LECTURE 21

Collective effects in multi-particle Beams:Wake functions and impedance

Wake fields and forces

Wake potentials and wake functions

Impedance; relation to wake functions

Longitudinal impedances in accelerators

Wake fields and forces

We've seen examples of the collective fields of the beam, and the forces they exert on individual particles. We'd like a general formalism to describe the effects of these collective forces on the trajectories of beam particles. This general formalism is provided by the concepts of *wake functions* and *impedance*.

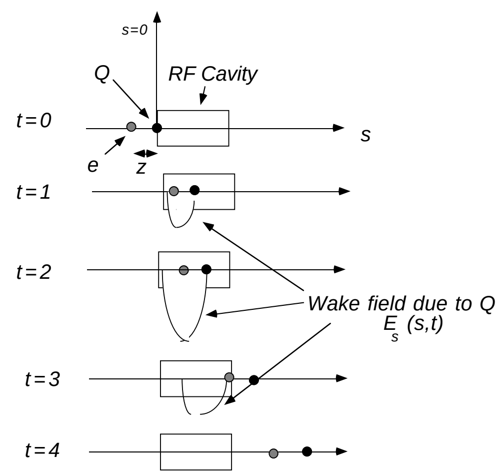
In general, the collective force will be the Lorentz force experienced by a particle in the collective fields. Let us consider the field produced by a single, highly relativistic, charged particle, of charge Q, traveling in the vacuum chamber of an accelerator. If we can find the fields due to this particle,

we can get the collective fields of the whole beam by superposition.

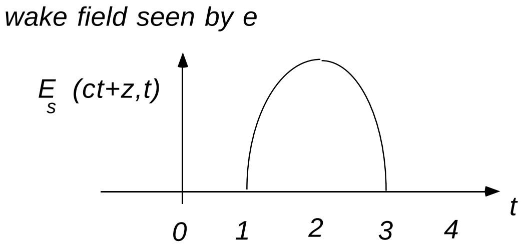
Because of the requirement that the fields satisfy the boundary conditions at the chamber walls, the general form of the fields may be quite complex. As it travels down the vacuum chamber, the particle may leave some fields behind it: these are often called its "wake fields". (Example: the particle travels through an rf cavity and excites it; the cavity continues to ring down after the particle has passed through. These beam-induced cavity fields are wake fields).

Suppressing for the moment the transverse variables, let's look at a specific component of the wake fields, the longitudinal electric field $E_s(s,t)$. Let the particle with charge Q be at s=0 at t=0. It is being followed by another particle, of charge e,

trailing a distance z behind. (z is defined to be negative for e trailing Q.) A cartoon of what happens is shown below:



As Q enters the rf cavity, a wake field develops behind it. The trailing charge e feels that wake field. The wake field gets bigger as Q gets further into the cavity, then drops off as Q exits. During its passage through the cavity, the charge e has felt a wakefield that varies with time:



Since the position of the charge Q is $s_Q=ct$, and e is always a distance $-z$ behind it, $z = s_e - ct$, and the charge e feels the wakefield $E_s(s_e,t) = E_s(ct+z,t)$, and experiences a force $F_s(ct+z,t) = eE_s(ct+z,t)$. The forces due to wakefields are generally never strong enough that the detailed variation with time will matter: they will be treated as impulses, and the only quantity of interest will be the force integrated along the trajectory. If L is the length of the rf cavity, then the integral over the cavity of the force is

$$\bar{F}_s(z) = \frac{1}{c}\int_{-\frac{z}{c}}^{L-\frac{z}{c}} dt F_s(ct+z,t) = \int_0^L ds F_s\left(s, \frac{s-z}{c}\right)$$

These integrated forces, which are only functions of the distance z between a trailing charge and the source of the wakefields, are called <u>wake potentials</u>. We now wish to rewrite Maxwell's equations in terms of wake potentials. This will give us general equations for the wake potentials, from which we'll be able to draw some conclusions.

