LECTURE 23

Collective instabilities

Types of instabilities

An instability driven by narrow-band rf cavities: the Robinson instability

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Types of instabilities

The various wake potentials we have discussed constitute forces on the beam; these forces will alter the trajectory equations of motion. Depending upon the phase relationship between the forces and the dynamical variables of the beam, the only result may be tune shifts and lattice function distortion, or a loss of stability may occur. In such cases, the beam is said to be subject to a collective instability.

Collective instabilities can be present in both bunched beams and unbunched beams, in either or both the transverse and the longitudinal planes. Because of the absence of synchrotron motion, the longitudinal and transverse dynamics of unbunched beams are quite different from that of bunched

beams, and the instability mechanisms are likewise quite different. Although the bunched beam case is more complex, we will start the discussion of instabilities with those of bunched beams, because they are by far the most common and important case.

An instability driven by narrow-band rf cavities: the Robinson instability

We have seen that the single strongest impedance in a machine is the fundamental narrow band rf cavity longitudinal impedance. The associated wake fields can cause an instability called the *Robinson instability*. This is one of

the most important instability mechanisms in accelerators. Fortunately, the control of this instability is relatively straightforward.

To see how this works, consider a “macroparticle”: a point charge of magnitude Ne, circulating in a synchrotron. This macroparticle will create a wake field when it passes through the rf cavity. The macroparticle undergoes synchrotron oscillations; the wake potentials introduce additional forces into the synchrotron equations of motion. These additional forces can lead to an instability.

The wake fields generated by the macroparticle can be expressed in terms of a voltage drop across the rf cavity. The voltage drop due to a pure harmonic current of the form

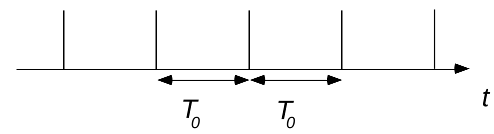
$I_0(t) = \tilde{I}_0(\omega)\cos(\omega t)$, from Lecture 21, p 19, can be written in terms of the cavity impedance as

$$\overline{E}_s(t) = -\tilde{I}_0(\omega)\cos(\omega t) Z_0^{\|}(\omega)$$

To use the above equation, we need to know the Fourier spectrum of the current, which consists of a single circulating macroparticle of charge *Ne*. Let us consider for the moment that the macroparticle is not undergoing synchrotron oscillations. The current due to the point charge *Ne* is a series of impulses, which occur at times $t = nT_0$, where n is the turn number, an integer running from $-\infty$ to ∞. This current can be represented as a sum of Dirac delta-functions

$$I_0(t) = Ne \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$$

in which T_0 is the revolution period. The sum is over all turns.



The Fourier transform of this is

$$\tilde{I}_0(\omega) = \int_{-\infty}^{\infty} dt \exp(-i\omega t) I_0(t) = Ne \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dt \exp(-i\omega t)\delta(t - nT_0)$$

$$= Ne \sum_{n=-\infty}^{\infty} \exp(-i\omega n T_0)$$

The Fourier transform has the form of a series of exponentials. We can convert this into a series of Dirac delta-functions, using a fundamental result from Fourier transform theory, called the Poisson sum formula:

$$\sum_{n=-\infty}^{\infty} \exp(inx) = 2\pi \sum_{p=-\infty}^{\infty} \delta(x - 2\pi p)$$

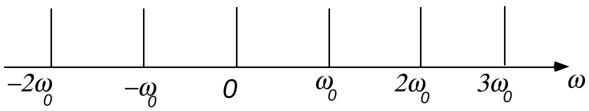
Using this, we have

$$\sum_{n=-\infty}^{\infty} \exp(-i\omega n T_0) = 2\pi \sum_{p=-\infty}^{\infty} \delta(\omega T_0 + 2\pi p) = \frac{2\pi}{T_0} \sum_{p=-\infty}^{\infty} \delta(\omega + p\omega_0)$$

in which $\omega_0 = \dfrac{2\pi}{T}$ is the revolution frequency. So, finally, we have

$$\tilde{I}_0(\omega) = \frac{2\pi Ne}{T_0} \sum_{p=-\infty}^{\infty} \delta(\omega + p\omega_0)$$

The Fourier spectrum of the current due to the circulating macroparticle is just a series of discrete lines at integral multiples of the revolution frequency.



