# LECTURE 22

## Collective effects in multi-particle beams:

Longitudinal impedances in accelerators

Transverse impedances in accelerators

Parasitic Losses



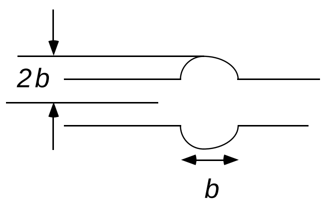
## Longitudinal impedances in accelerators (continued)

### The broad-band resonator model

The vacuum chamber of a typical accelerator is not a perfectly smooth round pipe. Diagnostic devices, such as beam position monitors, are typically sprinkled throughout the machine; these devices may have pickup plates and thus deviate from a cylindrical geometry. Special magnets, such as kickers and septa for injection and extraction, or wigglers and undulators, may have irregular apertures. Special devices such as separators, and the transitions into and out of rf cavities, also represent changes in the dimensions of the vacuum chamber.

A very crude model for these discontinuities in the vacuum chamber's dimensions is to consider them to be small resonant cavities, of the following generic form





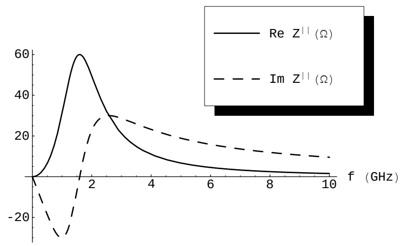

Such a cavity has a radius $2b$ and a resonant frequency of order $\omega_R = c/b$. In travelling past this cavity, the beam wake fields that penetrate the cavity are left behind as the beam exits the cavity: this constitutes an energy loss to the beam. This may be estimated by computing the stored energy in the cavity due to the beam's fields. A roughly equal amount of energy at $\omega > c/b$ propagates down the pipe with the beam. Equating the total



energy lost by the beam to the integrated power loss on the cavity impedance gives a crude estimate of the cavity impedance close to resonance: about 60 Ω. Examination of the response of the beam to the cavity at low frequencies then shows that the effective $Q$ is close to 1. This is the basis of the *broad-band resonator model*. In this model, the generic "cavity" is treated as a single, low-$Q$ resonator ($Q$=1), with a resonant frequency $\omega_R = c/b$, where $b$ is the radius of the vacuum chamber, and a shunt impedance $R_s$=60 Ω. From the general form for a cavity resonator, the impedance is then

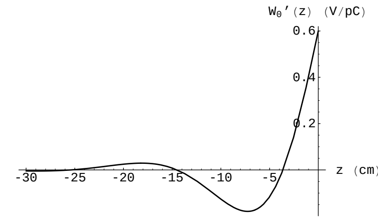
$$Z_0^{\|}(\omega) = \frac{60 \quad \Omega}{1 + i\left(\frac{c}{b\omega} - \frac{b\omega}{c}\right)}$$

A plot is given below, for the case $b$=3 cm:




The impedance is peaked at a high frequency, about 2 GHz. It tends to be mostly resistive there, and mostly inductive at low frequencies. The wake function is






It has quite a short range, because of the low $Q$. Its amplitude at small $z$ is comparable to the wake function from a narrow band resonator.

The broad band resonator model is not very accurate for frequencies above cutoff $\omega >> c/b$ ($|z| << b$), so the details near $z$=0 are wrong. Nevertheless, the model is useful for rough estimates.



We'll see later (perhaps) that the effect of the longitudinal impedance $Z_0^{\|}$ on the dynamics of the beam scales like $Z_0^{\|}(\omega)/n(\omega)$, where $n(\omega) = \omega/\omega_0$, with $\omega_0$ being the revolution frequency: $\omega_0 = 2\pi/T = 2\pi c/C$. A broad band resonator thus gives a contribution to the total $Z_0^{\|}/n$ of the machine equal to $\left(Z_0^{\|}/n\right)_{bb} \approx \dfrac{R_s\omega_0}{\omega_R} = \dfrac{377b}{C}$ Ω. A well-designed machine will have a broad band impedance of no more than about $\left(Z_0^{\|}/n\right)_{\max} \approx 1$ Ω. Thus, the maximum number of generic broad-band cavities allowed per unit length is about



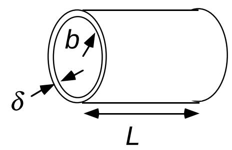
$$\frac{n_{bb\,cavity}}{C} \leq \frac{1}{377b}.$$

For example, for $b$=3 cm, we need to have less than about 1 such cavity every 11 m. This gives a crude estimate of the required "smoothness" of the machine's vacuum chamber.