Maxwell's equations:

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad \vec{\nabla}\bullet\vec{E} = \frac{\rho}{\varepsilon_0} \quad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} \quad \vec{\nabla}\bullet\vec{B} = 0$$

Use the Lorentz force $\vec{F} = e\left(\vec{E} + \vec{v}\times\vec{B}\right)$ to eliminate $\vec{E}$:

$$\vec{\nabla}\times\left(\frac{\vec{F}}{e} - \vec{v}\times\vec{B}\right) = -\frac{\partial\vec{B}}{\partial t} \quad \vec{\nabla}\bullet\left(\frac{\vec{F}}{e} - \vec{v}\times\vec{B}\right) = \frac{\rho}{\varepsilon_0}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\vec{F}}{e} - \vec{v}\times\vec{B}\right) \quad \vec{\nabla}\bullet\vec{B} = 0$$

These may be simplified using vector identities and the fact that $\vec{v} = c\hat{s}$

$$\vec{\nabla}\times\left(\vec{v}\times\vec{B}\right) = \vec{B}\bullet\vec{\nabla}v - \vec{v}\bullet\vec{\nabla}\vec{B} + \vec{v}\left(\vec{\nabla}\bullet\vec{B}\right) - \vec{B}\left(\vec{\nabla}\bullet\vec{v}\right) = -c\frac{\partial\vec{B}}{\partial s}$$

This follows since $\vec{v}$ is a constant, so $\vec{\nabla}v = \vec{\nabla}\bullet\vec{v} = 0$, $\vec{\nabla}\bullet\vec{B} = 0$ from Maxwell, and $-\vec{v}\bullet\vec{\nabla}\vec{B} = -c\hat{s}\bullet\vec{\nabla}\vec{B} = -c\frac{\partial\vec{B}}{\partial s}$

$$\vec{\nabla}\bullet\left(\vec{v}\times\vec{B}\right)=\vec{B}\bullet\vec{\nabla}\times\vec{v}-\vec{v}\bullet\vec{\nabla}\times\vec{B}=-c\hat{s}\bullet\left(\mu_0\vec{J}+\frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\vec{F}}{e}-\vec{v}\times\vec{B}\right)\right)$$

$$=-c\left(\mu_0 J_s+\frac{1}{ec^2}\frac{\partial F_s}{\partial t}\right)$$

in which $\vec{\nabla}\times\vec{v}=0$ since $\vec{v}$ is a constant.

Then, using $J_s=\rho c$, we find

$$\vec{\nabla}\times\left(\frac{\vec{F}}{e}\right)=-\frac{\partial\vec{B}}{\partial t}-c\frac{\partial\vec{B}}{\partial s}\qquad\vec{\nabla}\bullet\left(\frac{\vec{F}}{e}\right)=\frac{\rho}{\varepsilon_0}-\mu_0\rho c^2-\frac{1}{ec}\frac{\partial F_s}{\partial t}=-\frac{1}{ec}\frac{\partial F_s}{\partial t}$$

Now we form the wake potentials by integration of the forces. In general, we have for any function *g* representing a field or force component

$$\bar{g}(z)=\int_0^L ds\,g\left(s,\frac{s-z}{c}\right)\qquad\frac{d\bar{g}(z)}{dz}=-\frac{1}{c}\int_{-L/2}^{L/2}ds\,g^{(0,1)}\left(s,\frac{s-z}{c}\right)$$

where $g^{(0,1)}(s,t)=\frac{\partial g(s,t)}{\partial t}$.

