LECTURE 24

Collective instabilities

Bunched beam instabilities driven by short-range wakefields:

Head-tail instabilities in synchrotrons

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Head-tail instabilities in synchrotrons

“Strong” head-tail instability

The “head-tail” instability is a transverse instability in which the transverse wake field generated by the head of a bunch exerts a force on the tail of the bunch. Such a condition may lead to unstable motion of the tail, resulting in breakup of the bunch.

It should be clear that such an instability will be driven most easily by short-range wakefields, which extend over a distance of order the length of the bunch. As we have seen, such

wakefields are generated by the relatively high frequency impedance of broad band resonators. We will take a very simple model for the wake function that drives the head tail instability, namely:

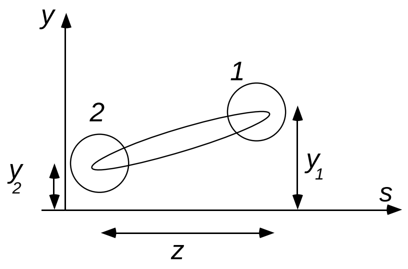
$$W_1(z) = \begin{cases} -W & \text{if } \; 0 > z > -\text{bunch length} \\ 0 & \text{otherwise} \end{cases}$$

The transverse wake potential generated by a total charge Q, undergoing vertical motion with a dipole moment $\langle y \rangle$, will then be (Lect 24, p. 16)

$$\overline{F}_y = eQW\langle y \rangle$$

(We’ll only discuss vertical oscillations here, but the treatment for the horizontal case is essentially identical).

We will use a “two-macroparticle” model for the beam. One macroparticle, labeled “1”, will represent the head of the beam, and the other, labeled “2”, will represent the tail of the beam. Each macroparticle contains charge $Ne/2$.



If we ignore wakefields, then each macroparticle can execute free betatron oscillations about y=0. If we focus on a particular point in the ring, then the transformation of y and y' at this point over n turns can be described by the matrix

$$\begin{pmatrix} y(n) \\ y'(n) \end{pmatrix} = \begin{pmatrix} \cos 2\pi n Q_y & \beta_y \sin 2\pi n Q_y \\ -\frac{1}{\beta_y}\sin 2\pi n Q_y & \cos 2\pi n Q_y \end{pmatrix} \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix}$$

For simplicity, we've taken $\alpha_y=0$. Now let there be an impedance at this point in the ring, which has the wake function W. Consider the effect of the wake field of particle 1 on particle 2. The wake potential generated by particle 1 is

$$\overline{F}_y = \frac{Ne^2}{2} W y_1$$

This will cause a kick to particle 2 equal to

$$\Delta y_2' = \frac{\overline{F}_y}{pv} = \frac{Ne^2}{2m_0c^2\gamma} W y_1$$

in which we've taken the particle velocity to be c. This obviously represents a coupling between the motion of the two particles via the wake function, and this will be the source of the instability.

From the matrix transformation above, we have, in the absence of wake fields,

$$y_1(n) = y_1(0)\cos 2\pi n Q_y + y_1'(0)\beta_y \sin 2\pi n Q_y$$

$$\frac{d^2 y_2(n)}{dn^2} = -\left(2\pi Q_y\right)^2 y_2(n)$$

where we've assumed $y_1'(0)=0$ for simplicity. Using

$$y_2(n) = y_{20}\cos\left(2\pi Q_y n\right) \qquad y_2'(n) = -\frac{y_{20}}{\beta_y}\sin\left(2\pi Q_y n\right)$$

$$\frac{dy_2(n)}{dn} = -2\pi Q_y y_{20}\sin\left(2\pi Q_y n\right) = 2\pi Q_y \beta_y y_2'(n)$$

we see that the wake fields modify the equation for $\frac{d^2 y_2(n)}{dn^2}$ to

$$\frac{d^2 y_2(n)}{dn^2} = -\left(2\pi Q_y\right)^2 y_2(n) + \frac{N 2\pi Q_y \beta_y e^2}{2m_0c^2\gamma} W y_1(n)$$

The solution of this equation, is

$$y_2(n) = y_2(0)\cos 2\pi n Q_y + \beta_y y_2'(0)\sin 2\pi n Q_y$$
$$+\frac{Ne^2 W\beta_y^2 \sin 2\pi n Q_y}{8\pi Q_y m_0 c^2\gamma} y_1'(0)$$
$$+\frac{Ne^2 W\beta_y}{4m_0c^2\gamma} n\left(y_1(0)\sin 2\pi n Q_y - y_1'(0)\frac{\beta_y}{\pi Q_y}\cos 2\pi n Q_y \right)$$

The last term grows with n, and represents the resonant response of the second particle to the driving force delivered by the first particle. It would seem that the tail of the bunch would rapidly be driven to large amplitudes and be lost. This, in fact, is what happens in linacs, where this instability is referred to as the *beam breakup instability*.