## Impedance of the resistive wall

A relativistic point charge travelling through a vacuum chamber with perfectly conducting walls leaves behind no wake fields, since the fields do not penetrate the chamber. No energy is dissipated in the walls. However, if the vacuum chamber walls have a finite conductivity, then energy will be dissipated by the beam's induced currents, and a wake field will be produced.

The full expression for the wake fields and wake potentials can only be obtained by solving Maxwell's equations in the resistive pipe. (See text, sec. 6.3.2) However, we can get a crude estimate of the impedance of the wall in the following simple picture:




Let the conductivity of the wall be $\sigma$. The current flowing in a section of the wall of length $L$ passes through an area



$A = 2\pi b\delta$, where $b$ is the pipe radius, and the skin depth is $\delta = \sqrt{\frac{2}{\sigma\mu\omega}}$. Thus, the resistance per unit length is

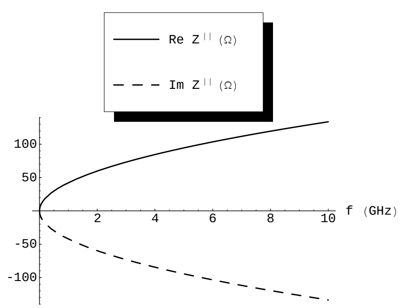
$$\frac{R_{wall}}{L} = \frac{1}{\sigma A} = \frac{1}{\sigma 2\pi b\delta} = \frac{1}{2\pi b}\sqrt{\frac{\mu\omega}{2\sigma}}$$

The full solution for the fields shows that the impedance is complex; the above is its real part. The complete expression, for a machine of circumference $C$, is

$$Z_0^{\|}(\omega) = C\frac{1 - i\,\mathrm{sgn}(\omega)}{2\pi b}\sqrt{\frac{\mu|\omega|}{2\sigma}}$$

The following plot shows the longitudinal resistive wall impedance, for $b$=3 cm, an aluminum wall, and $C$=750 m.

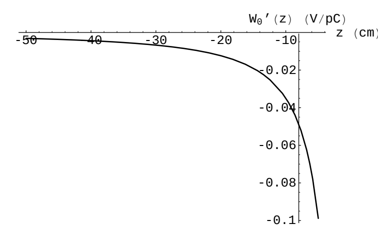





The associated wake field can be established by an inverse Fourier transform and is given by the equation $W_0'(z) = -C\frac{c}{4\pi b}\sqrt{\frac{c\mu}{\pi\sigma}}\frac{1}{\sqrt{|z|^3}}$

It is plotted in the next figure

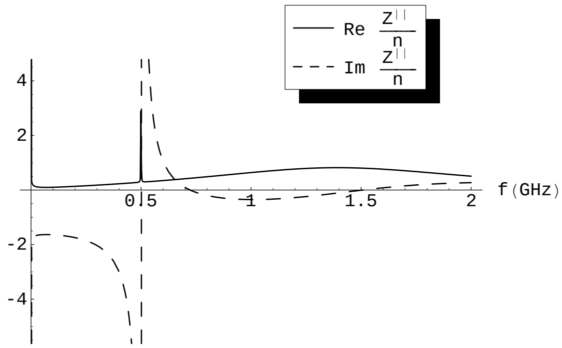





The slow decay of the resistive wall wake function with $z$ leads to a long tail.

Total longitudinal impedance:

The next plot shows the total impedance $\left(Z_0^{\|}/n\right)$ as a function of frequency for 4 narrow band cavities at 500 MHz (with the parameters given in the previous numerical example) and 50 generic broad band resonators. The resistive wall impedance is also included, although it is small: a few tenths of an ohm.






