\gamma}$, we have

$$\mathsf{L}_{\text{round}} = f\,\frac{BN\xi\gamma}{r_0\beta^*}$$

This shows that, for a fixed beam-beam parameter, the luminosity is proportional to the number of particles per bunch, the energy, and the number of bunches, and inversely proportional β at the interaction point. For flat beams, for which the beam-beam parameter is $\xi_y \approx \frac{N r_0 \beta_y^*}{2\pi\sigma_x\sigma_y\gamma}$, the luminosity is

$$\mathsf{L}_{\text{flat}} = f\,\frac{BN^2}{4\pi\sigma_x\sigma_y} = f\,\frac{BN\xi_y\gamma}{2r_0\beta_y^*}$$

Examples:

Tevatron collider

f=47 kHz, B=36, N=2x10^{11}, ξ=0.01, γ=1066, r_0=1.53x10^{-18} m, β^*=35 cm=>$\mathsf{L}_{\text{round}}$=6.8x10^{32} cm^{-2} s^{-1}

CESR

f=390 kHz, B=45, N=2x10^{11}, ξ=0.05, γ=10^4, r_0=2.82 x10^{-15} m, β^*=1.8 cm=>L_{flat}=1.8x10^{33} cm^{-2} s^{-1}