LECTURE 26

Collective instabilities;

Rigid beam transverse multibunch instability

The macroparticle model used in the previous lecture can be applied to the important case of multiple bunches in a common vacuum chamber. Long-range wakefields will couple the motion of the bunches together and can lead to tune shifts and instabilities.

As we saw above, the wake fields generated by the macroparticle can be expressed in terms of a transverse integrated force exerted at the location of the impedance.

$$\vec{\bar{F}}_\perp(t) = ieI_m(t)mr^{m-1}\left(\hat{r}\cos m\phi - \hat{\phi}\sin m\phi\right)Z_m^\perp(\omega)$$

For m=1, and in the vertical direction, we have

$$\bar{F}_y(t) = ieI_1(t)Z_1^\perp(\omega)$$

To use the above equation, we need to know the Fourier spectrum of the dipole moment of the current. As discussed in Lecture 25, the wake force is

$$\bar{F}_y(t) = \frac{iNe^2}{2T_0}\sum_{p=-\infty}^{\infty} \tilde{y}_0 \exp\left(-i\left(p+Q_y\right)\omega_0 t\right)Z_1^\perp\left(\left(p+Q_y\right)\omega_0\right)$$

$$+\tilde{y}_0^* \exp\left(-i\left(p-Q_y\right)\omega_0 t\right)Z_1^\perp\left(\left(p-Q_y\right)\omega_0\right)$$

in which $\tilde{y} = y + i\beta_y y'$. Using the symmetry property

$$Z_1^\perp(\omega) = -Z_1^{\perp *}(-\omega).$$

The integrated force, summed over all harmonics, can be written as

$$\bar{F}_y(t) = \frac{iNe^2}{2T_0}\tilde{y}_0\sum_{p=-\infty}^{\infty} \exp\left(-i\left(p+Q_y\right)\omega_0 t\right)Z_1^\perp\left(\left(p+Q_y\right)\omega_0\right) + c.c.$$

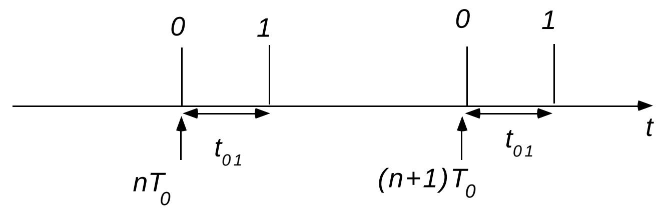
c.c represents the complex conjugate-dropped for now, added back in equation of motion

This is the integrated force due to a single macroparticle. Suppose now that we have 2 bunches (macroparticles), of equal charge. We'll label the first bunch 0, and the second (trailing) bunch 1. The wake force due to bunch 0 can be written as

$$\bar{F}_{y0}(t) = \frac{iNe^2}{2T_0}\tilde{y}_0(t)\sum_{p=-\infty}^{\infty} \exp\left(-ip\omega_0 t\right)Z_1^\perp\left(\left(p+Q_y\right)\omega_0\right)$$

in which $\tilde{y}(t) = \tilde{y}_0 \exp\left(-iQ_y\omega_0 t\right)$

Suppose that bunch 1 trails bunch 0 by the time interval $t = t_{01}$.



Since it arrives at the impedance at $t = nT_0 + t_{01}$, its current is given by

$$I_0(t) = \frac{Ne}{T_0}\sum_{p=-\infty}^{\infty} \exp\left(-ip\omega_0\left(t-t_{01}\right)\right)$$

and its betatron oscillation can be written as

$$\tilde{y}_1(t) = \tilde{y}_{10} \exp\left(-iQ_y\omega_0\left(t - t_{01}\right)\right)$$

so the force created by its wake is given by

$$\overline{F}_{y1}(t) = \frac{iNe^2}{2T_0}\tilde{y}_{10} \exp\left(-iQ_y\omega_0\left(t - t_{01}\right)\right) \times$$

$$\sum_{p=-\infty}^{\infty} \exp\left(-ip\omega_0\left(t - t_{01}\right)\right) Z_1^{\perp}\left(\left(p + Q_y\right)\omega_0\right)$$