In linacs, the instability can be controlled by arranging for the head and the tail of the bunch to have different betatron frequencies, so the resonant response is not realized. This is done by introducing an energy spread into the beam, correlated with position in the bunch. Chromaticity will then produce a tune dependence on position in the bunch, and the growth of the instability can be limited. This procedure is referred to as “BNS damping”.

In a synchrotron, there is a natural mechanism for suppression of the instability: synchrotron motion. The macroparticles 1 and 2 exchange places every 1/2 of a synchrotron period. This tends to limit the growth of the instability. At large enough currents, however, it can still occur.

To see when the instability develops, we have to analyze the coupled motion of the two macroparticle. Such an analysis can be simplified by the following transformation. We define the following complex variable

$$\tilde{y} = y + i\frac{dy/dn}{2\pi Q_y} = y + i\beta_y y'$$

In terms of this variable, the transformation through n turns (in the absence of wakefields) can be written very simply as

$$\tilde{y}(n) = \exp\left(-2\pi i n Q_y\right)\tilde{y}(0)$$

So the motion of both macroparticles, ignoring the wake effects, can be written in matrix form as

$$\begin{pmatrix}\tilde{y}_1(n)\\ \tilde{y}_2(n)\end{pmatrix} = \exp\left(-2\pi i n Q_y\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\tilde{y}_1(0)\\ \tilde{y}_2(0)\end{pmatrix}$$

In the presence of wakefields, we have solved for the motion of particle 2 when it is in the tail of the bunch:

$$\begin{aligned}y_2(n) &= y_2(0)\cos 2\pi n Q_y + \beta_y y_2'(0)\sin 2\pi n Q_y\\ &+ \frac{Ne^2 W\beta_y^2 \sin 2\pi n Q_y}{8\pi Q_y m_0 c^2\gamma} y_1'(0)\\ &+ \frac{Ne^2 W\beta_y}{4 m_0 c^2 \gamma} n\left(y_1(0)\sin 2\pi n Q_y - y_1'(0)\frac{\beta_y}{\pi Q_y}\cos 2\pi n Q_y\right)\end{aligned}$$

We want to write this in terms of the $\tilde{y}$ variables. Using the definition given above, we have

$$\tilde{y}_2(n) = \exp\left(-2\pi i Q_y n\right)\left(\begin{aligned}&\tilde{y}_2(0) + \tilde{y}_1(0) i n \frac{Ne^2 W\beta_y}{4 m_0 c^2\gamma} + \\ &\frac{Ne^2 W\beta_y \tilde{y}_1^*(0)\left(-1 + \exp\left(4\pi i Q_y n\right)\right)}{16\pi Q_y m_0 c^2\gamma}\end{aligned}\right)$$

If we retain only the resonant term (the one proportional to n), which will dominate after many turns, then we have

$$\tilde{y}_2(n) \approx \exp\left(-2\pi i Q_y n\right)\left(\tilde{y}_2(0) + i n \tilde{y}_1(0)\frac{Ne^2 W\beta_y}{4 m_0 c^2\gamma}\right)$$

We can now write the solution for the motion of both macroparticles, including the resonant term produced by the wake field, in matrix form as

$$\begin{pmatrix} \tilde{y}_1(n) \\ \tilde{y}_2(n) \end{pmatrix} = \exp(-2\pi i n Q_y) \begin{pmatrix} 1 & 0 \\ in\dfrac{Ne^2 W\beta_y}{4m_0c^2\gamma} & 1 \end{pmatrix} \begin{pmatrix} \tilde{y}_1(0) \\ \tilde{y}_2(0) \end{pmatrix}$$

As mentioned above, this will be correct for about 1/2 of a synchrotron oscillation period; then the roles of particles 1 and 2 will reverse. Thus, we need to look at the above expression for $n = \dfrac{1}{2Q_s}$, where Q_s is the synchrotron tune. This is

$$\begin{pmatrix} \tilde{y}_1\left(\dfrac{1}{2Q_s}\right) \\ \tilde{y}_2\left(\dfrac{1}{2Q_s}\right) \end{pmatrix} = \exp\left(-i\frac{\pi Q_y}{Q_s}\right) \begin{pmatrix} 1 & 0 \\ i\mathrm{T} & 1 \end{pmatrix} \begin{pmatrix} \tilde{y}_1(0) \\ \tilde{y}_2(0) \end{pmatrix}$$

in which

$$\mathrm{T} = \frac{Ne^2 W\beta_y}{8Q_s m_0 c^2\gamma}$$

is a positive dimensionless parameter. For the second half of the synchrotron period, particle 2 drives particle 1; so we have

$$\begin{pmatrix} \tilde{y}_1\left(\dfrac{1}{Q_s}\right) \\ \tilde{y}_2\left(\dfrac{1}{Q_s}\right) \end{pmatrix} = \exp\left(-i\frac{\pi Q_y}{Q_s}\right) \begin{pmatrix} 1 & i\mathrm{T} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{y}_1\left(\dfrac{1}{2Q_s}\right) \\ \tilde{y}_2\left(\dfrac{1}{2Q_s}\right) \end{pmatrix}$$