Using the other form for the average (from p. 6)

$$\bar{g}(z)=\frac{1}{c}\int_{-\frac{z}{c}}^{L-\frac{z}{c}}dt\,g(ct+z,t)$$

, we have

$$\frac{d\bar{g}(z)}{dz}=g(0,-z/c)-g(L,(L-z)/c)+\frac{1}{c}\int_{-\frac{z}{c}}^{L-\frac{z}{c}}dt\,g^{(1,0)}(ct+z,t)$$

$$=\frac{1}{c}\int_{-\frac{z}{c}}^{L-\frac{z}{c}}dt\,g^{(1,0)}(ct+z,t)=\int_{-L/2}^{L/2}ds\,g^{(1,0)}\left(s,\frac{s-z}{c}\right)$$

where $g^{(1,0)}(s,t)=\frac{\partial g(s,t)}{\partial s}$, provided the wake fields go to zero at the ends of the impedance, or are only a function of *s*-*ct*.

Then

$$\int_{-L/2}^{L/2}ds\left(-\frac{\partial\vec{B}}{\partial t}-c\frac{\partial\vec{B}}{\partial s}\right)=c\frac{\partial\bar{\vec{B}}}{\partial z}-c\frac{\partial\bar{\vec{B}}}{\partial z}=0$$

$$\int_{-L/2}^{L/2}ds\left(\frac{1}{e}\frac{\partial F_s}{\partial s}+\frac{1}{ec}\frac{\partial F_s}{\partial t}\right)=\frac{1}{e}\frac{\partial\bar{F}_s}{\partial z}-\frac{1}{e}\frac{\partial\bar{F}_s}{\partial z}=0$$

and we find

$$\vec{\nabla}\times\bar{\vec{F}}=0\qquad\frac{\partial\bar{F}_x}{\partial x}+\frac{\partial\bar{F}_y}{\partial y}=\vec{\nabla}_\perp\bullet\bar{\vec{F}}_\perp=0$$

More vector calculus: Since $\vec{\nabla}\times\bar{\vec{F}}=0$, we can write $\bar{\vec{F}}=-\vec{\nabla}V$, where *V* is a scalar potential. The transverse part of $\bar{\vec{F}}$ is given by $\bar{\vec{F}}_\perp=-\vec{\nabla}_\perp V$, and the longitudinal part by $\bar{F}_s=-\frac{\partial V}{\partial z}$.

Panofsky-Wentzel theorem:

$$\frac{\partial \vec{\bar{F}}_\perp}{\partial z} = -\vec{\nabla}_\perp \frac{\partial V}{\partial z} = \vec{\nabla}_\perp \bar{F}_s$$

This theorem relates the longitudinal gradient of the transverse wake potential to the transverse gradient of the longitudinal wake potential.

Wake functions

Since $\vec{\nabla}_\perp \bullet \vec{\bar{F}}_\perp = \vec{\nabla}_\perp^2 V = 0$, the transverse part of V is a solution to the two-dimensional LaPlace equation. In (r,ϕ) cylindrical

coordinates, *if the boundary conditions are axisymmetric*, the solution for V can be written in the form

$$V(r,\phi,z) = e\sum_m W_m(z) r^m \left(Q_m \cos m\phi + \tilde{Q}_m \sin m\phi\right)$$

with

$$Q_m = \int_0^{2\pi} d\phi' \int_0^\infty dr' r'^{m+1} \cos m\phi' \rho(r',\phi')$$

$$\tilde{Q}_m = \int_0^{2\pi} d\phi' \int_0^\infty dr' r'^{m+1} \sin m\phi' \rho(r',\phi')$$

The coefficients $W_m(z)$ are called the *wake functions*. They depend only on the details of the environment in which the beam is travelling (e.g, structure of an rf cavity it may be

passing through). The Q_m coefficients are the moments of the charge distribution of the beam that is producing the wakefields. If Q is the total charge, then

$$Q_0 = \int_0^{2\pi} d\phi' \int_0^\infty dr' r' \rho(r',\phi') = Q$$

$$Q_1 = \int_0^{2\pi} d\phi' \int_0^\infty dr' r' (r' \cos m\phi') \rho(r',\phi') = Q\langle x \rangle$$

$$\tilde{Q}_1 = \int_0^{2\pi} d\phi' \int_0^\infty dr' r' (r' \sin m\phi') \rho(r',\phi') = Q\langle y \rangle$$