Bunch 0 arrives at the impedance at time $t = ... - T_0, 0, T_0, ...$ and feels the total wake force

$$\overline{F}_{y0,n} = \overline{F}_{y0}\left(nT_0\right) + \overline{F}_{y1}\left(nT_0\right) =$$

$$= \frac{iNe^2}{2T_0}\tilde{y}_{00} \exp\left(-i2\pi Q_y n\right) \sum_{p=-\infty}^{\infty} Z_1^{\perp}\left(\left(p + Q_y\right)\omega_0\right) +$$

$$\frac{iNe^2}{2T_0}\tilde{y}_{01} \exp\left(-i2\pi Q_y\left(n - \frac{t_{01}}{T_0}\right)\right) \sum_{p=-\infty}^{\infty} \exp\left(ip\omega_0 t_{01}\right) Z_1^{\perp}\left(\left(p + Q_y\right)\omega_0\right)$$

Let us define

$$\hat{y}_0(n) = \tilde{y}_{00} \exp\left(-2\pi i Q_y n\right) = \tilde{y}_0(n)$$

$$\hat{y}_1(n) = \tilde{y}_{01} \exp\left(-2\pi i Q_y\left(n - \frac{t_{01}}{T_0}\right)\right) = \tilde{y}_1\left(nT_0\right)$$

$\hat{y}_0(n),\ \hat{y}_1(n)$ are the $\hat{y}$ variables of bunch 0,1 when bunch 0 crosses the location of the impedance. These are sometimes called the “snapshot” position of the bunch. $\hat{y}_0(n),\ \hat{y}_1(n)$ describe the bunch displacements and slopes, not at the same location, but at the same time at different locations (the location of the impedance, for bunch 0; a distance ct_{01} behind bunch 0, for bunch 1).

Then we have

$$F_{y0,n} = \frac{iNe^2}{2T_0}\left(\begin{array}{l} \hat{y}_0(n) \displaystyle\sum_{p=-\infty}^{\infty} Z_1^{\perp}\left(\left(p + Q_y\right)\omega_0\right) + \\ \hat{y}_1(n) \displaystyle\sum_{p=-\infty}^{\infty} \exp\left(2\pi i p \frac{t_{01}}{T_0}\right) Z_1^{\perp}\left(\left(p + Q_y\right)\omega_0\right) \end{array}\right)$$

We now insert this into the betatron equation of motion. The unperturbed betatron equation for the 0th bunch, written in terms of turn number, has the form

$$\frac{d\hat{y}_0}{dn} = -2\pi i Q_y \hat{y}_0$$

The effect of the integrated force is to produce a change in $\hat{y}_0$ given by $\Delta\hat{y}_0 = i\beta_y \Delta y_0' = i\beta_y \frac{\overline{F}_{y0,n}}{pv} = i\beta_y \frac{\overline{F}_{y0,n}}{m_0 c^2 \gamma}$. Hence the equation of motion becomes

$$\frac{d\hat{y}_0}{dn} = -2\pi i Q_y \hat{y}_0 + i\beta_y\left(\frac{iNe^2}{2m_0c^2\gamma T_0}\left(\hat{y}_0 A - \hat{y}_0^* A^* + \hat{y}_1 B - \hat{y}_1^* B^*\right)\right)$$

$$A = \sum_{p=-\infty}^{\infty} Z_1^{\perp}\left(\left(p+Q_y\right)\omega_0\right)$$

$$B = \sum_{p=-\infty}^{\infty} \exp\left(2\pi i p\frac{t_{01}}{T_0}\right) Z_1^{\perp}\left(\left(p+Q_y\right)\omega_0\right)$$

Now let us consider the motion of bunch 1. Since bunch 1 trails bunch 0, it crosses the impedance at the time $nT_0 + t_{01}$ and feels the force

$$\overline{F}_{y1,n} = \overline{F}_{y0}\left(nT_0 + t_{01}\right) + \overline{F}_{y1}\left(nT_0 + t_{01}\right) =$$
$$\frac{iNe^2}{2T_0}\tilde{y}_0(n)\exp\left(-i2\pi Q_y\frac{t_{01}}{T_0}\right)\sum_{p=-\infty}^{\infty}\exp\left(-ip\omega_0 t_{01}\right)Z_1^{\perp}\left(\left(p+Q_y\right)\omega_0\right) +$$
$$\frac{iNe^2}{2T_0}\tilde{y}_1(n)\sum_{p=-\infty}^{\infty}Z_1^{\perp}\left(\left(p+Q_y\right)\omega_0\right)$$
$$= \frac{iNe^2}{2T_0}\exp\left(-i2\pi Q_y\frac{t_{01}}{T_0}\right)\left(\hat{y}_0 B' + \hat{y}_1 A\right)$$
$$B' = \sum_{p=-\infty}^{\infty}\exp\left(-i2\pi p\frac{t_{01}}{T_0}\right)Z_1^{\perp}\left(\left(p+Q_y\right)\omega_0\right)$$

