LECTURE 25

Collective instabilities;

Rigid beam transverse instability

This is the transverse analog of the Robinson instability

To see how this works, consider a “macroparticle”: a point charge of magnitude *Ne*, circulating in a synchrotron. This macroparticle will create a wake field when it passes through an impedance. The macroparticle undergoes betatron oscillations; the wake potentials introduce additional forces into the betatron equations of motion. These additional forces can lead to an instability.

The wake fields generated by the macroparticle can be expressed in terms of a transverse integrated force exerted at the location of the impedance. The transverse integrated force due to a pure harmonic current, with an *m*th moment, of the form , $I_m(t) = \tilde{I}_m(\omega)\cos(\omega t)$ from Lecture 21, p 25, can be written in terms of the impedance as

$$\bar{\vec{F}}_\perp(t) = ieI_m(t)mr^{m-1}\left(\hat{r}\cos m\phi - \hat{\phi}\sin m\phi\right)Z_m^\perp(\omega)$$

For *m*=1, and in the vertical direction, we have

$$\bar{F}_y(t) = ieI_1(t)Z_1^\perp(\omega)$$

To use the above equation, we need to know the Fourier spectrum of the dipole moment of the current, which consists of a single circulating macroparticle of charge *Ne*. Let us consider for the moment the monopole current of a macroparticle that is

not undergoing betatron oscillations. As discussed above in Lecture 23, the current due to the point charge *Ne* is a series of impulses, which occur at times $t = nT_0$, where n is the turn number, an integer running from $-\infty$ to ∞. This current can be represented as a sum of Dirac delta-functions

$$I_0(t) = Ne\sum_{n=-\infty}^{\infty}\delta(t - nT_0)$$

in which T_0 is the revolution period.

As explained in Lecture 23, this can be written in terms of harmonics as

$$I_0(t) = \frac{Ne}{T_0}\sum_{p=-\infty}^{\infty}\exp(-ip\omega_0 t)$$

Now let us consider the dipole moment of the current. For a particle executing a betatron oscillation, we have $I_1(t) = I_0(t)y(t)$, where $y(t)$ has the form of a betatron oscillation, evaluated at the location of the impedance. Thus

$$y(t) = y_0\cos\left(Q_y\omega_0 t\right) + y_0'\beta_y\sin\left(Q_y\omega_0 t\right)$$

$$y'(t) = -\frac{y_0}{\beta_y}\sin\left(Q_y\omega_0 t\right) + y_0'\cos\left(Q_y\omega_0 t\right).$$

Then we have

$$I_1(t) = I_0(t)y(t) =$$

$$\frac{Ne}{T_0}\left(y_0 \cos\left(Q_y\omega_0 t\right) + y_0'\beta_y \sin\left(Q_y\omega_0 t\right)\right)\sum_{p=-\infty}^{\infty}\exp(-ip\omega_0 t)$$

$$= \frac{Ne}{2T_0}\sum_{p=-\infty}^{\infty}\left(y_0 + iy_0'\beta_y\right)\exp\left(-i\left(p+Q_y\right)\omega_0 t\right)$$

$$+\left(y_0 - iy_0'\beta_y\right)\exp\left(-i\left(p-Q_y\right)\omega_0 t\right)$$

The integrated force, summed over all harmonics, will then be

$$\overline{F}_y(t) = \frac{iNe^2}{2T_0}\sum_{p=-\infty}^{\infty}\left(y_0 + iy_0'\beta_y\right)\exp\left(-i\left(p+Q_y\right)\omega_0 t\right)Z_1^{\perp}\left(\left(p+Q_y\right)\omega_0\right)$$

$$+\left(y_0 - iy_0'\beta_y\right)\exp\left(-i\left(p-Q_y\right)\omega_0 t\right)Z_1^{\perp}\left(\left(p-Q_y\right)\omega_0\right)$$

$$= \frac{iNe^2}{2T_0}\sum_{p=-\infty}^{\infty}\tilde{y}_0\exp\left(-i\left(p+Q_y\right)\omega_0 t\right)Z_1^{\perp}\left(\left(p+Q_y\right)\omega_0\right)$$

$$+\tilde{y}_0^*\exp\left(-i\left(p-Q_y\right)\omega_0 t\right)Z_1^{\perp}\left(\left(p-Q_y\right)\omega_0\right)$$

using $\tilde{y} = y + i\beta_y y'$

We want to evaluate this on each turn, that is, at $t=nT_0$.