At high frequencies, the impedance is mostly resistive, dominated by the broad band resonators. Near the frequency of the rf cavities, they dominate. At low frequencies, the impedance is mostly inductive, due to the broad band resonators.

One type of longitudinal impedance that we have not discussed here is the *longitudinal space charge impedance*. The wake functions are derivable from the longitudinal space charge forces, which result from variations in the longitudinal charge density. Like transverse space charge forces, the wake functions and the impedance decrease with $1/\gamma^2$, and so are inconsequential for high energy electron machines, but may play an important role in relatively low energy (1-10 GeV) proton machine. Longitudinal space charge is discussed in the text, sec 6.2.1.



## Transverse impedance in accelerators

The principal sources of transverse impedances in accelerators are similar to the longitudinal ones that we have just discussed. There will be transverse impedance associated with narrow band rf cavities, broad band resonators, and the resistive wall.

Narrow-band transverse impedance

For any mode *m*, the transverse and longitudinal rf cavity impedances are related by

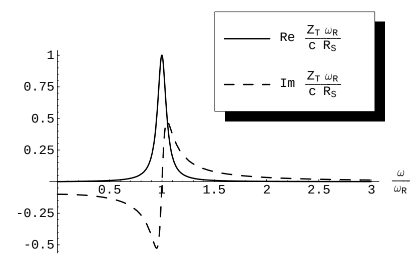
$$Z_m^{\|}(\omega) = \frac{\omega}{c} Z_m^{\perp}(\omega)$$

Thus



$$Z_1^{\perp}(\omega) = \frac{c}{\omega} Z_1^{\|}(\omega) = \frac{c}{\omega} \frac{R_S^{\perp}}{1 + iQ^{\perp}\left(\dfrac{\omega_R^{\perp}}{\omega} - \dfrac{\omega}{\omega_R^{\perp}}\right)}$$

in which the parameters $R_S^{\perp}, Q^{\perp}$ and $\omega_R^{\perp}$ now refer to a transverse cavity mode, that is, one for which the fields produce transverse forces. A plot of the transverse impedance is:

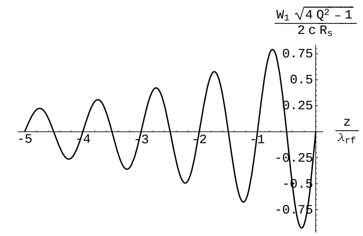


The wake function for this impedance can be obtained by taking a Fourier transform. The result, for $z<0$, is

$$W_1(z) = \frac{2cR_s^{\perp}}{\sqrt{4\left(Q^{\perp}\right)^2 - 1}} \exp\left(\frac{\omega_R^{\perp} z}{2cQ^{\perp}}\right) \sin\left[\frac{\omega_R^{\perp} z}{c}\sqrt{1 - \frac{1}{4\left(Q^{\perp}\right)^2}}\right]$$

Plot of $\frac{W_1(z)\sqrt{4\left(Q^{\perp}\right)^2 - 1}}{2cR_s^{\perp}}$ vs $\frac{z}{\lambda_{rf}}$ for $Q^{\perp}=10$.






As for the longitudinal case, the wakefield oscillates in $z$ with a wavelength equal to $\lambda_{rf}$; it is damped to $1/e$ in a distance

$$\frac{Q^{\perp}}{\pi}\lambda_{rf}.$$

Transverse broad-band resonators

We can model the transverse effects of a generic cavity in the machine with the same broad-band resonator model we used in



the longitudinal plane. To relate the transverse impedance of a broad-band resonator to the longitudinal impedance, we use the approximate result quoted in Lecture 24:

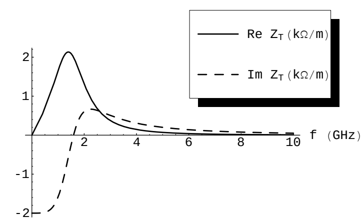
$$Z_1^{\perp}(\omega) \approx \frac{cZ_0^{\parallel}(\omega)}{\omega b^2}.$$

The broad-band transverse impedance is then

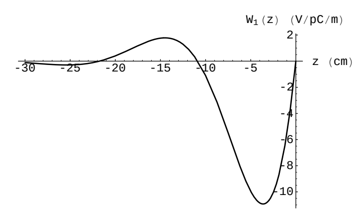
$$Z_1^{\perp}(\omega) \approx \frac{c}{\omega}\frac{60 \quad \Omega/b^2}{1 + i\left(\frac{c}{b\omega} - \frac{b\omega}{c}\right)}$$

A plot is given below, for the case $b$=3 cm:






At low frequencies, the imaginary part dominates. The wake function is



It has quite a short range, because of the low $Q$.