We now insert this into the betatron equation of motion for bunch 1. The unperturbed betatron equation for bunch 1, written in terms of turn number, has the form

$$\frac{d\hat{y}_1}{dn} = -2\pi i Q_y \hat{y}_1$$

The effect of the integrated force is to produce a change in $\hat{y}_1$ given by

$$\Delta\hat{y}_1 = i\beta_y \Delta y_1' \exp\left(2\pi i Q_y\frac{t_{01}}{T_0}\right) = i\beta_y\frac{\overline{F}_{y1,n}}{m_0c^2\gamma}\exp\left(2\pi i Q_y\frac{t_{01}}{T_0}\right).$$ Hence the equation of motion becomes

$$\frac{d\hat{y}_1}{dn} = -2\pi i Q_y \hat{y}_1 + i\beta_y\left(\frac{iNe^2}{2m_0c^2\gamma T_0}\left(\hat{y}_1 A - \hat{y}_1^* A^* + \hat{y}_0 B' - \hat{y}_0^* B'^*\right)\right)$$

This, and the equation for bunch 0, are a set of coupled differential equations, for the 2 bunches. We can rewrite these equations as

$$\frac{d\hat{y}_0}{dn} = -2\pi i Q_y \hat{y}_0 - \hat{y}_0\Gamma_A + \hat{y}_0^*\Gamma_A{}^* - \hat{y}_1\Gamma_B + \hat{y}_1^*\Gamma_B{}^*$$
$$\frac{d\hat{y}_1}{dn} = -2\pi i Q_y \hat{y}_1 - \hat{y}_1\Gamma_A + \hat{y}_1^*\Gamma_A{}^* - \hat{y}_0\Gamma_{B'} + \hat{y}_0^*\Gamma_{B'}{}^*$$

in which

$$\Gamma_A = \frac{ANe^2\beta_y}{2m_0c^2\gamma T_0} \quad \Gamma_B = \frac{BNe^2\beta_y}{2m_0c^2\gamma T_0} \quad \Gamma_{B'} = \frac{B'Ne^2\beta_y}{2m_0c^2\gamma T_0}$$

We will treat the wake effects as a small perturbation: that is, we assume that

$\frac{|\Gamma|}{2\pi Q_y} << 1$, and take the motion of the two bunches to have the form

$$\hat{y}_{0,1}(n) = \hat{y}_{00,01} \exp(-i\Omega n)$$

with $\Omega = 2\pi Q_y + \delta, \ |\delta| << 1$

In this case, the complex conjugate terms in the above equations have the approximate forms

$$\hat{y}^*_{0,1}(n) \approx \hat{y}^*_{00,01} \exp\left(i2\pi Q_y n\right) \approx \hat{y}_{0,1}(n)\left(\frac{\hat{y}^*_{00,01}}{\hat{y}_{00,01}} \exp\left(i4\pi Q_y n\right)\right)$$

For $\frac{|\Gamma|}{2\pi Q_y} << 1$, these rapidly oscillating terms may be omitted from the equations, which then simplify to the set of coupled equations

$$\frac{d\hat{y}_0}{dn} = -2\pi i Q_y \hat{y}_0 - \hat{y}_0 \Gamma_A - \hat{y}_1 \Gamma_B$$

$$\frac{d\hat{y}_1}{dn} = -2\pi i Q_y \hat{y}_1 - \hat{y}_1 \Gamma_A - \hat{y}_0 \Gamma_{B'}$$

or, in matrix form,

$$\frac{d\vec{\hat{y}}'}{dn} = \mathbf{M}\vec{\hat{y}}, \quad \mathbf{M} = \begin{pmatrix} -2\pi i Q_y - \Gamma_A & -\Gamma_B \\ -\Gamma_{B'} & -2\pi i Q_y - \Gamma_A \end{pmatrix}$$