The overall matrix for one synchrotron period is the product:

$$\begin{pmatrix} \tilde{y}_1\left(\dfrac{1}{Q_s}\right) \\ \tilde{y}_2\left(\dfrac{1}{Q_s}\right) \end{pmatrix} = \exp\left(-i\frac{2\pi Q_y}{Q_s}\right) \begin{pmatrix} 1-\mathrm{T}^2 & i\mathrm{T} \\ i\mathrm{T} & 1 \end{pmatrix} \begin{pmatrix} \tilde{y}_1(0) \\ \tilde{y}_2(0) \end{pmatrix}$$

The requirement for stability over many synchrotron periods is that the absolute value of the trace of the matrix should be less than 2. So we have the requirement

$$\left|2-\mathrm{T}^2\right| \le 2 \Rightarrow \mathrm{T} = \frac{Ne^2 W\beta_y}{8Q_s m_0 c^2\gamma} \le 2$$

If the single bunch intensity exceeds the threshold

$$N_{th} = \frac{16Q_s m_0 c^2\gamma}{e^2 W\beta_y}$$

the beam will very rapidly become unstable. This type of instability is referred to as the “strong head-tail instability”, or sometimes the “transverse mode-coupling instability (TMCI)”. The latter designation comes from the fact that at the instability threshold, the oscillation frequencies of the normal modes of the two macroparticles become equal.

Example

In Lecture 22, p 20, we estimated the transverse wake function from a broad band resonator to be about 10 V/pC/m. Suppose that there are 50 such objects in CESR. What is the threshold intensity for the strong head-tail instability?

We'll take β_y=20 m, Q_s =0.052, W=5x10^{14} V/C/m, γ=10^4. We find N_{th} = 2.55x10^{12} (160 ma) per bunch.

Below threshold, the motion of the normal modes can be complex. To see how the normal mode frequencies become equal at the instability threshold, we must perform a normal mode analysis.

The normal modes are defined as those linear combinations of $(\tilde{y}_1, \tilde{y}_2)$ which are decoupled after every synchrotron oscillation period. The normal modes, $\vec{\zeta}_1, \vec{\zeta}_2$, satisfy the equations

$$\Lambda \vec{\zeta}_i = \lambda_i \vec{\zeta}_i$$

in which $\Lambda = \mathbf{S}^{-1}\mathbf{MS}$, and $\mathbf{S}$ is the matrix which diagonalizes

$$\mathbf{M} = \begin{pmatrix} 1-T^2 & iT \\ iT & 1 \end{pmatrix}.$$

The eigenvalues are given by the secular equation

$$|\mathbf{M} - \mathbf{I}\lambda| = \begin{vmatrix} 1-T^2-\lambda & iT \\ iT & 1-\lambda \end{vmatrix} = 0$$

This is

$$\left(1-\mathrm{T}^2-\lambda\right)(1-\lambda)+\mathrm{T}^2=0$$

Since the matrix has a unit determinant, the diagonalized matrix will also, so we have $\lambda_1\lambda_2 = 1, \quad \lambda_{1,2} = \exp(\mp i\phi)$

and

$$\left(1-T^2-\exp(i\phi)\right)\left(1-\exp(i\phi)\right)+T^2=0$$

$$-\exp(i\phi)-\left(1-T^2\right)\left(\exp(i\phi)\right)+\exp(2i\phi)=0$$

$$\left(1-T^2\right)=\exp(-i\phi)-1+\exp(i\phi)=2\cos\phi-1$$

$$T^2=2-2\cos\phi \Rightarrow T=2\sin\frac{\phi}{2}$$

The normal mode eigenvectors in the $(\tilde{y}_1, \tilde{y}_2)$ basis are

$$\vec{\zeta}_1 = \begin{pmatrix} -\exp\left(-i\frac{\phi}{2}\right) \\ 1 \end{pmatrix}, \quad \vec{\zeta}_2 = \begin{pmatrix} \exp\left(i\frac{\phi}{2}\right) \\ 1 \end{pmatrix}$$

That is,

$$\zeta_1 = \tilde{y}_2 - \exp\left(-i\frac{\phi}{2}\right)\tilde{y}_1, \quad \zeta_2 = \tilde{y}_2 + \exp\left(i\frac{\phi}{2}\right)\tilde{y}_1$$

The matrix which diagonalizes $\mathbf{M}$ is

$$\mathbf{S} = \begin{pmatrix} -\exp\left(-i\frac{\phi}{2}\right) & \exp\left(i\frac{\phi}{2}\right) \\ 1 & 1 \end{pmatrix}$$