The current for the circulating macroparticle may now be written in terms of pure harmonic components as

$$I(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \exp(i\omega t)\tilde{I}_0(\omega)=\frac{Ne}{T_0}\sum_{p=-\infty}^{\infty}\int_{-\infty}^{\infty} d\omega \exp(i\omega t)\delta(\omega+p\omega_0)$$

$$=\frac{Ne}{T_0}\sum_{p=-\infty}^{\infty}\exp(-ip\omega_0 t)$$

The wake voltage, summed over all harmonics, will then be

$$\overline{E}_s(t)=-\frac{Ne}{T_0}\sum_{p=-\infty}^{\infty}\exp(ip\omega_0 t)Z_0^{\parallel}(p\omega_0)$$

We now let the macroparticle execute synchrotron oscillations. The synchrotron oscillations will introduce *additional frequency components* into the Fourier spectrum. The equations for small-amplitude synchrotron motion, from Lecture 10, p. 16, can be written as

$$\Delta t_n=(\Delta t)_{\max}\cos 2\pi Q_s n+(\Delta E)_{\max}\beta_L\sin 2\pi Q_s n$$
$$=A\cos(2\pi Q_s n+\Phi)$$
$$\Delta E_n=(\Delta E)_{\max}\cos 2\pi Q_s n-\frac{(\Delta t)_{\max}}{\beta_L}\sin 2\pi Q_s n$$
$$=-\frac{A}{\beta_L}\sin(2\pi Q_s n+\Phi)$$

in which

$$A^2=(\Delta t)_{\max}^2+(\Delta E)_{\max}^2\beta_L^2,$$
$$\tan\Phi=-\frac{(\Delta E)_{\max}\beta_L}{(\Delta t)_{\max}}$$

The current associated with the macroparticle now consists of series of impulses at $t=nT_0+\Delta t_n$, rather than at $t=nT_0$. The current is

$$I_0(t)=Ne\sum_{n=-\infty}^{\infty}\delta(t-nT_0-\Delta t_n)$$

The Fourier transform is

$$\tilde{I}_0(\omega)=Ne\sum_{n=-\infty}^{\infty}\exp\left(-i\omega\left(nT_0+\Delta t_n\right)\right)$$

Identity:

$$\exp(-ix\cos\phi)=\sum_{l=-\infty}^{\infty} i^{-l}J_l(x)\exp(il\phi)$$

so

$$\tilde{I}_0(\omega)=Ne\sum_{n=-\infty}^{\infty}\exp\left(-i\omega\left(nT_0+A\cos(2\pi Q_s n+\Phi)\right)\right)$$

$$=Ne\sum_{n=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} i^{-l}J_l(\omega A)\exp\left(-i\omega nT_0+l\left(2\pi Q_s n+\Phi\right)\right)$$

Using the Poisson sum formula again gives

$$\sum_{n=-\infty}^{\infty} \exp\left(-in\left(\omega T_0 - 2\pi Q_s l\right)\right) = 2\pi \sum_{p=-\infty}^{\infty} \delta\left(\omega T_0 - 2\pi Q_s l + 2\pi p\right)$$

$$= \frac{2\pi}{T_0} \sum_{p=-\infty}^{\infty} \delta\left(\omega - l\omega_s + p\omega_0\right)$$

so

$$\tilde{I}_0(\omega) = \frac{2\pi Ne}{T_0} \sum_{p=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} i^{-l} J_l(\omega A) \exp(il\Phi) \delta\left(\omega - l\omega_s + p\omega_0\right)$$

Each discrete revolution harmonic line in the Fourier spectrum of the current due to the circulating macroparticle (l=0 in the above series) acquires a series of “synchrotron sideband” lines, spaced at multiples of the synchrotron

frequency $\omega_s = Q_s \omega_0$ on either side of the l=0 lines. For small amplitude synchrotron oscillations, $\omega A << 1$,

$$J_0(\omega A) \approx 1,$$

$$J_1(\omega A) \approx \frac{\omega A}{2},$$

and

$$\tilde{I}_0(\omega) \approx \frac{2\pi Ne}{T_0} \sum_{p=-\infty}^{\infty} \delta\left(\omega + p\omega_0\right) + \frac{\omega A}{2i} \begin{pmatrix} \exp(i\Phi)\delta\left(\omega - \omega_s + p\omega_0\right) \\ + \exp(-i\Phi)\delta\left(\omega + \omega_s + p\omega_0\right) \end{pmatrix}$$