$$\overline{F}_y(nT_0) = \overline{F}_{y,n} =$$

$$\frac{iNe^2}{2T_0}\begin{pmatrix}\exp\left(-i2\pi Q_y n\right)\tilde{y}_0\sum_{p=-\infty}^{\infty}Z_1^{\perp}\left(\left(p+Q_y\right)\omega_0\right) \\ +\exp\left(i2\pi Q_y n\right)\tilde{y}_0^*\sum_{p=-\infty}^{\infty}Z_1^{\perp}\left(\left(p-Q_y\right)\omega_0\right)\end{pmatrix}$$

Then, we can use the symmetry property

$$Z_1^{\perp}(\omega) = -Z_1^{\perp *}(-\omega).$$

So that

$$\sum_{p=-\infty}^{\infty}Z_1^{\perp}\left(\left(p-Q_y\right)\omega_0\right) = -\sum_{p=-\infty}^{\infty}Z_1^{\perp *}\left(\left(p+Q_y\right)\omega_0\right)$$

and we have

$$\overline{F}_{y,n} = \frac{iNe^2}{2T_0}\begin{pmatrix}\tilde{y}(n)\sum_{p=-\infty}^{\infty}Z_1^{\perp}\left(\left(p+Q_y\right)\omega_0\right) \\ -\tilde{y}^*(n)\sum_{p=-\infty}^{\infty}Z_1^{\perp *}\left(\left(p+Q_y\right)\omega_0\right)\end{pmatrix}$$

This can be written as

$$\overline{F}_{y,n} = -\frac{Ne^2}{T_0}\times$$

$$\left(y(n)\sum_{p=-\infty}^{\infty}\mathrm{Im}\left[Z_1^{\perp}\left(\left(p+Q_y\right)\omega_0\right)\right] + \beta_y y'(n)\sum_{p=-\infty}^{\infty}\mathrm{Re}\left[Z_1^{\perp}\left(\left(p+Q_y\right)\omega_0\right)\right]\right)$$

We now insert this into the betatron equations of motion. The unperturbed betatron equations, written in terms of turn number, have the form

$$\frac{dy'}{dn} = -2\pi Q_y \frac{y(n)}{\beta_y} \quad \frac{dy}{dn} = 2\pi Q_y \beta_y y'(n)$$

The effect of the integrated force is to produce a change in y' given by $\Delta y' = \frac{\bar{F}_{y,n}}{pv} = \frac{\bar{F}_{y,n}}{m_0 c^2 \gamma}$. Hence the equation of motion becomes

$$\frac{dy'}{dn} = -2\pi Q_y \frac{y(n)}{\beta_y} - \frac{Ne^2}{m_0 c^2 \gamma T_0}\left(Ay(n) + B\beta_y y'(n)\right)$$

$$A = \sum_{p=-\infty}^{\infty} \operatorname{Im}\left[Z_1^\perp\left(\left(p+Q_y\right)\omega_0\right)\right] \quad B = \sum_{p=-\infty}^{\infty} \operatorname{Re}\left[Z_1^\perp\left(\left(p+Q_y\right)\omega_0\right)\right]$$

and so

$$\frac{d^2y}{dn^2} = 2\pi Q_y \beta_y \frac{dy'(n)}{dn}$$

$$= -\left(2\pi Q_y\right)^2 y(n) - \frac{2\pi Q_y \beta_y Ne^2}{m_0 c^2 \gamma T_0}\left(Ay(n) + B\beta_y y'(n)\right)$$

$$= -\left(\left(2\pi Q_y\right)^2 + \frac{2\pi Q_y \beta_y Ne^2}{m_0 c^2 \gamma T_0} A\right) y(n) - \frac{\beta_y Ne^2}{m_0 c^2 \gamma T_0} B \frac{dy}{dn}$$

The A coefficient is associated with a tune shift; the B coefficient can cause damping or an instability. This equation has the general form

$$\frac{d^2y}{dn^2} = -2\alpha \frac{dy}{dn} - \left(2\pi Q_y'\right)^2 y$$

which is the equation of a damped harmonic oscillator, with a solution (for $\alpha << 2\pi Q_y'$)

$$y(n) \propto \exp\left(-\alpha + 2\pi i Q_y'\right) n$$

By comparing with the equation above, we see that the damping rate (per turn) is

$$\alpha = \frac{\beta_y Ne^2}{2m_0 c^2 \gamma T_0} B = \frac{\beta_y Ne^2}{2m_0 c^2 \gamma T_0} \sum_{p=-\infty}^{\infty} \operatorname{Re}\left[Z_1^\perp\left(\left(p+Q_y\right)\omega_0\right)\right]$$

and the frequency is given by

$$\left(2\pi Q_y'\right)^2 = \left(2\pi Q_y\right)^2 + \frac{2\pi Q_y \beta_y Ne^2}{m_0 c^2 \gamma T_0} A$$

