LECTURE 3

Particle trajectory equations (continued)

Expansion of the fields about the reference orbit:

$$B_y = B_0 + B'x + \Delta B_y(x,y,s)$$
$$B_x = B'y + \Delta B_x(x,y,s)$$
$$B_s = \Delta B_s(x,y,s)$$

In these equations, we explicitly include only the *idealized* normal dipole and quadrupole fields. $\Delta\vec{B}(x,y,s)$ represents additional idealized fields, due, for example, to skew quadrupoles, sextupoles, or solenoids, or the deviations between the true magnetic field (including errors, misalignments, fringe fields, etc.) and the idealized fields.

Then

$$x'' = \frac{1}{\rho}\left(1+\frac{x}{\rho}\right) - \left(1+\frac{x}{\rho}\right)^2 \frac{e}{p}\left[B_0 + B'x + \Delta B_y(x,y,s)\right]$$
$$+ y' \frac{e\Delta B_s(x,y,s)}{p}\left(1+\frac{x}{\rho}\right)$$

$$y'' = \left(1+\frac{x}{\rho}\right)^2 \frac{e}{p}\left[B'y + \Delta B_x(x,y,s)\right] - x' \frac{e\Delta B_s(x,y,s)}{p}\left(1+\frac{x}{\rho}\right)$$

On the reference orbit: $x = x' = x'' = y = y' = y'' = \Delta\vec{B} = 0$

$$\frac{1}{\rho} = \frac{eB_0}{p_0}$$

Define

$k = \dfrac{eB'}{p_0} = \dfrac{B'}{B_0\rho}$, $k[\mathrm{m}^{-2}] = 0.2998\dfrac{B'[\mathrm{T/m}]}{p_0[\mathrm{GeV/c}]}$ = quadrupole strength;

Substitute:

$$x'' = \frac{1}{\rho} + \frac{x}{\rho^2} - \frac{p_0}{p}\left(1+\frac{x}{\rho}\right)^2\left(\frac{1}{\rho} + kx + \frac{\Delta B_y(x,y,s)}{B_0\rho}\right)$$
$$+y'\frac{p_0}{p}\frac{\Delta B_s(x,y,s)}{B_0\rho}\left(1+\frac{x}{\rho}\right)$$
$$y'' = \frac{p_0}{p}\left(1+\frac{x}{\rho}\right)^2\left(ky + \frac{\Delta B_x(x,y,s)}{B_0\rho}\right) - x'\frac{p_0}{p}\frac{\Delta B_s(x,y,s)}{B_0\rho}\left(1+\frac{x}{\rho}\right)$$

Expand, keeping only terms to second order in products of x, y, x', y', and $\Delta\vec{B}$

$$x'' = \frac{1}{\rho}\left(1-\frac{p_0}{p}\right) - x\frac{p_0}{p}\left(\left(2-\frac{p}{p_0}\right)\frac{1}{\rho^2} + k\right) - x^2\frac{p_0}{p}\left(\frac{2k}{\rho} + \frac{1}{\rho^3}\right)$$
$$-\frac{p_0}{p}\left(\frac{\Delta B_y(x,y,s)}{B_0\rho}\left(1+\frac{2x}{\rho}\right) - y'\frac{\Delta B_s(x,y,s)}{B_0\rho}\right)$$
$$y'' = y\frac{p_0}{p}k + xy\frac{p_0}{p}\left(\frac{2k}{\rho}\right) + \frac{p_0}{p}\left(\frac{\Delta B_x(x,y,s)}{B_0\rho}\left(1+\frac{2x}{\rho}\right) - x'\frac{\Delta B_s(x,y,s)}{B_0\rho}\right)$$