The current is

$$I(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \exp(i\omega t) \tilde{I}_0(\omega) =$$

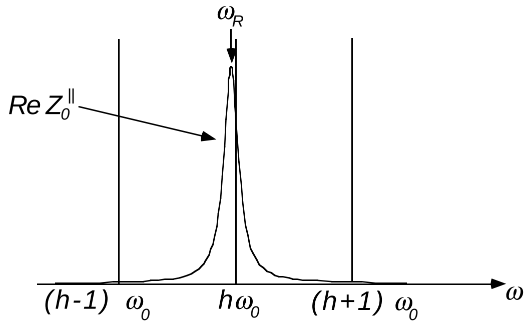
$$= \frac{Ne}{T_0} \left[\begin{array}{l} \sum_{p=-\infty}^{\infty} \exp\left(-ip\omega_0 t\right) \\ + \frac{A}{2i} \begin{pmatrix} \left(-p\omega_0 + \omega_s\right) \exp(i\Phi) \exp\left(i\left(-p\omega_0 + \omega_s\right)t\right) \\ + \left(-p\omega_0 - \omega_s\right)\left(\exp(-i\Phi)\right) \exp\left(i\left(-p\omega_0 - \omega_s\right)t\right) \end{pmatrix} \end{array} \right]$$

The wake voltage, summed over all harmonics, will then be

$$\bar{E}_s(t) = -\frac{Ne}{T_0} \sum_{p=-\infty}^{\infty} \exp\left(ip\omega_0 t\right) Z_0^{\parallel}\left(p\omega_0\right)$$

$$+ \frac{A}{2i} \begin{pmatrix} \left(p\omega_0 + \omega_s\right) \exp(i\Phi) \exp\left(i\left(p\omega_0 + \omega_s\right)t\right) Z_0^{\parallel}\left(p\omega_0 + \omega_s\right) \\ + \left(p\omega_0 - \omega_s\right)\left(\exp(-i\Phi)\right) \exp\left(i\left(p\omega_0 - \omega_s\right)t\right) Z_0^{\parallel}\left(p\omega_0 - \omega_s\right) \end{pmatrix}$$

Let the resonant frequency of the narrow-band impedance $Z_0^{\parallel}$ be ω_R. Let the closest harmonic line to this frequency have the harmonic number h, so $\omega_R \approx h\omega_0$. The width of the resonance is of order $\frac{\omega_R}{2Q} \approx \frac{h\omega_0}{2Q}$. Provided that $\frac{h}{Q}$<<1, as is typical for narrow-band resonators, only the contributions at $p = \pm h$, close to the resonant frequency, will be important.



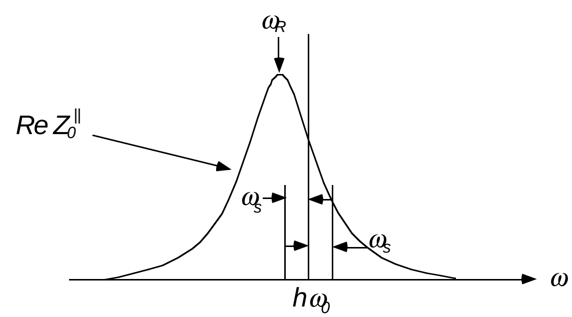
Thus, the narrow-band wake voltage can be written as

$$\bar{E}_s(t) = -\frac{Ne}{T_0}\Big(\exp(ih\omega_0 t)Z_0^{\parallel}(h\omega_0) + \exp(-ih\omega_0 t)Z_0^{\parallel}(-h\omega_0)\Big)$$
$$+\frac{A}{2i}\begin{pmatrix}(h\omega_0+\omega_s)\exp(i\Phi)\exp\big(i(h\omega_0+\omega_s)t\big)Z_0^{\parallel}(h\omega_0+\omega_s)\\ +(h\omega_0-\omega_s)(\exp(-i\Phi))\exp\big(i(h\omega_0-\omega_s)t\big)Z_0^{\parallel}(h\omega_0-\omega_s)\end{pmatrix}$$
$$+\frac{A}{2i}\begin{pmatrix}(-h\omega_0+\omega_s)\exp(i\Phi)\exp\big(i(-h\omega_0+\omega_s)t\big)Z_0^{\parallel}(-h\omega_0+\omega_s)\\ +(-h\omega_0-\omega_s)(\exp(-i\Phi))\exp\big(i(-h\omega_0-\omega_s)t\big)Z_0^{\parallel}(-h\omega_0-\omega_s)\end{pmatrix}$$