and so on. In terms of the wake functions, the wake potentials can be written as

$$\vec{\bar{F}}_\perp = -\vec{\nabla}_\perp V \Rightarrow$$

$$\vec{\bar{F}}_{\perp,m} = -emW_m(z) r^{m-1} \begin{pmatrix} \hat{r}\left(Q_m \cos m\phi + \tilde{Q}_m \sin m\phi\right) \\ -\hat{\phi}\left(Q_m \sin m\phi - \tilde{Q}_m \cos m\phi\right) \end{pmatrix}$$

$$\bar{F}_{s,m} = -\frac{\partial V_m}{\partial z} = -eW_m'(z) r^m \left(Q_m \cos m\phi + \tilde{Q}_m \sin m\phi\right)$$

The index m describes the transverse variation of the wake potentials. For the longitudinal potential, m=0 is constant, m=1 varies linearly with x and y, etc.

$$\bar{F}_{s,0} = -eQW_0'(z)$$

$$\bar{F}_{s,1} = -eW_1'(z) r\left(Q_1 \cos\phi + \tilde{Q}_1 \sin\phi\right) = -eQW_1'(z)\left(\langle x \rangle x + \langle y \rangle y\right)$$

For m=0, there is no transverse potential. For m=1, the transverse potential is constant, but depends on the dipole moments of the source beam:

$$\vec{F}_{\perp,1} = -eW_1(z)\left(\hat{r}\left(Q_1\cos\phi + \tilde{Q}_1\sin\phi\right) - \hat{\phi}\left(Q_1\sin\phi - \tilde{Q}_1\cos\phi\right)\right)$$

$$= -eW_1(z)\left(Q_1\hat{x} + \tilde{Q}_1\hat{y}\right) = -eQW_1(z)\left(\langle x\rangle\hat{x} + \langle y\rangle\hat{y}\right)$$

The units of the wake functions depend on the index m. The units of W_0' are V/C, and of W_1' are V/(C-meter); the units of W_m are V/(C-meter$^{(2m-1)}$).

If we know the wake functions $W_m(z)$, then we can find all the components of the integrated forces on a particle due to wake fields, and we can construct the trajectory equations and solve for the particle's motion.

The wake functions have a number of important general properties, of which one of the most important is that $W_m(z) = 0 \quad \text{for} \quad z > 0$. This follows from causality: the wake fields cannot exist in front of the particle.

There are some simple, crude scaling rules for wake potentials associated with cavity structures of a size similar to the vacuum chamber radius b. Since W_m depends only on the environment of the beam, and b is the only dimension in that envirnoment, W_m must scale like $1/b^{2m-1}$ and W_m' as $1/b^{2m}$. The transverse wake potentials scale roughly as $\left(\frac{a}{b}\right)^{2m-1}$, and the longitudinal wake potentials scale roughly as $\left(\frac{a}{b}\right)^{2m}$, where a is a measure

of the beam size. Since typically $a \ll b$, higher m potentials tend to be less important.

The detailed determination of wake functions is a complex business, usually only done numerically for realistic cases. However, we can find the wake functions for some simple situations by introducing the concept of impedance. In addition to allowing estimate of wake functions, this concept is an extremely useful way to understand the effects of wake fields and collective effects in general. The connection between wake functions and impedance is described in what follows.

Impedance

The impedance is related to the fields produced by a pure harmonic current distribution. Any general current distribution $I(s,t)$, can be Fourier decomposed into harmonics of the

form $I_0(s,t) = \tilde{I}_0(k,\omega)\exp(i(ks-\omega t))$. (The 0 subscript corresponds to a current with no x-y dependence). Consider an rf cavity, or other source of impedance, of length L, through which this harmonic of the beam current flows. We *define* the longitudinal impedance $Z_0^{\parallel}(\omega)$ of that cavity as given by the relation

$$\bar{E}_s(s,t) = -I_0(s,t)Z_0^{\parallel}(\omega)$$

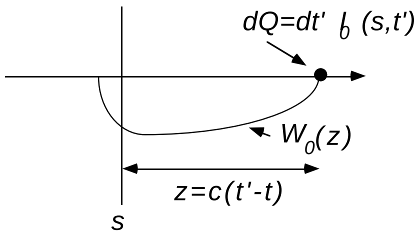
where $\bar{E}_s(s,t)$ is the integral over L of the longitudinal electric wake field (i.e, the voltage), produced by the current $I_0(s,t)$.