Numerical example:

If a bunch passing through this broad band resonator has 2x10[11] particles, and it is off-axis in the cavity by 2 cm in $x$, then the transverse deflecting (integrated) force it produces at z~-5 cm is

$$\left|\frac{\overline{F}_x}{e}\right| \approx Q_0 W_1(-5\,\mathrm{cm})\langle x\rangle = (Ne)\times 11\times 10^{12}\times 0.02 \ \ \frac{\mathrm{V}}{\mathrm{C}}$$
$$\approx (2\times 10^{11}\times 1.6\times 10^{-19})\times 2.2\times 10^{11} \ \ \mathrm{V}$$
$$\approx 7 \ \ \mathrm{kV}$$



The change in the $x$-trajectory slope of a particle following in the wake is

$$\Delta x' = \frac{\overline{F}_x}{pv} = \frac{\overline{F}_x}{m_0 c^2 \beta^2 \gamma}$$

For an electron in CESR, with $\gamma$=10[4], we find

$$\Delta x' = \frac{7 \ \ \mathrm{keV}}{511\times 10^3 \times 10^4 \ \ \mathrm{eV}} = 1.4 \ \ \mu\mathrm{rad}$$

This is the effective peak dipole kick applied to a trailing particle by the wakefield of the bunch.



Transverse impedance of the resistive wall

The resistive wall transverse impedance can be obtained from the resistive wall longitudinal impedance using the approximate relation from Lecture 24:

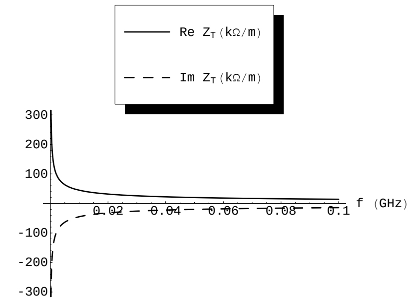
$$Z_1^{\perp}(\omega) \approx \frac{c Z_0^{\parallel}(\omega)}{\omega b^2}$$

Using the expression given above for the resistive wall longitudinal impedance, we have for the transverse impedance, for a machine of circumference $C$,

$$Z_1^{\perp}(\omega) = C\frac{1 - i\,\mathrm{sgn}(\omega)}{\pi b^3}\sqrt{\frac{\mu c^2}{2|\omega|\sigma}}$$



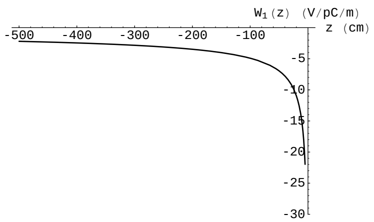
in which an extra factor of has been inserted in the numerator (the approximate relation given above is only good to a factor of two in this case). The following plot shows the transverse resistive wall impedance, for $b$=3 cm, an aluminum wall, and $C$=750 m.



This impedance is quite strong at low frequencies.

The associated wake field is given by $W_1(z) = -\frac{cC}{\pi b^3}\sqrt{\frac{c\mu}{\pi\sigma}}\frac{1}{\sqrt{|z|}}$

It is plotted in the next figure



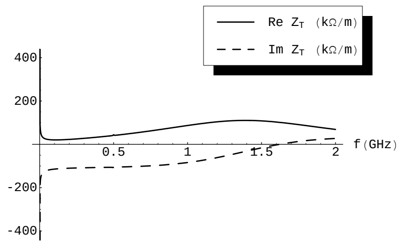

The decay of the resistive wall transverse wake function with *z* is even slower than that of the longitudinal resistive wall. The



very long tail can be important in driving transverse instabilities in which multiple bunches are coupled together.