There will be a set of normal modes ζ_m, for which the equations of motion decouple:

$$\vec{\hat{y}} = \mathbf{S}\vec{\zeta}$$

The normal mode eqations are

$$\mathbf{S}\frac{d\vec{\zeta}'}{dn} = \mathbf{MS}\vec{\zeta} \qquad \frac{d\vec{\zeta}'}{dn} = \mathbf{S}^{-1}\mathbf{MS}\vec{\zeta} = \Lambda\vec{\zeta}$$

The matrix $\Lambda = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{pmatrix}$

in which λ_0 and λ_1 are the eigenvalues of the matrix **M**. For the matrix given above, the eigenvalues are

$$\lambda_i = -2\pi i Q_y - \Gamma_A \pm \sqrt{\Gamma_B \Gamma_{B'}}$$

The eigenvectors in the $(\hat{y}_0, \hat{y}_1)$ basis are

$$\vec{\zeta}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{\frac{B}{B'}} \\ 1 \end{pmatrix}, \quad \vec{\zeta}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -\sqrt{\frac{B}{B'}} \\ 1 \end{pmatrix}$$

We have, for each normal mode, the equation

$$\frac{d\zeta_i'}{dn} = \lambda_i \zeta_i$$

Assuming a solution of the form

$\zeta_i(n) = \tilde{\zeta}_{i0} \exp\left(-in\Omega_i\right)$. Using

$$\frac{d\zeta_i'}{dn} = -i\Omega_i \zeta_i = \lambda_i \zeta_i.$$

The normal mode frequencies are

$$\Omega_i = i\lambda_i = 2\pi Q_y - i\Gamma_A \pm i\sqrt{\Gamma_B \Gamma_{B'}}$$

Using the definitions of Γ and A, B from above, these become

$$\Omega_i - 2\pi Q_y = -i\left(\Gamma_A \mp \sqrt{\Gamma_B \Gamma_{B'}}\right)$$
$$= -\frac{i\beta_y N e^2}{2 m_0 c^2 \gamma T_0}\left(A \pm \sqrt{BB'}\right)$$
$$= -\frac{i\beta_y N e^2}{2 m_0 c^2 \gamma T_0} \times$$
$$\left[\sum_{p=-\infty}^{\infty} Z_1^{\perp}\left(\left(p+Q_y\right)\omega_0\right) \pm \sqrt{\sum_{p=-\infty}^{\infty}\sum_{p'=-\infty}^{\infty} \exp\left(2\pi i(p-p')\frac{t_{01}}{T_0}\right) Z_1^{\perp}\left(\left(p+Q_y\right)\omega_0\right) Z_1^{\perp}\left(\left(p'+Q_y\right)\omega_0\right)}\right]$$

Consider the special case when $t_{01} = \frac{T_0}{2}$. Then

$$\exp\left(2\pi i(p-p')\frac{t_{01}}{T_0}\right) = \exp\left(\pi i(p-p')\right) = (-1)^p(-1)^{p'}$$

$$\Omega_i - 2\pi Q_y = -\frac{i\beta_y N e^2}{2 m_0 c^2 \gamma T_0}\left[\sum_{p=-\infty}^{\infty}\left(1 \pm (-1)^p\right) Z_1^{\perp}\left(\left(p+Q_y\right)\omega_0\right)\right]$$

The eigenvectors are

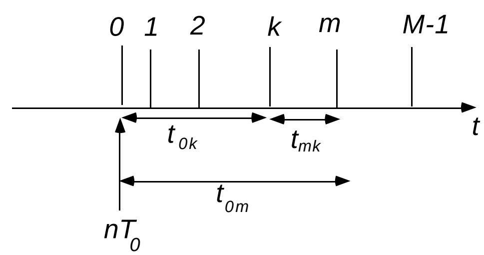
$$\vec{\zeta}_0 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix},\ \vec{\zeta}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix} \Rightarrow$$

$$\zeta_0 = \hat{y}_0 + \hat{y}_1, \quad \zeta_1 = -\hat{y}_0 + \hat{y}_1$$

In the sum mode, both bunches oscillate in phase; in the difference mode, the two bunches oscillate out of phase.