The wake potentials correspond to fields produced by a point charge. How do we relate these to fields produced by a current, such as $I_0(s,t)$? Use the principle of superposition:

The longitudinal wake potential corresponding to m=0, produced by an element of charge dQ, is

$$d\overline{F}_s = ed\overline{E}_s = -eW_0'(z)dQ$$

To find the integrated field for a current distribution $I_0(s,t)$, we need to write dQ in terms of $I_0(s,t)$, and integrate over the whole current distribution.



Focus on a particular longitudinal position s. The element of charge passing this point at time t' is $dQ = I_0(s,t')dt'$. At a later time t, (shown in the figure above), the wake function at s due to this element of charge is $W_0(z)$, where z is the distance from s to the location of dQ at t: $z = c(t'-t)$. Thus, we have

$$d\overline{E}_s = -W_0'(c(t'-t))I_0(s,t')dt'$$

To find the total integrated longitudinal field, we integrate over all earlier times t'

$$\overline{E}_s(s,t) = -\int_{-\infty}^{t} dt' W_0'(c(t'-t))I_0(s,t') = -\int_{-\infty}^{\infty} dt' W_0'(c(t'-t))I_0(s,t')$$

where we can extend the integral to +∞ since $W_0(z) = 0 \quad \text{for} \quad z > 0$. Then we change variables to $z = c(t'-t)$ to get

$$\overline{E}_s(s,t) = -\frac{1}{c}\int_{-\infty}^{\infty} dz W_0'(z) I_0(s, \frac{z}{c} + t)$$

Now using the harmonic form $I_0(s,t) = \tilde{I}_0(k,\omega)\exp(i(ks-\omega t))$, we find

$$\overline{E}_s(s,t) = -\frac{\tilde{I}_0(k,\omega)}{c}\int_{-\infty}^{\infty} dz W_0'(z)\exp\left(i\left(ks - \omega\left(\frac{z}{c}+t\right)\right)\right)$$

$$= -\frac{\tilde{I}_0(k,\omega)}{c}\exp(i(ks-\omega t))\int_{-\infty}^{\infty} dz W_0'(z)\exp\left(-i\frac{\omega z}{c}\right)$$

$$= -\frac{I_0(s,t)}{c}\int_{-\infty}^{\infty} dz W_0'(z)\exp\left(-i\frac{\omega z}{c}\right)$$

Comparing with the defining equation relating the integrated field to the impedance, given above, we see that the impedance is related to the wake function by

$$Z_0^{\parallel}(\omega) = \frac{1}{c}\int_{-\infty}^{\infty} dz W_0'(z)\exp\left(-i\frac{\omega z}{c}\right)$$

That is, the impedance is just the Fourier transform of the wake function. Similar discussions for $m>0$ (corresponding to currents with some transverse spatial dependence) generalize the above relation to

$$Z_m^{\|}(\omega)=\frac{1}{c}\int_{-\infty}^{\infty}dz W_m'(z)\exp\left(-i\frac{\omega z}{c}\right)$$

Also, for $m>0$, we can define a transverse impedance by

$$\vec{F}_{\perp}(s,t)=ieI_m(s,t)mr^{m-1}\left(\hat{r}\cos m\phi-\hat{\phi}\sin m\phi\right)Z_m^{\perp}(\omega)$$

with $I_m(s,t)$ the mth moment of the current distribution. The transverse impedance relates to the transverse wake function by

$$Z_m^{\perp}(\omega)=\frac{i}{c}\int_{-\infty}^{\infty}dz W_m(z)\exp\left(-i\frac{\omega z}{c}\right)$$