After some algebra, and using $\left[Z_0^{\parallel}(h\omega_0)\right]^* = Z_0^{\parallel}(-h\omega_0)$, we find

$$\bar{E}_s(t) = -\frac{2Ne}{T_0}\left(\begin{matrix}\cos(h\omega_0 t)\,\mathrm{Re}\big[Z_0^{\parallel}(h\omega_0)\big]-\sin(h\omega_0 t)\,\mathrm{Im}\big[Z_0^{\parallel}(h\omega_0)\big]- \\ \frac{A}{2}\begin{pmatrix}\cos(-h\omega_0 t+\omega_s t+\Phi)\,\mathrm{Im}\big[Z_0^{\parallel}(h\omega_0-\omega_s)\big](h\omega_0-\omega_s)+\\ \cos(h\omega_0 t+\omega_s t+\Phi)\,\mathrm{Im}\big[Z_0^{\parallel}(h\omega_0+\omega_s)\big](h\omega_0+\omega_s)-\\ \sin(-h\omega_0 t+\omega_s t+\Phi)\,\mathrm{Re}\big[Z_0^{\parallel}(h\omega_0-\omega_s)\big](h\omega_0-\omega_s)+\\ \sin(h\omega_0 t+\omega_s t+\Phi)\,\mathrm{Re}\big[Z_0^{\parallel}(h\omega_0+\omega_s)\big](h\omega_0+\omega_s)\end{pmatrix}\end{matrix}\right)$$

The synchrotron oscillations of the macroparticle are responsible for the frequency components in this expression at $\omega_0 \pm \omega_s$, which sample the impedance at these frequencies:



We want to find to find energy change per turn produced by this wake voltage: then we can insert this energy change per turn into the synchrotron equations of motion and look for a solution. If t=0 is the time when the macroparticle is at the rf cavity, then at a later time $t = nT_0 + \Delta t_n$, the wake voltage is $\bar{E}_s(nT_0 + \Delta t_n) = \bar{E}_{s,n}$, where n is the turn number. To lowest order in the synchrotron oscillation amplitude, making use of the fact that h is an integer, and using $h\omega_0 >> \omega_s$, we have

$$\bar{E}_{s,n} \approx \bar{E}_s(nT_0) = -\frac{2Ne}{T_0}\left(\begin{array}{l} \mathrm{Re}\left[Z_0^{\parallel}(h\omega_0)\right] - \\ \dfrac{h\omega_0\beta_L\Delta E_n \,\mathrm{Re}\left[Z_0^{\parallel}(h\omega_0+\omega_s) - Z_0^{\parallel}(h\omega_0-\omega_s)\right]}{2} \\ + h\omega_0\Delta t_n \,\mathrm{Im}\left[Z_0^{\parallel}(h\omega_0) - \dfrac{\left(Z_0^{\parallel}(h\omega_0+\omega_s) + Z_0^{\parallel}(h\omega_0-\omega_s)\right)}{2}\right] \end{array} \right)$$

The first term in brackets corresponds to the parasitic energy loss, which we have already discussed. The second term represents a dynamic effect: the wake energy change is proportional to the energy difference from the synchronous

particle. This term has the potential to produce damping or growth of the synchrotron oscillation, depending on the sign of the coefficient of ΔE_n. For example, if the sign is such that the energy change due to the wake is the same as that of ΔE_n, then ΔE_n can grow without bound and we have a instability. If the signs are opposite, then the wake potential will act to damp synchrotron oscillations.