This is also the transformation matrix from the eigenvectors to $\vec{\tilde{y}}$:

$$\vec{\tilde{y}} = \mathbf{S}\vec{\zeta}$$

To find the normal mode frequencies, we must Fourier analyze the time dependence of the eignemodes. The time evolution of the eigenvectors can be obtained from the results given above. Over the first half of a synchrotron period, the motion of $\vec{\tilde{y}}$ is given by

$$\begin{pmatrix}\tilde{y}_1(n)\\ \tilde{y}_2(n)\end{pmatrix} = \exp\left(-2\pi i n Q_y\right)\begin{pmatrix}1 & 0\\ 2inTQ_s & 1\end{pmatrix}\begin{pmatrix}\tilde{y}_1(0)\\ \tilde{y}_2(0)\end{pmatrix}\Rightarrow$$

$$\vec{\tilde{y}}(n) = \exp\left(-2\pi i n Q_y\right)\mathbf{M}_1(n)\vec{\tilde{y}}(0)$$

while over the second half of the period, we have

$$\begin{pmatrix}\tilde{y}_1(n)\\ \tilde{y}_2(n)\end{pmatrix} = \exp\left(-2\pi i n Q_y\right)\begin{pmatrix}1 & 2i\left(n-\frac{1}{2Q_s}\right)\mathrm{T}Q_s\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ i\mathrm{T} & 1\end{pmatrix}\begin{pmatrix}\tilde{y}_1(0)\\ \tilde{y}_2(0)\end{pmatrix}$$

$$= \exp\left(-2\pi i n Q_y\right)\begin{pmatrix}1+T^2\left(1-2nQ_s\right) & iT\left(2nQ_s-1\right)\\ i\mathrm{T} & 1\end{pmatrix}\begin{pmatrix}\tilde{y}_1(0)\\ \tilde{y}_2(0)\end{pmatrix}\Rightarrow$$

$$\vec{\tilde{y}}(n) = \exp\left(-2\pi i n Q_y\right)\mathbf{M}_2(n)\vec{\tilde{y}}(0)$$

The time evolution of the eigenvectors is then

$$\vec{\tilde{y}}(n) = \mathbf{S}\vec{\zeta}(n)\Rightarrow$$

$$\vec{\tilde{y}}(n) = \exp\left(-2\pi i n Q_y\right)\mathbf{M}_T(n)\vec{\tilde{y}}(0) = \exp\left(-2\pi i n Q_y\right)\mathbf{M}_T(n)\mathbf{S}\vec{\zeta}(0)\Rightarrow$$

$$\vec{\zeta}(n) = \exp\left(-2\pi i n Q_y\right)\mathbf{S}^{-1}\mathbf{M}_T(n)\mathbf{S}\vec{\zeta}(0)$$

where

$$\mathbf{M}_T(n) = \begin{Bmatrix}\mathbf{M}_1(n) & 0 < n < 1/2Q_s\\ \mathbf{M}_2(n) & 1/2Q_s < n < 1/Q_s\end{Bmatrix}$$

and $\mathbf{M}_T(n) = \mathbf{M}$. Let $\mathbf{Z}(n) = \mathbf{S}^{-1}\mathbf{M}_T(n)\mathbf{S}$; then

$$\vec{\zeta}(n) = \exp\left(-2\pi i n Q_y\right)\mathbf{Z}(n)\vec{\zeta}(0)$$

We know that, by the definition of the eigenvectors, after 1 synchrotron oscillation period

$$\mathbf{Z}\left(\frac{1}{Q_s}\right) = \begin{pmatrix}\exp(-i\phi) & 0\\ 0 & \exp(i\phi)\end{pmatrix} = \Lambda$$

and after *m* synchrotron oscillations

$$\mathbf{Z}\left(\frac{m}{Q_s}\right) = \begin{pmatrix}\exp(-im\phi) & 0\\ 0 & \exp(im\phi)\end{pmatrix} = \Lambda^m$$

Thus, we can write in general that

$$\vec{\zeta}(n) = \Omega(n)\vec{\zeta}(0)$$

$$\Omega(n) = \exp\left(-2\pi i n Q_y\right)\mathbf{Z}\left(n-\frac{m}{Q_s}\right)\Lambda^m \text{ for } \frac{m}{Q_s} < n < \frac{m+1}{Q_s}$$

To see the spectral content of the normal mode oscillations, we can Fourier analzye this expression. We use the general relations

$$y(n) = \int_{-\infty}^{\infty} dh \exp(2\pi i h n)\tilde{y}(h) \qquad \tilde{y}(h) = \int_{-\infty}^{\infty} dn \exp(-2\pi i h n) y(n)$$

Thus, extending the range to all times $\vec{\zeta}(n) = \Omega(n)\vec{\zeta}(-\infty)$,

$$\tilde{\Omega}(h) = \sum_{m=-\infty}^{\infty} \tilde{\Omega}_m(h)$$

$$\tilde{\Omega}_m(h) = \int_{m/Q_s}^{m+1/Q_s} dn \exp(-2\pi i h n) \exp\left(-2\pi i n Q_y\right) \mathbf{Z}\left(n - \frac{m}{Q_s}\right)\Lambda^m$$