The general case

Let there be M bunches in the machine, with the labels $y_0, y_1, \ldots, y_{M-1}$. Let the time separation between the bunches be as shown below



Following from above, the force due to the mth bunch, is given by

$$\bar{F}_{ym}(t) = \frac{iNe^2}{2T_0}\tilde{y}_{m0}\exp\left(-2\pi i Q_y \omega_0 (t - t_{0m})\right)\times$$
$$\sum_{p=-\infty}^{\infty}\exp\left(-ip\omega_0(t - t_{0m})\right)Z_1^{\perp}\left(\left(p + Q_y\right)\omega_0\right)$$

The force on the kth bunch due to the mth bunch is

$$\bar{F}_{yk,m}(n) = \bar{F}_{ym}(nT_0 + t_{0k}) =$$
$$\frac{iNe^2}{2T_0}\tilde{y}_{m0}\exp\left(-2\pi i Q_y\left(n - \frac{t_{0m} - t_{0k}}{T_0}\right)\right)\times$$
$$\sum_{p=-\infty}^{\infty}\exp\left(i2\pi p\left(\frac{t_{mk}}{T_0}\right)\right)Z_1^{\perp}\left(\left(p + Q_y\right)\omega_0\right)$$

in which

$$t_{mk} = t_{0m} - t_{0k}$$

Using $\hat{y}_m(n) = \tilde{y}_m\exp\left(-2\pi i Q_y\left(n - \frac{t_{0m}}{T_0}\right)\right)$, the force on the kth bunch is

$$\bar{F}_{yk,m}(n) = \frac{iNe^2}{2T_0}\tilde{y}_m(n)\exp\left(-2\pi i Q_y\left(\frac{t_{0k}}{T_0}\right)\right)\times$$
$$\sum_{p=-\infty}^{\infty}\exp\left(i2\pi p\left(\frac{t_{mk}}{T_0}\right)\right)Z_1^{\perp}\left(\left(p + Q_y\right)\omega_0\right)$$

The total force on the kth bunch is

$$\bar{F}_{yk}(n) = \frac{iNe^2}{2T_0}\exp\left(-2\pi i Q_y\left(\frac{t_{0k}}{T_0}\right)\right)\times\sum_{m=0}^{M-1}\tilde{y}_m(n)$$
$$\sum_{p=-\infty}^{\infty}\exp\left(i2\pi p\left(\frac{t_{mk}}{T_0}\right)\right)Z_1^{\perp}\left(\left(p + Q_y\right)\omega_0\right)$$

We now insert this into the betatron equation of motion for bunch k. The unperturbed betatron equation for bunch k, written in terms of turn number, has the form

$$\frac{d\hat{y}_k}{dn} = -2\pi i Q_y\hat{y}_k$$

The effect of the integrated force is to produce a change in $\hat{y}_k$ given by

$$\Delta\hat{y}_k = i\beta_y\Delta y'_k\exp\left(2\pi i Q_y\frac{t_{0k}}{T_0}\right) = i\beta_y\frac{\bar{F}_{yk}(n)}{m_0c^2\gamma}\exp\left(2\pi i Q_y\frac{t_{0k}}{T_0}\right).$$ Hence the equation of motion becomes

$$\frac{d\hat{y}_k}{dn} = -2\pi i Q_y\hat{y}_k - \left(\begin{array}{c}\frac{N\beta_y e^2}{2m_0c^2\gamma T_0}\times\sum_{m=0}^{M-1}\tilde{y}_m(n)\\ \sum_{p=-\infty}^{\infty}\exp\left(i2\pi p\left(\frac{t_{mk}}{T_0}\right)\right)Z_1^{\perp}\left(\left(p + Q_y\right)\omega_0\right)\end{array}\right)$$

This is a set of M coupled differential equations for the M bunches. In matrix form, it can be written as

$$\frac{d\vec{y}}{dn} = \mathbf{M}\vec{y},$$

$$\mathbf{M}_{km} = -2\pi i Q_y \delta_{km} - \frac{N\beta_y e^2}{2m_0c^2\gamma T_0}\sum_{p=-\infty}^{\infty}\exp\left(i2\pi p\left(\frac{t_{mk}}{T_0}\right)\right)Z_1^{\perp}\left(\left(p+Q_y\right)\omega_0\right)$$

There will be a set of M normal modes ζ_m, for which the equations of motion decouple:

$$\vec{y} = \mathbf{S}\vec{\zeta}$$

$$\mathbf{S}\frac{d\vec{\zeta}'}{dn} = \mathbf{MS}\vec{\zeta} \qquad \frac{d\vec{\zeta}'}{dn} = \mathbf{S}^{-1}\mathbf{MS}\vec{\zeta} = \Lambda\vec{\zeta}$$

The matrix $\Lambda_{ij} = \delta_{ij}\lambda_i$ contains the eigenvalues λ_i of the matrix **M**.