Total impedance: The next plot shows the total impedance $Z_1^{\perp}$ as a function of frequency for 50 generic broad band resonators, and the resistive wall. Narrow band cavities are not included; generally they do not play an important role, unless they have very strong transverse deflecting modes.






The real part is dominated by broad band resonators at high frequencies, and the resistive wall at low frequencies. The imaginary part is mostly due to the broad band resonators except at very low frequencies, where the resistive wall takes off.

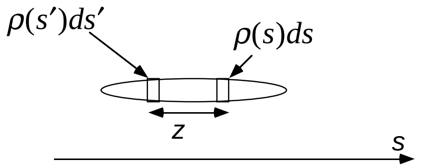
Transverse space charge can also be considered to be a source of impedance. The wake functions are derivable from the



transverse space charge forces, obtained in lecture 23. The wake functions and the impedance decrease with $1/\gamma^2$, and so are inconsequential for high energy electron machines, but may play an important role in relatively low energy (1-10 GeV) proton machines.

## Parasitic Losses

When a bunch passes through a cavity or other source of longitudinal impedance in a machine and generates longitudinal wakefields, these fields will tend to decelerate the bunch itself. Such energy losses are called *parasitic losses*. Consider an extended charge distribution $\rho(s)$ passing through a cavity. An increment of charge $dq_1 = \rho(s)ds$ in the front of the bunch produces a longitudinal wake function $W_0'(z)$, which is seen by an element of charge $dq_2 = \rho(s')ds'$ later in the bunch.




The incremental wake potential seen by $dq_2$ due to $dq_1$ is

$$d^2\overline{F}_s = -dq_1 dq_2 W_0'(z)$$

The total change in the energy of the bunch is

$$\Delta E = -\int d^2F_s = \int_{-\infty}^{\infty} ds'\rho(s')\int_{s'}^{\infty} ds\rho(s)W_0'(s'-s)$$



Suppose that the bunch is very, very short: much shorter than the distance scale over which $W_0'(z)$ varies. Then

$$\Delta E \cong -W_0'(0_-)\int_{-\infty}^{\infty} ds'\rho(s')\int_{s'}^{\infty} ds\rho(s)$$

in which $W_0'(0_-)$ is the value of the longitudinal wake function at a very small distance from $z$=0. Then, if we make the substitution

$$u(s') = \int_{s'}^{\infty}\rho(s)ds \qquad du = -\rho(s')ds'$$

$$u(-\infty) = \int_{-\infty}^{\infty}\rho(s)ds = q$$



where $q$ is the total charge, then

$$\int_{-\infty}^{\infty} ds'\rho(s')\int_{s'}^{\infty} ds\rho(s) = -\int_{q}^{0} udu = \frac{q^2}{2}$$

and $\Delta E \cong -\frac{q^2}{2}W_0'(0_-)$

We see that for a very short bunch (i.e., a point charge), the energy lost in passing through an impedance is *one-half* of the product of the charge squared with the longitudinal wake field produced by the point charge at $z$=0_. This is called the *fundamental theorem of beam loading* . For an rf cavity, from the expression given above for the wake function, we have for



the parasitic energy loss of a point charge in the cavity

$$\Delta E = -\frac{q^2}{2}\frac{R_s\omega_R}{Q} = -q^2k$$

in which $k = \frac{R_s\omega_R}{2Q}$ is called the *loss factor* of the cavity. If the cavity can oscillate in modes other than the fundamental, there will be a $k$ for each mode. Each $k$ will give the energy deposited into that mode by a point charge travelling through the cavity, and will also be related, by $2k = W_0'(0_-)$, to the wake function associated with the impedance of that mode.

Example: consider the 500 MHz narrow-band rf cavity discussed earlier. The loss factor is

$$k=\frac{\omega_R R_s}{2Q}=\frac{\pi\times10^9\times8\times10^6}{2\times32000}\approx4\times10^{11}\frac{\text{J}}{\text{C}^2}=0.4\frac{\text{V}}{\text{pC}}$$

So a beam with a very short bunch and a charge of $2\times10^{11}\times1.6\times10^{-19}=3.2\times10^4$ pC will loose about 12.8 keV in the cavity on each passage.