Using

$$2\pi Q_y' = \sqrt{\left(2\pi Q_y\right)^2 + \Delta} = 2\pi Q_y \sqrt{1 + \frac{\Delta}{\left(2\pi Q_y\right)^2}} \approx 2\pi Q_y + \frac{\Delta}{4\pi Q_y}$$

$$\Rightarrow \delta Q_y = \frac{\Delta}{8\pi^2 Q_y}$$

we get

$$\delta Q_y = \frac{\beta_y Ne^2}{4\pi m_0 c^2 \gamma T_0} A = \frac{\beta_y Ne^2}{4\pi m_0 c^2 \gamma T_0} \sum_{p=-\infty}^{\infty} \operatorname{Im}\left[Z_1^\perp\left(\left(p+Q_y\right)\omega_0\right)\right]$$

Example: the transverse resistive wall instability. The impedance is (Lecture 19, p 23)

$$Z_1^{\perp}(\omega) = C\frac{1 - i\,\mathrm{sgn}(\omega)}{\omega\pi b^3}\sqrt{\frac{|\omega|\mu c^2}{2\sigma}}$$

$$\sum_{p=-\infty}^{\infty}\mathrm{Re}\left[Z_1^{\perp}\left(\left(p+Q_y\right)\omega_0\right)\right] = \frac{C}{\pi b^3}\sqrt{\frac{\mu c^2}{2\omega_0\sigma}}\sum_{p=-\infty}^{\infty}\frac{\sqrt{|p+Q_y|}}{p+Q_y}$$

$$= \frac{C}{\pi b^3}\sqrt{\frac{\mu c^2}{\omega_0\sigma}}f\left(\Delta_\beta\right), \quad f\left(\Delta_\beta\right) = \frac{1}{\sqrt{2}}\sum_{p=-\infty}^{\infty}\frac{\sqrt{|p+\Delta_\beta|}}{p+\Delta_\beta}, \quad Q_y = n + \Delta_\beta$$

in which n is the integral part of the tune.

The damping rate per turn is

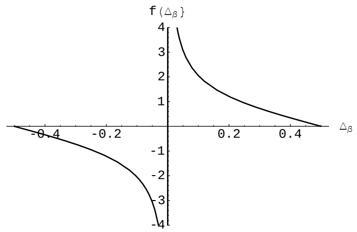
$$\alpha = \frac{\beta_y N e^2 c^2}{2\pi b^3 \gamma m_0 c^2}\sqrt{\frac{\mu}{\omega_0\sigma}}f\left(\Delta_\beta\right)$$

The function

$$f\left(\Delta_\beta\right) = \frac{1}{\sqrt{2}}\sum_{p=-\infty}^{\infty}\frac{\sqrt{|p+\Delta_\beta|}}{p+\Delta_\beta} = \frac{1}{\sqrt{2}}\left(\frac{\Delta_\beta}{\left|\Delta_\beta\right|^{3/2}} + \zeta\left(\frac{1}{2}, 1+\Delta_\beta\right) - \zeta\left(\frac{1}{2}, 1-\Delta_\beta\right)\right)$$

($\zeta(a,b)$ is the zeta function)

is shown below



For damping, we need to have α>0, that is, $f\left(\Delta_\beta\right)$>0, which requires $\Delta_\beta > 0$ Since $\Delta_\beta = Q_y - n$, to damp the transverse resistive wall instability, the fractional part of the machine tune should be below the half-integer, that is, in the range $0 < \Delta_\beta < 0.5$

For CESR parameters: Q_y=9.58, Δ_β=-0.42, $f(\Delta_\beta)$=–0.27, β_y=20 m, N=2x10^{11}, b=2.5 cm, γ=10^4, T_0=2.5 µs, σ=3.5x10^7 Ω^{-1}m^{-1} (aluminum), we find a growth time of $\frac{T_0}{\alpha} = -0.67$ s. (not a very strong instability).