As in the two-bunch case, the normal mode frequencies are given by the eigenvalues of **M**:

$$\Omega_i = i\lambda_i$$

For any bunch spacing and impedance, the matrix given above may be diagonalized numerically and the normal mode frequencies obtained. However, a general analytical solution for the normal mode frequencies for M bunches is only possible in special cases.

For example, suppose that the M bunches are uniformly distributed around the ring.

Then, we can write

$$t_{0m} = \frac{(m-1)}{M}T_0$$

and

$$\mathbf{M}_{km} = -2\pi i Q_y \delta_{km} - \frac{N\beta_y e^2}{2m_0c^2\gamma T_0}\sum_{p=-\infty}^{\infty}\exp\left(i2\pi p\left(\frac{m-k}{M}\right)\right)Z_1^{\perp}\left(\left(p+Q_y\right)\omega_0\right)$$

By analogy with the 2-bunch case, the matrix which gives the normal modes has the form

$$S_{ab} = \frac{1}{\sqrt{M}}\exp\left(\frac{2\pi iab}{M}\right)$$

and the eigenvalue matrix is then

$$\Lambda_{ab} = \sum_{k=0}^{M-1}\sum_{m=0}^{M-1}S_{ak}^{-1}M_{km}S_{mb}$$

$$= -2\pi i Q_y \delta_{ab} - \frac{N\beta_y e^2}{2m_0c^2\gamma T_0}\sum_{p=-\infty}^{\infty}Z_1^{\perp}\left(\left(p+Q_y\right)\omega_0\right)\times$$

$$\frac{1}{M}\sum_{m=0}^{M-1}\sum_{k=0}^{M-1}\exp\left(-\frac{2\pi iam}{M}\right)\exp\left(i2\pi p\left(\frac{m-k}{M}\right)\right)\exp\left(\frac{2\pi ikb}{M}\right)$$

Using the identity

$$\sum_{m=0}^{M-1}\exp\left(2\pi im\frac{i-j}{M}\right) = M\delta_{i-j,\,rM}$$

where r is any integer, we find

$$\frac{1}{M}\sum_{m=0}^{M-1}\sum_{k=0}^{M-1}\exp\left(-\frac{2\pi iam}{M}\right)\exp\left(i2\pi p\left(\frac{m-k}{M}\right)\right)\exp\left(\frac{2\pi ikb}{M}\right)$$
$$=\frac{1}{M}\sum_{m=0}^{M-1}\exp\left(2\pi im\frac{p-a}{M}\right)\sum_{k=0}^{M-1}\exp\left(2\pi ik\left(\frac{b-p}{M}\right)\right)$$
$$=\delta_{p,\ b+rM}\sum_{m=0}^{M-1}\exp\left(2\pi im\frac{b-a}{M}\right)=M\delta_{p,\ b+rM}\delta_{b,a}$$

so the eigenvalues are

$$\lambda_m=-2\pi iQ_y-\frac{NM\beta_y e^2}{2m_0c^2\gamma T_0}\sum_{r=-\infty}^{\infty}Z_1^{\perp}\left(\left(rM+m+Q_y\right)\omega_0\right)$$

and the normal mode frequencies are

$$\Omega_m-2\pi Q_y=-\frac{iNM\beta_y e^2}{2m_0c^2\gamma T_0}\sum_{r=-\infty}^{\infty}Z_1^{\perp}\left(\left(rM+m+Q_y\right)\omega_0\right)$$

The tune shift and damping rate for mode m are related to $\Omega_m-2\pi Q_y$ by

$$\Omega_m-2\pi Q_y=2\pi\Delta Q_m-i\alpha_m$$

so the tune shift is

$$\Delta Q_m=\frac{\beta_y NMe^2}{4\pi m_0c^2\gamma T_0}\mathrm{Im}\left[\sum_{r=-\infty}^{\infty}Z_1^{\perp}\left(\left(rM+m+Q_y\right)\omega_0\right)\right]$$

and the damping rate is

$$\alpha_m=\frac{\beta_y NMe^2}{2m_0c^2\gamma T_0}\mathrm{Re}\left[\sum_{r=-\infty}^{\infty}Z_1^{\perp}\left(\left(rM+m+Q_y\right)\omega_0\right)\right]$$