The third term comes from the fact that

$$\sin\left(h\omega_0\left[nT_0 + \Delta t_n\right]\right) = \sin\left(2\pi nh + h\omega_0\Delta t_n\right) \approx h\omega_0\Delta t_n.$$

This term corresponds to a wake voltage proportional to the time difference: this will lead to a *frequency shift*. We now insert this into the synchrotron equations of motion. From Lecture 10, p.16: the longitudinal equations of motion are

$$\frac{d\Delta t_n}{dn} = 2\pi Q_s \beta_L \Delta E_n$$

$$\frac{d\Delta E_n}{dn} = -\frac{2\pi Q_s}{\beta_L}\Delta t_n$$

Inserting the wake energy changes into the equation of motion gives

$$\frac{d\Delta E_n}{dn} = e\overline{E}_{s,n} - \frac{2\pi Q_s}{\beta_L}\Delta t_n =$$

$$= \Delta E_n \frac{Ne^2 h\omega_0 \beta_L \,\mathrm{Re}\left[Z_0^{\|}(h\omega_0+\omega_s) - Z_0^{\|}(h\omega_0-\omega_s)\right]}{T_0}$$

$$-\Delta t_n \left(\frac{2\pi Q_s}{\beta_L} + \frac{2Ne^2}{T_0} h\omega_0 \,\mathrm{Im}\left(Z_0^{\|}(h\omega_0) - \frac{\left(Z_0^{\|}(h\omega_0+\omega_s) + Z_0^{\|}(h\omega_0-\omega_s)\right)}{2} \right) \right)$$

Differentiating once and using $\frac{d\Delta t_n}{dn} = 2\pi Q_s \beta_L \Delta E_n$ gives

$$\frac{d^2\Delta E_n}{dn^2} = \frac{d\Delta E_n}{dn} \frac{Ne^2 h\omega_0 \beta_L \,\mathrm{Re}\left[Z_0^{\|}(h\omega_0+\omega_s) - Z_0^{\|}(h\omega_0-\omega_s)\right]}{T_0}$$

$$-\Delta E_n \left((2\pi Q_s)^2 + \frac{Ne^2 4\pi Q_s \beta_L}{T_0} h\omega_0 \,\mathrm{Im}\left(Z_0^{\|}(h\omega_0) - \frac{\left(Z_0^{\|}(h\omega_0+\omega_s) + Z_0^{\|}(h\omega_0-\omega_s)\right)}{2} \right) \right)$$

This equation has the general form

$$\frac{d^2\Delta E_n}{dn^2} = -2\alpha \frac{d\Delta E_n}{dn} - (2\pi Q_s')^2 \Delta E_n$$

which is the equation of a damped harmonic oscillator, with a solution (for $\alpha << 2\pi Q_s'$)

$$\Delta E_n \propto \exp(-\alpha + 2\pi i Q_s')$$

By comparing with the equation above, we see that the damping rate is

$$\alpha = -\frac{Ne^2 h\omega_0 \beta_L \,\mathrm{Re}\left[Z_0^{\|}(h\omega_0+\omega_s) - Z_0^{\|}(h\omega_0-\omega_s)\right]}{2T_0}$$

and the frequency is given by

$$(2\pi Q_s')^2 = (2\pi Q_s)^2 + \frac{Ne^2 4\pi Q_s \beta_L}{T_0} h\omega_0 \,\mathrm{Im}\left(Z_0^{\|}(h\omega_0) - \frac{\left(Z_0^{\|}(h\omega_0+\omega_s) + Z_0^{\|}(h\omega_0-\omega_s)\right)}{2} \right)$$

Damping rate

From Lecture 10, p 16, we have

$$\beta_L = \frac{\eta_C h\lambda}{2\pi\beta_s^2 E_s c Q_s} = \frac{\eta_C T_0}{2\pi\beta_s^3 E_s Q_s}$$

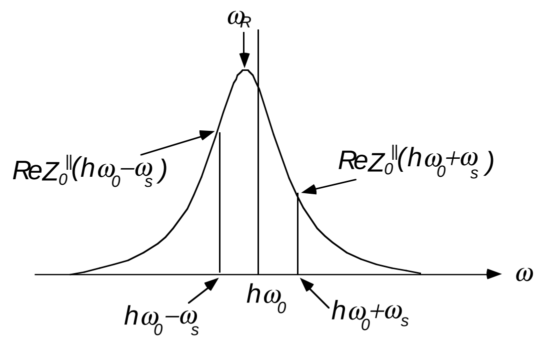
so that

$$\alpha = -\frac{Ne^2 h\omega_0 \eta_C}{4\pi\beta_s^3 E_s Q_s} \mathrm{Re}\left[Z_0^{\|}(h\omega_0+\omega_s) - Z_0^{\|}(h\omega_0-\omega_s)\right]$$