Using $n' = n - \dfrac{m}{Q_s}$

$$\tilde{\Omega}_m(h) = \int_0^{1/Q_s} dn' \exp\left(2\pi i\left(n' + \frac{m}{Q_s}\right)\left(-h - Q_y\right)\right)\mathbf{Z}(n')\Lambda^m$$

$$= \tilde{\mathbf{Z}}(h)\exp\left(\frac{-2\pi i m\left(h + Q_y\right)}{Q_s}\right)\Lambda^m$$

$$\tilde{\mathbf{Z}}(h) = \int_0^{1/Q_s} dn' \exp\left(-2\pi i n'\left(h + Q_y\right)\right)\mathbf{Z}(n')$$

Thus

$$\tilde{\Omega}(h) = \sum_{m=-\infty}^{\infty} \tilde{\Omega}_m(h) = \tilde{\mathbf{Z}}(h)\left(\sum_{m=-\infty}^{\infty} \exp\left(\frac{-2\pi i m\left(h + Q_y\right)}{Q_s}\right)\Lambda^m\right)$$

Each element of the matrix sum in brackets has the form

$$\sum_{m=-\infty}^{\infty} \exp\left(\frac{-2\pi i m\left(h + Q_y \pm \frac{\phi Q_s}{2\pi}\right)}{Q_s}\right)$$

Using the Poisson sum formula

$$\sum_{n=-\infty}^{\infty} \exp(inx) = 2\pi \sum_{p=-\infty}^{\infty} \delta(x - 2\pi p)$$

we have

$$\sum_{m=-\infty}^{\infty} \exp\left(\frac{-2\pi i m\left(h + Q_y \pm \frac{\phi Q_s}{2\pi}\right)}{Q_s}\right)$$

$$= 2\pi \sum_{p=-\infty}^{\infty} \delta\left(\frac{2\pi\left(h + Q_y \pm \frac{\phi Q_s}{2\pi}\right)}{Q_s} - 2\pi p\right)$$

We can transform the δ-function as follows:

$$\delta\left(\frac{2\pi\left(h + Q_y \pm \frac{\phi Q_s}{2\pi}\right)}{Q_s} - 2\pi p\right) = \frac{Q_s}{2\pi}\delta\left(h - Q_s p + Q_y \pm \frac{\phi Q_s}{2\pi}\right)$$

So we have

$$\tilde{\Omega}(h) = Q_s \sum_{p=-\infty}^{\infty} \tilde{\mathbf{Z}}(h)\Delta(p,h) \quad \Delta(p,h) = \begin{pmatrix} \delta\big(h - h_+(p)\big) & 0 \\ 0 & \delta\big(h - h_-(p)\big) \end{pmatrix}$$

$$h_\pm(p) = pQ_s - Q_y \mp \frac{\phi Q_s}{2\pi}$$

and so

$$\Omega(n) = \int_{-\infty}^{\infty} dh \exp(2\pi i h n)\tilde{\Omega}(h)$$

$$= Q_s \sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} dh \exp(2\pi i h n)\tilde{\mathbf{Z}}(h) \begin{pmatrix} \delta\big(h - h_+\big) & 0 \\ 0 & \delta\big(h - h_-\big) \end{pmatrix}$$

$$= Q_s \sum_{p=-\infty}^{\infty} \begin{pmatrix} \exp(2\pi i h_+ n)\tilde{Z}_{11}(h_+) & \exp(2\pi i h_- n)\tilde{Z}_{12}(h_-) \\ \exp(2\pi i h_+ n)\tilde{Z}_{21}(h_+) & \exp(2\pi i h_- n)\tilde{Z}_{22}(h_-) \end{pmatrix}$$

The general term has the form

$$\exp(2\pi i h_\pm n)\tilde{Z}_{ij}(h_\pm) = \exp\left(2\pi i n\left(pQ_s - Q_y \mp \frac{\phi Q_s}{2\pi}\right)\right) \times$$

$$\int_0^{1/Q_s} dn' \exp\left(-2\pi i n'\left(pQ_s \mp \frac{\phi Q_s}{2\pi}\right)\right) Z_{ij}(n')$$

$$\Omega_{ij}(n) = Q_s \exp\big(-2\pi i n Q_y \mp i n \phi Q_s\big)$$

$$\sum_{p=-\infty}^{\infty} \exp(2\pi i n p Q_s) \int_0^{1/Q_s} dn' \exp(-2\pi i n' p Q_s \pm i n' \phi Q_s) Z_{ij}(n')$$

or

$$\Omega(n) = Q_s \exp\big(-2\pi i n Q_y\big) \sum_{p=-\infty}^{\infty} \int_0^{1/Q_s} dn' \mathbf{Z}(n') \Pi_p(n,n')$$

$$\Pi_p(n,n') = \begin{pmatrix} \exp\big(i(n'-n)Q_s(\phi - 2\pi p)\big) & 0 \\ 0 & \exp\big(-i(n'-n)Q_s(\phi + 2\pi p)\big) \end{pmatrix}$$