The eigenmodes are

$$\zeta_b=\sum_{a=0}^{M-1}S_{ba}^{-1}\hat{y}_a=\frac{1}{\sqrt{M}}\sum_{a=0}^{M-1}\exp\left(-\frac{2\pi iab}{M}\right)\hat{y}_a$$

The damping rate (or instability growth rate, if it is negative) for the multibunch instability is proportional to the total number of bunches, that is, the total current. The impedance is sampled at frequencies spaced by $M\omega_0$, rather than ω_0, as in the single bunch case. If the frequency structure of the impedance is much broader than $M\omega_0$, then the sparse sampling roughly cancels the factor of M in front, and the damping or growth rates are roughly the same for multiple bunches as for one bunch.

(This is because the wakefields for a broadband impedance are short range, and do not couple the bunches together).

But if the impedance is narrow-band compared to $M\omega_0$ (long-range wakefield), then the bunches are strongly coupled and the multibunch growth rates can be M times larger than for a single bunch.

Example: the transverse resistive wall instability. The impedance is (Lecture 19, p 23)

$$Z_1^{\perp}(\omega)=C\frac{1-i\,\mathrm{sgn}(\omega)}{\omega\pi b^3}\sqrt{\frac{|\omega|\mu c^2}{2\sigma}}$$

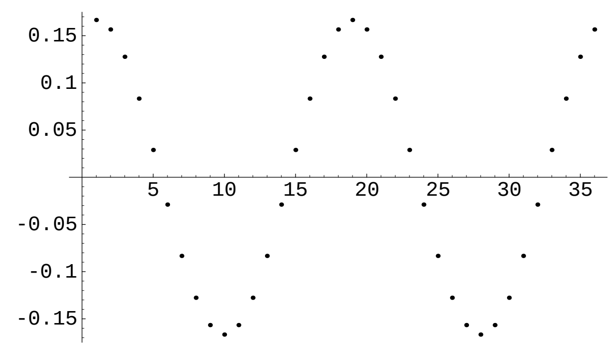
The impedance enters the damping rate in the form

$$\sum_{p=-\infty}^{\infty} \mathrm{Re}\left[Z_1^{\perp}\left(\left(pM+m+Q_y\right)\omega_0\right)\right] = \frac{C}{\pi b^3}\sqrt{\frac{\mu c^2}{2\omega_0\sigma}}\sum_{p=-\infty}^{\infty}\frac{\sqrt{\left|pM+m+Q_y\right|}}{pM+m+Q_y}$$

The multibunch mode which is most strongly driven will be the one for which the denominator is the smallest. The denominator is $pM+m+n+\Delta_\beta$, in which n is the integral part of the tune. Consider, for example, the Tevatron Collider, with M=36 bunches, and an integral tune of n=19. The denominator will be $36p+m+19+\Delta_\beta$, which is just Δ_β for p=-1 if the mode number is m=17. Thus, the mode m=17 will be the dominant multibunch mode. The snapshot mode pattern for m=17,

$$\hat{y}_a = \frac{1}{6}\sum_{a=0}^{M-1}\exp\left(\frac{17\pi i a}{18}\right)$$

is shown below:



This is a low frequency oscillation, which can be easily damped with a narrow band feedback system.

The damping rate per turn is

$$\alpha = \frac{M\beta_y N e^2 c^2}{2\pi b^3 \gamma m_0 c^2}\sqrt{\frac{\mu}{\omega_0\sigma}}f\left(\Delta_\beta\right)$$

in which $f\left(\Delta_\beta\right)$ is the function defined in Lecture 25. Taking the fractional tune to be Δ_β=-0.4, and with other parameters for the Tevatron as follows:

β_y=100 m, N=10^{11}, b=2.5 cm, γ=10^3, T_0=21 μs,

σ=3.5x10^7 Ω^{-1}m^{-1} (aluminum), we find a damping time of $\frac{T_0}{\alpha} = -3.2$ s. (a weak instability). This is a gross overestimate, in fact, since most of the Tevatron vacuum chamber is cold, and the resistance is therefore much less than assumed above.