In general, for a bunch of finite length, the parasitic energy loss will be less than for a point charge. The loss can be computed from the relation given above

$$\Delta E=-\int_{-\infty}^{\infty}ds'\rho(s')\int_{-\infty}^{\infty}ds\rho(s)W_0'(s'-s)$$

in which the lower integration limit has been extended to $-\infty$, since $W_0(z)$=0 for $z$>0. Then, introducing the longitudinal



impedance $Z_0^{\parallel}(\omega)=\frac{1}{c}\int_{-\infty}^{\infty}dzW_0'(z)\exp\left(-i\frac{\omega z}{c}\right)$, this expression can be transformed into

$$\Delta E=-\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\left|\tilde\rho(\omega)\right|^2\mathrm{Re}\left(Z_0^{\parallel}(\omega)\right)$$

where $\tilde\rho(\omega)=\int_{-\infty}^{\infty}ds\exp(-i\omega s)\rho(s)$ is the Fourier transform of the longitudinal charge density. For a Gaussian bunch of charge $Ne$ and rms length $\sigma_s$, we have

$$\tilde\rho(\omega)=Ne\exp\left(-\frac{\omega^2\sigma_s^2}{2c^2}\right)$$



so the parasitic loss is

$$\Delta E=-\frac{(Ne)^2}{2\pi}\int_{-\infty}^{\infty}d\omega\exp\left(-\frac{\omega^2\sigma_s^2}{c^2}\right)\mathrm{Re}\left(Z_0^{\parallel}(\omega)\right)$$

For a narrow band resonator in a synchrotron, the wake field may last more than one revolution. In this case, the wake fields from previous bunch passages must be included in the calculation of the parasitic energy loss. The expression for the energy loss in this case becomes

$$\Delta E=-\int_{-\infty}^{\infty}ds'\rho(s')\int_{-\infty}^{\infty}ds\rho(s)\sum_{k=-\infty}^{\infty}W_0'(kC+s'-s)$$



where $C$ is the circumference. For a point charge $q$, this is just

$$\Delta E=-q^2\sum_{k=-\infty}^{\infty}W_0'(kC).$$

It turns out that this sum can be done analytically for the case of a resonator wake function, as given in Lecture 24, p. 33, for the case of $Q$>>1, and for the on-resonance case $\omega_R=h\omega_0$:

$$\Delta E=-q^2\frac{cR_s}{C}\frac{\pi h}{Q}\frac{\exp\left(\frac{\pi h}{Q}\right)+1}{\exp\left(\frac{\pi h}{Q}\right)-1}$$

For $\frac{\pi h}{Q}<<1$, this becomes just $\Delta E=-q^2\frac{2cR_s}{C}$

Example: for the 500 MHz narrow-band rf cavity discussed earlier, if we evaluate the sum over wake functions on previous turns, we find, for $C = h\lambda_{rf}$, with $h$=1281, and with

$$q = 3.2\times 10^4 \ \text{pC},$$

$$\Delta E = -q^2 \frac{2cR_s}{C} \approx 200 \ \text{keV}$$

So the effects of the previous turns' wakes in the cavity in fact is much larger than the $k$=0 term, which was estimated above at about 12.8 keV.

The parasitic energy loss for a bunch of finite length, with longitudinal charge density $\rho(s)$, including the effect of multiple turns, in terms of the impedance, is



$$\Delta E = -\frac{\omega_0}{2\pi} \sum_{p=-\infty}^{\infty} \left|\tilde{\rho}(p\omega_0)\right|^2 \mathrm{Re}\left(Z_0^{\|}(p\omega_0)\right)$$

For a point charge $q$, this becomes

$$\Delta E = -q^2 \frac{\omega_0}{2\pi} \sum_{p=-\infty}^{\infty} \mathrm{Re}\left(Z_0^{\|}(p\omega_0)\right)$$

and for a narrow band impedance, with $p=h$, $Q$>>1 and $h/Q$<<1, and $h\omega_0 = \omega_R$, we have

$$\Delta E = -q^2 \frac{\omega_0 R_s}{\pi}$$

in agreement with the sum over wake functions given above.