This defines the frequency structure of the normal mode coupling matrix. After carrying out the integration, we find

$$\Omega(n) = \exp\big(-2\pi i n Q_y\big) \sum_{p=-\infty}^{\infty} \Omega_p(n)$$

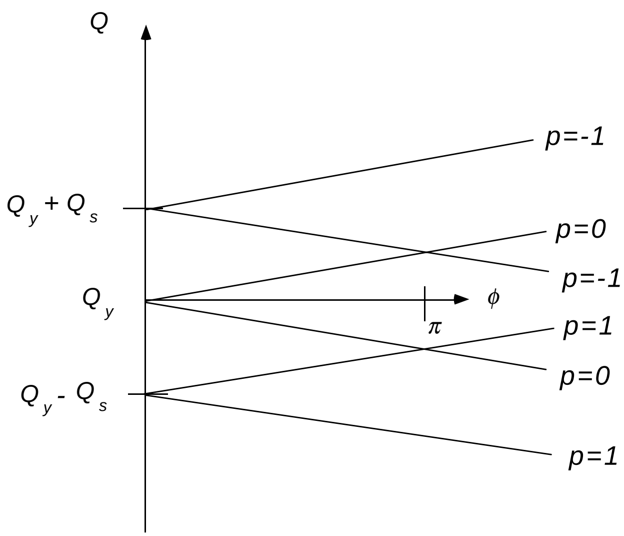
$$\Omega_{p,22}(n) = \exp\left(inQ_s(2\pi p + \phi)\right)\frac{4(-1)^p \sin\frac{\phi}{2}\tan\frac{\phi}{2}}{(\phi + 2p\pi)^2}$$

$$\Omega_{p,12}(n) = -2\exp\left(inQ_s(2\pi p + \phi)\right)\frac{\left(-1+(-1)^p\exp\left(\frac{i\phi}{2}\right)\right)^2\left(-1+\exp(i\phi)\right)}{\left(1+\exp(i\phi)\right)(\phi+2p\pi)^2}$$

$$\Omega_{p,11}(n) = \exp\left(inQ_s(2\pi p - \phi)\right)\frac{4(-1)^p \sin\frac{\phi}{2}\tan\frac{\phi}{2}}{(\phi - 2p\pi)^2}$$

$$\Omega_{p,21}(n) = 2\exp\left(inQ_s(2\pi p - \phi)\right)\frac{\left(-1+(-1)^p\exp\left(-\frac{i\phi}{2}\right)\right)^2\left(-1+\exp(i\phi)\right)}{\left(1+\exp(i\phi)\right)(\phi-2p\pi)^2}$$

The Fourier components are at frequencies $Q_y - pQ_s \pm \frac{Q_s\phi}{2\pi}$ for the two modes. As we approach the instability, $\phi \to \pi$, and the frequencies of the two modes approach each other (from adjacent sidebands). Hence the term, “mode coupling instability”. When both modes have the same frequency, resonant growth is rapid. The instability growth time is of order the synchrotron oscillation period.



Head-tail instability

There is another, weaker form of head-tail instability, which is referred to as simply the head-tail instability (without the adjective “strong”). This phenomenon is similar to the one that we have just described; however, it arises from the dependence of the betatron tune on energy (through the chromaticity). In contrast to the case for the “strong” head-tail instability, there is no sudden onset of the instability at a particular intensity: rather there is a characteristic growth time for the instability. This growth time may be very long, in which case the instability will never be seen. In practice, the growth time needs only be longer than transverse damping times (from synchrotron radiation, or from feedback systems) for the instability to be unimportant.

The growth rate is proportional to the chromaticity of the machine: hence, to suppress the instability, small values of the chromaticity are desirable. The control of this instability is one of the principal reasons for the use of sextupoles as chromatic correctors in high-energy accelerators.

The dependence of the vertical betatron tune on relative momentum deviation δ is

$$Q_y(\delta) = Q_{y0} + \xi_y \delta$$

where ξ_y is the vertical chromaticity. Consider the two macroparticles, representing the head and tail of the bunch. These particles are undergoing synchrotron oscillations, so the energy is a function of turn number, and hence so is the vertical tune:

$$Q_y(n) = Q_{y0} + \xi_y \delta(n)$$

Ignore the wake field effects for the moment, and focus on the motion of macroparticle 1. The equations of motion for a constant tune have the form

$$\frac{dy}{dn} = 2\pi Q_y \beta_y y' \quad \frac{dy'}{dn} = -\frac{2\pi Q_y}{\beta_y} y$$

For a variable Q, we have

$$\frac{dy}{dn} = 2\pi Q_y(n) \beta_y y' \quad \frac{dy'}{dn} = -\frac{2\pi Q_y(n)}{\beta_y} y$$