If we know the impedance, then we can find the wake functions by inverse Fourier transform:

$$W_m'(z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega Z_m^{\|}(\omega)\exp\left(i\frac{\omega z}{c}\right)$$

$$W_m(z)=-\frac{i}{2\pi}\int_{-\infty}^{\infty}d\omega Z_m^{\perp}(\omega)\exp\left(i\frac{\omega z}{c}\right)$$

from which we also see that $Z_m^{\|}(\omega)=\frac{\omega}{c}Z_m^{\perp}(\omega)$.

The fact that $W_m(z)$ is real leads to the following relations:

$$\left[Z_m^{\|}(\omega)\right]^*=Z_m^{\|}(-\omega)\qquad\left[Z_m^{\perp}(\omega)\right]^*=-Z_m^{\perp}(-\omega)$$

which in turn imply that

$\mathrm{Re}\left[Z_m^{\|}(\omega)\right]$ and $\mathrm{Im}\left[Z_m^{\perp}(\omega)\right]$ are even in ω

$\mathrm{Im}\left[Z_m^{\|}(\omega)\right]$ and $\mathrm{Re}\left[Z_m^{\perp}(\omega)\right]$ are odd in ω

The longitudinal impedance associated with wake potentials that do not vary with x and y is $Z_0^{\|}(\omega)$. This is often referred to as “the” longitudinal impedance. For $m=0$, the transverse wake potentials are zero. The first nonzero transverse wake potentials correspond to $m=1$. The corresponding transverse impedance, $Z_1^{\perp}(\omega)$, is often referred to as “the” transverse impedance. Typically, higher m impedances are less important in machines than the $m=0$ and $m=1$ pieces.

For a given general cavity structure, there is no precise general connection between $Z_0^{\|}(\omega)$ and $Z_1^{\perp}(\omega)$. However, for cavity structures of a size similar to the vacuum chamber radius b, we’ve seen from dimensional analysis that $W_m'\sim 1/b^{2m}$; so

$Z_m^{\|}\sim\frac{b}{c}W_m'\sim 1/b^{2m-1}$. Thus $\frac{Z_m^{\|}}{Z_0^{\|}}\approx 1/b^{2m}$, so

$$Z_1^{\|}(\omega)\approx\frac{Z_0^{\|}(\omega)}{b^2}$$

From the relation given above between transverse and longitudinal impedances, we get

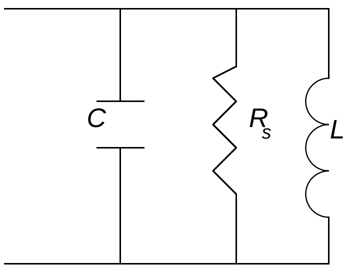
$$Z_1^{\perp}(\omega)\approx\frac{cZ_0^{\|}(\omega)}{\omega b^2}$$

The unit of longitudinal impedance $Z_m^{\|}(\omega)$ is Ω/meter^{2m}, and of transverse impedance $Z_m^{\perp}(\omega)$ is Ω/meter$^{(2m-1)}$

Longitudinal impedances in accelerators

RF cavities.

This is typically the dominant contribution to the longitudinal machine impedance. We can model an rf cavity as a parallel RLC circuit



The longitudinal impedance of this circuit is

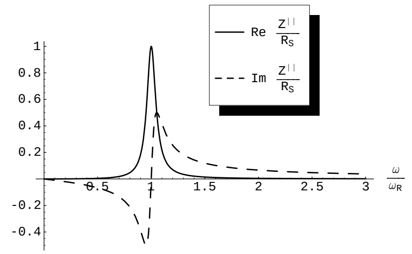
$$\frac{1}{Z_0^{\|}} = \frac{1}{R_s} + \frac{i}{\omega L} - i\omega C$$

which can be written in terms of the resonant frequency $\omega_R = \frac{1}{\sqrt{LC}}$, the Q-value $Q = \frac{R_s}{\omega_R L}$ and the cavity shunt impedance R_s:

$$Z_0^{\|}(\omega) = \frac{R_s}{1 + iQ\left(\frac{\omega_R}{\omega} - \frac{\omega}{\omega_R}\right)}$$