The damping rate must be positive for damping; a negative damping rate corresponds to exponential growth. This growth is called the Robinson instability. To avoid the instability, above transition, (when η_C>0) we require that

$$\mathrm{Re}\left[Z_0^{\|}(h\omega_0 - \omega_s)\right] > \mathrm{Re}\left[Z_0^{\|}(h\omega_0 + \omega_s)\right].$$

This condition is called the *Robinson criterion.* It is achieved in practice by tuning the cavity resonant frequency ω_R to be slightly lower than $h\omega_0$, as shown below:

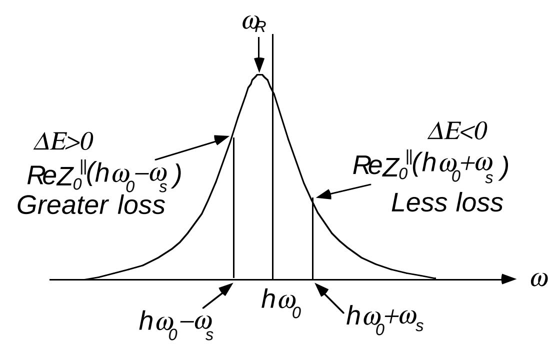


A simple physical picture can be provided to qualitatively explain the Robinson criterion. The synchrotron oscillations effectively introduced additional frequency components into

the macroparticle current, one at ω_s above $h\omega_0$, and one at ω_s below $h\omega_0$. The slip factor relates frequency to ΔE, via

$$\frac{\Delta\omega}{\omega} \approx -\eta_C \frac{\Delta E}{E}.$$

The frequency component at $h\omega_0 - \omega_s$ thus is associated with a positive ΔE. If the energy loss due to the wake is greater at this frequency than at the frequency $h\omega_0 + \omega_s$, for which ΔE is negative, then the wake energy loss will tend to reduce the energy more when ΔE is positive than when ΔE is negative, providing damping.



Below transition, (when η_C<0), the Robinson criterion reverses, to become

$$\mathrm{Re}\left[Z_0^{\|}(h\omega_0 - \omega_s)\right] < \mathrm{Re}\left[Z_0^{\|}(h\omega_0 + \omega_s)\right].$$

Synchrotron oscillation tune shift

The synchrotron oscillation tune in the presence of the wake field is Q_s'; if the tune shift is $\delta Q_s = Q_s' - Q_s$, we have, for a small quantity Δ,

$$2\pi Q_s' = \sqrt{(2\pi Q_s)^2 + \Delta} = 2\pi Q_s \sqrt{1 + \frac{\Delta}{(2\pi Q_s)^2}} \approx 2\pi Q_s + \frac{\Delta}{4\pi Q_s},$$

$$\Rightarrow \delta Q_s = \frac{\Delta}{8\pi^2 Q_s}$$

Comparing this with the equation on p. 29, we have

$$\delta Q_s \approx \frac{Ne^2 \beta_L}{2\pi T_0} h\omega_0 \,\mathrm{Im}\left(Z_0^{\|}(h\omega_0) - \frac{\left(Z_0^{\|}(h\omega_0 + \omega_s) + Z_0^{\|}(h\omega_0 - \omega_s) \right)}{2} \right)$$

Substituting for β_L from above gives for the synchrotron oscillation tune shift due to a narrow-band cavity:

$$\delta Q_s \approx \frac{Ne^2 \eta_C h\omega_0}{(2\pi)^2 \beta_s^3 E_s Q_s} \mathrm{Im}\left(Z_0^{\|}(h\omega_0) - \frac{\left(Z_0^{\|}(h\omega_0 + \omega_s) + Z_0^{\|}(h\omega_0 - \omega_s) \right)}{2} \right)$$

The first term in the brackets is a “static” effect (it does not involve the synchrotron motion of the macroparticle); it is called “potential well distortion”. The slope of the wake field voltage adds to the slope of the rf voltage, thereby changing

the oscillation frequency. Since it does not involve coherent motion of the macroparticle, this piece of the tune shift is incoherent, and can cause reduction or growth of the bunch length.