The solution can be written in matrix form as

$$\begin{pmatrix} y(n) \\ y'(n) \end{pmatrix} = \begin{pmatrix} \cos 2\pi \int_0^n dn Q_y(n) & \beta_y \sin 2\pi \int_0^n dn Q_y(n) \\ -\frac{1}{\beta_y} \sin 2\pi \int_0^n dn Q_y(n) & \cos 2\pi \int_0^n dn Q_y(n) \end{pmatrix} \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix}$$

Or, in terms of the $\tilde{y}$ variable introduced earlier

$$\tilde{y}(n) = \tilde{y}(0) \exp\left(-2\pi i \int_0^n dn Q_y(n) \right)$$

The integral can be written as

$$\int_0^n dn Q_y(n) = n Q_{y0} + \xi_y \int_0^n dn \delta(n)$$

From Lecture 10, p 15, we have

$$\frac{d\Delta t_n}{dn} \approx \frac{C\eta_C}{c} \delta \Rightarrow \delta \approx \frac{1}{C\eta_C} c \frac{d\Delta t_n}{dn} = -\frac{1}{C\eta_C} \frac{dz(n)}{dn}$$

for β=1 particles, in which $z = -c\Delta t_n$ is the longitudinal distance from the synchronous particle. Then

$$\int_0^n \delta(n) dn = -\frac{1}{C\eta_C} \int_0^n \frac{dz(n)}{dn} dn = -\frac{z(n) - z(0)}{C\eta_C}$$

Let macroparticle 1 be undergoing a synchrotron oscillation of the form

$$z_1(n) = z_0 \sin(2\pi Q_s n).$$

Then

$$\tilde{y}_1(n) = \tilde{y}_1(0)\exp\Big(-i\big(2\pi Q_{y0} n - \chi \sin(2\pi Q_s n)\big)\Big)$$

in which $\chi = \dfrac{2\pi \xi_y}{C\eta_C} z_0$

We see that the betatron phase is modulated according to the relative longitudinal position of the macroparticle. Typically, the modulation amplitude χ (called the "head-tail phase") is small. For example, for z_0=1 cm, η_C=0.01, C=750 m, and ξ_y = -5, we have $\chi \approx$ -0.04.

The modulation of the tune by the head-tail phase is the mechanism behind the head-tail instability. The modulation

allows a slow growth of unstable motion at any beam intensity, despite the fact that the macroparticles exchange places during the synchrotron oscillations.

Macroparticle 2 is undergoing a synchrotron oscillation also, but it's at the other end of the bunch; so

$$z_2(n) = -z_0 \sin(2\pi Q_s n)$$

and the transverse motion of macroparticle 2 is

$$\tilde{y}_2(n) = \tilde{y}_2(0)\exp\Big(-i\big(2\pi Q_{y0} n + \chi \sin(2\pi Q_s n)\big)\Big)$$

From above, we have

$$\frac{d\tilde{y}_2}{dn} = -2\pi i Q_y(n)\tilde{y}_2 \quad = -\big(2\pi Q_{y0} + 2\pi\chi Q_s \cos(2\pi Q_s n)\big) i\tilde{y}_2$$

Now we want to include the wake field. From above,

$$\tilde{y} = y + i\frac{dy/dn}{2\pi Q_y}, \qquad \frac{dy(n)}{dn} = 2\pi Q_y \beta_y y' \Rightarrow \tilde{y} = y + i\beta_y y'$$

$$\Delta y_2' = \frac{Ne^2}{2m_0c^2\gamma} W y_1 \Rightarrow \Delta\tilde{y} = i\beta_y \Delta y' = \frac{i\beta_y Ne^2}{2m_0c^2\gamma} W y_1$$

so the equation of motion for macroparticle 2 is

$$\frac{d\tilde{y}_2}{dn} = -\big(2\pi Q_{y0} + 2\pi\chi Q_s \cos(2\pi Q_s n)\big) i\tilde{y}_2 + \frac{i\beta_y Ne^2}{2m_0c^2\gamma} W y_1$$

To solve this equation, we use the result given above for $\tilde{y}_1(n)$,

$$\tilde{y}_1(n) = \tilde{y}_1(0)\exp\Big(-i\big(2\pi Q_{y0} n - \chi \sin(2\pi Q_s n)\big)\Big)$$

and use $y_1(n) = \dfrac{1}{2}\big(\tilde{y}_1(n) + \tilde{y}_1^*(n)\big)$

Then we take as a trial solution for $\tilde{y}_2(n)$ the form

$$\tilde{y}_2(n) = \hat{y}_2(n)\exp\Big(-i\big[2\pi Q_{y0} n + \chi \sin(2\pi Q_s n)\big]\Big)$$

in which $\hat{y}_2(n)$ is a slowly varying complex function of n. The equation above then gives

$$\frac{d\hat{y}_2}{dn} \approx i\tilde{y}_1(0)\frac{Ne^2 W\beta_y}{4m_0c^2\gamma}\exp\big(2i\chi \sin(2\pi Q_s n)\big)$$

$$+ i\tilde{y}_1^*(0)\frac{Ne^2 W\beta_y}{4m_0c^2\gamma}\exp\big(4\pi i n Q_y\big)$$