For large Q, this impedance is sharply peaked and real at ω_R. For $|\omega| << \omega_R$, it is mostly negative imaginary ("inductive"); for $|\omega| >> \omega_R$, it is mostly positive imaginary ("capacitive").

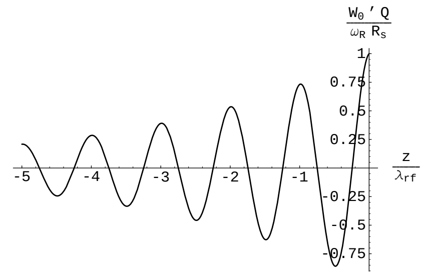
Plot of $\frac{Z_0^{\|}\left(\frac{\omega}{\omega_R}\right)}{R_s}$ vs $\frac{\omega}{\omega_R}$ for Q=10.



The wake function for this impedance can be obtained by taking a Fourier transform of the impedance. The result, for z<0, is

$$W_0'(z) = \frac{\omega_R R_s}{Q} \exp\left(\frac{\omega_R z}{2cQ}\right)\left(\cos\left[\frac{\omega_R z}{c}\sqrt{1-\frac{1}{4Q^2}}\right] + \frac{\sin\left[\frac{\omega_R z}{c}\sqrt{1-\frac{1}{4Q^2}}\right]}{\sqrt{4Q^2-1}}\right)$$

Plot of $\frac{W_0'(z)Q}{\omega_R R_s}$ vs $\frac{z}{\lambda_{rf}}$ for Q=10.



For Q>>1, this simplifies to $W_0'(z) = \frac{\omega_R R_s}{Q}\exp\left(\frac{\omega_R z}{2cQ}\right)\cos\left[\frac{\omega_R z}{c}\right]$

The wakefield oscillates in z with a wavelength equal to λ_{rf}; it is damped to $1/e$ in a distance $\frac{Q}{\pi}\lambda_{rf}$.

Numerical example:

Suppose that I have a copper pillbox rf cavity in an accelerator, with a resonant frequency of 500 MHz. For operation as an accelerating cavity, the cavity radius should be R=23 cm, and the length should be L=(2/3)R=15 cm, giving a transit time factor of 0.9. Such a pillbox cavity has a Q of about 32,000, a shunt impedance per unit length of about 52 MΩ/m, and a shunt impedance R_s = 8 MΩ. This is also the value of the longitudinal impedance, on resonance.

The wake function has an amplitude equal to

$$\frac{\omega_R R_s}{Q} = \frac{\pi\times 10^9 \times 8\times 10^6}{32000} \approx 7.9\times 10^{11}\ \frac{\text{V}}{\text{C}}$$

It decays from this value over a length of about

$$\frac{Q}{\pi}\lambda_{rf} = \frac{32000}{\pi}\times 0.6\ \text{ m} \approx 6\text{ km}$$

which is about 8 turns in CESR.

If a bunch passing through this cavity has 2x10[11] particles, the longitudinal wake potential it creates in the rf cavity is

$$\left|\frac{\overline{F_s}}{e}\right| \approx Q_0\frac{\omega_R R_s}{Q} = (Ne)\times 7.9\times 10^{11}\ \frac{\text{V}}{\text{C}}$$

$$= (2\times 10^{11}\times 1.6\times 10^{-19})\times 7.9\times 10^{11}\ \text{ V}$$

$$\approx 25\ \text{ kV}$$

This is the effective peak decelerating or accelerating voltage applied to a trailing particle by the wakefield of the bunch.