The other terms are dynamic effects, which will appear as a coherent synchrotron oscillation tune shift, but will not affect the bunch length. The total coherent tune shift is δQ_s.

Example.

Consider the standard expression for the narrow band resonator impedance: $Z_0^{\|}(\omega) = \dfrac{R_s}{1 + iQ\left(\dfrac{\omega_R}{\omega} - \dfrac{\omega}{\omega_R}\right)}$

If we let $\Delta = h\omega_0 - \omega_R$ and take Δ and ω_s both to be much less than the resonator width $\dfrac{\omega_R}{2Q}$, then we have

$$\alpha \approx \Delta \frac{4Ne^2 \eta_C Q^2 R_S}{\pi h E_s}$$

$$\frac{\delta Q_s}{Q_s} \approx -\Delta \frac{6Ne^2 \eta_C Q^3 R_S}{\pi^2 h^2 E_s}$$

Consider the 500 MHz copper cavity again, operating in CESR for which $\eta_C \approx 0.01$, h=1281, and E_s =5.2 GeV. Let the “macroparticle” contain $2\text{x}10^{11}$ electrons. If we let the detuning parameter $\Delta = \frac{\omega_R}{20Q}$ (i.e., detune about 1/10 of the width of the resonance), then we find from the above formula that α=0.00245 (this is the damping rate *per turn*) and $\delta Q_s = -0.03Q_s$. The synchrotron tune is shifted down by about 3%. The energy damping time is $\tau = \frac{T_0}{\alpha}$ =1 ms. Note that this is much more rapid than synchrotron radiation damping. Conversely, if the detuning has the wrong sign, the instability growth rate can be much faster than the radiation damping time.

The considerations given above refer to any impedance, not just a narrow-band one. However, a broad band impedance

will sample many revolution harmonics of the current, so that the sum over harmonics becomes effectively an integral over the frequency spectrum of the impedance. The real part of the longitudinal impedance is an even function of ω (this follows from the fact that $W'_m(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega Z_m^{\|}(\omega)\exp\left(i\frac{\omega z}{c}\right)$ is real), and so $\int_{-\infty}^{\infty} d\omega\omega\,\mathrm{Re}\left(Z_0^{\|}(\omega)\right) = 0$. Broad band impedances do not lead to Robinson-type instabilities.

Parasitic losses will cause a shift of the synchronous phase. This can be evaluated by writing the equation for the energy oscillations in an rf cavity driven by a sinusoidal voltage: (Lecture 10, pg 16):

$$\frac{d\Delta E_n}{dn} = eV\left(\sin(\omega t_n) - \sin(\omega t_s)\right)$$

and adding the parasitic loss term (pg. 20 above):

$$\bar{E}_{s,n} \approx -\frac{2Ne}{T_0}\mathrm{Re}\left[Z_0^{\|}(h\omega_0)\right]$$

giving

$$\frac{d\Delta E_n}{dn} = -\frac{2Ne^2}{T_0}\mathrm{Re}\left[Z_0^{\|}(h\omega_0)\right]$$
$$+eV\left(\sin(h\omega_0 t_s) + \cos(h\omega_0 t_s)h\omega_0\Delta t_n - \sin(h\omega_0 t_{s0})\right)$$
$$= eV\left(\sin(\phi_s) - \sin(\phi_{s0})\right) - \frac{2Ne^2}{T_0}\mathrm{Re}\left[Z_0^{\|}(h\omega_0)\right]$$

in which $\phi_s = h\omega_0 t_s$, $\phi_{s0} = h\omega_0 t_{s0}$. The synchronous phase is determined by the condition

$$eV\left(\sin(\phi_s) - \sin(\phi_{s0})\right) = \frac{2Ne^2}{T_0}\mathrm{Re}\left[Z_0^{\|}(h\omega_0)\right]$$

If $\phi_s = \phi_{s0} + \delta\phi_s$, in which $\delta\phi_s << 1$, then

$$\sin(\phi_s) = \sin(\phi_{s0} + \delta\phi_s) \approx \sin(\phi_{s0}) + \delta\phi_s\cos(\phi_{s0})$$

so

$$\delta\phi_s = \frac{2Ne\,\mathrm{Re}\left[Z_0^{\parallel}(h\omega_0)\right]}{VT_0\cos(\phi_{s0})}$$

This is the shift in the synchronous phase produced by the wakefield.