The second term is a rapidly varying function of n and may be dropped. We then expand the exponential (since χ<<1) and solve the differential equation

$$\frac{d\hat{y}_2}{dn} = \tilde{y}_1(0)\frac{Ne^2 W\beta_y}{4m_0c^2\gamma}\left(i - 2\chi\sin(2\pi Q_s n)\right)$$

The solution is

$$\hat{y}_2(n) = \hat{y}_2(0) + \tilde{y}_1(0)\frac{i\beta_y W N e^2\left(n + \frac{i\chi}{\pi Q_s}\left(1-\cos(2\pi Q_s n)\right)\right)}{4m_0c^2\gamma}$$

The solution can be written in matrix form, for the first half of the synchrotron period, as

$$\begin{pmatrix}\hat{y}_1(n)\\ \hat{y}_2(n)\end{pmatrix} = \begin{pmatrix}1 & 0\\ 2iTQ_s\left(n + \frac{i\chi}{\pi Q_s}\left(1-\cos(2\pi Q_s n)\right)\right) & 1\end{pmatrix}\begin{pmatrix}\hat{y}_1(0)\\ \hat{y}_2(0)\end{pmatrix}$$

After 1/2 of a synchrotron oscillation period, we have

$$\begin{pmatrix}\hat{y}_1\left(\frac{1}{2Q_S}\right)\\ \hat{y}_2\left(\frac{1}{2Q_S}\right)\end{pmatrix} = \begin{pmatrix}1 & 0\\ iT\left(1+\frac{4i\chi}{\pi}\right) & 1\end{pmatrix}\begin{pmatrix}\hat{y}_1(0)\\ \hat{y}_2(0)\end{pmatrix}$$

For the second half of the synchrotron period, particle 2 drives particle 1; so we have

$$\begin{pmatrix}\hat{y}_1\left(\frac{1}{Q_s}\right)\\ \hat{y}_2\left(\frac{1}{Q_s}\right)\end{pmatrix} = \begin{pmatrix}1 & iT\left(1+\frac{4i\chi}{\pi}\right)\\ 0 & 1\end{pmatrix}\begin{pmatrix}\hat{y}_1\left(\frac{1}{2Q_s}\right)\\ \hat{y}_2\left(\frac{1}{2Q_s}\right)\end{pmatrix}$$

The overall matrix for one synchrotron period is the product:

$$\begin{pmatrix}\hat{y}_1\left(\frac{1}{Q_s}\right)\\ \hat{y}_2\left(\frac{1}{Q_s}\right)\end{pmatrix} = \begin{pmatrix}1-T^2\left(1+\frac{4i\chi}{\pi}\right)^2 & iT\left(1+\frac{4i\chi}{\pi}\right)\\ iT\left(1+\frac{4i\chi}{\pi}\right) & 1\end{pmatrix}\begin{pmatrix}\hat{y}_1(0)\\ \hat{y}_2(0)\end{pmatrix}$$

Since the matrix has determinant=1, the eigenvalues of this matrix have the form $\lambda_1\lambda_2 = 1, \quad \lambda_{1,2} = \exp(\mp i\phi)$, in which ϕ is

complex. Following the same argument as in the discussion of the strong head-tail instability, we conclude that

$$\sin\frac{\phi}{2} = \frac{T}{2}\left(1+\frac{4i\chi}{\pi}\right)$$

For the case of T<<1, we have

$$\phi \approx T + \frac{4i\chi T}{\pi}$$

and the eigenvalues are

$$\lambda_{1,2} = \exp\left(\mp i\left(T + \frac{4i\chi T}{\pi}\right)\right)$$

$$= \exp(\mp iT)\exp\left(\pm\frac{4\chi T}{\pi}\right)$$

The real part, which is related to the modulation of the tune, gives unstable growth of one of the eigenmodes (and damping of the other). The growth rate per synchrotron period is $\frac{4\chi \mathrm{T}}{\pi}$, so the growth rate per unit time is

$$\frac{1}{\tau} = \frac{4\chi \mathrm{T}}{T_s \pi} = \frac{\chi}{T_s \pi} \frac{Ne^2 W \beta_y}{2Q_s m_0 c^2 \gamma} = \frac{1}{T_0} \frac{Ne^2 W \beta_y \chi}{2\pi m_0 c^2 \gamma}$$

Example:

We'll take β_y=20 m, χ=-0.04, W=5x10^{14} V/C/m, γ=10^4, C=750 m, T_0=2.5 μs, N=2x10^{11}. We find τ= 6.3 ms.

Although it appears that the growth rate is zero only for zero chromaticity, in fact a more sophisticated analysis shows that the growth rate of the (-) mode (corresponding to positive chromaticity) is smaller than given by the above formula. Hence most machines are operated with a small positive chromaticity.