Introducing the relative momentum deviation $\delta = \frac{p-p_0}{p_0} << 1$,

we have, to second order in products of x, y, x', y', δ, and $\Delta\vec{B}$

$$x'' + x\left(k(1-\delta) + \frac{1-2\delta}{\rho^2}\right) = \frac{\delta(1-\delta)}{\rho} - \frac{\Delta B_y(x,y,s)}{B_0\rho}\left(1-\delta+\frac{2x}{\rho}\right)$$
$$+y'(1-\delta)\frac{\Delta B_s(x,y,s)}{B_0\rho} - x^2\left(\frac{2k}{\rho} + \frac{1}{\rho^3}\right) + \frac{y^2k}{2\rho}$$
$$y'' - yk(1-\delta) = \frac{\Delta B_x(x,y,s)}{B_0\rho}\left(1-\delta+\frac{2x}{\rho}\right) - x'(1-\delta)\frac{\Delta B_s(x,y,s)}{B_0\rho} + xy\left(\frac{2k}{\rho}\right)$$
$$l' = \left(1+\frac{x}{\rho}\right)$$

The linear terms in these equations, with $\Delta\vec{B} = 0$ form the basis for ideal linear optics. The nonlinear terms are generally treated as perturbations to the linear motion.

Note:

Trajectory equations are written in curvilinear coordinate system, but the field expansion is based on solution to LaPlace's equation in Cartesian coordinates.

We should really solve LaPlace's equation in the reference orbit coordinate system, and use that expansion for the fields. See Brown and Servranckx, "First and Second Order Charged Particle Optics", in "Physics of High Energy Accelerators", AIP #127 (1985) pp. 64-138.

Result: additional nonlinear term $\frac{y^2k}{2\rho}$ in the x'' equation above.

The effects of skew quadrupoles, sextupoles, and solenoids can be treated by including the appropriate idealized fields in $\Delta\vec{B}$.

Example: skew quadrupole

$\Delta B_y = -\tilde{B}'y; \quad \Delta B_x = \tilde{B}'x; \quad k = \frac{1}{\rho} = 0.$ Define

$\tilde{k} = \frac{e\tilde{B}'}{p_0} = \frac{\tilde{B}'}{B_0\rho}, \; \tilde{k}[\mathrm{m}^{-2}] = 0.2998\frac{\tilde{B}'[\mathrm{T/m}]}{p_0[\mathrm{GeV/c}]}$ = skew quadrupole strength

$$x'' = y\tilde{k}(1-\delta); \qquad y'' = x\tilde{k}(1-\delta)$$

The x and y motions are coupled

Example: solenoid

$$\Delta B_y = -\frac{1}{2}\frac{\partial B_s}{\partial s}y; \quad \Delta B_x = -\frac{1}{2}\frac{\partial B_s}{\partial s}x; \quad \Delta B_s = B_s; \quad k = \frac{1}{\rho} = 0.$$

Define

$r = \frac{eB_s}{p_0} = \frac{B_s}{B_0\rho}, \; r[\mathrm{m}^{-1}] = 0.2998\frac{B_s[\mathrm{T}]}{p_0[\mathrm{GeV/c}]}$ = solenoid strength

$$r' = \frac{e}{p_0}\frac{\partial B_s}{\partial s} = \frac{1}{B_0\rho}\frac{\partial B_s}{\partial s}, \; r'[\mathrm{m}^{-2}] = 0.2998\frac{\frac{\partial B_s}{\partial s}\left[\frac{\mathrm{T}}{\mathrm{m}}\right]}{p_0[\mathrm{GeV/c}]}$$

$$x'' = (1-\delta)\left(\frac{r'}{2}y + ry'\right); \quad y'' = -(1-\delta)\left(\frac{r'}{2}x + rx'\right)$$

Again, the x and y motions are coupled

(Note that the r' terms are only non-zero in the solenoid ends)

Example: sextupole

$$\Delta B_y = \frac{B''}{2}(x^2 - y^2); \quad \Delta B_x = B''xy; \quad k = \frac{1}{\rho} = 0.$$

Define

$m = \frac{eB''}{p_0} = \frac{B''}{B_0\rho}, \; m[\mathrm{m}^{-3}] = 0.2998\frac{B''[\mathrm{T/m}^2]}{p_0[\mathrm{GeV/c}]}$=sextupole strength

$$x'' = -\frac{m}{2}(x^2 - y^2)(1-\delta); \quad y'' = mxy(1-\delta)$$

The equations are nonlinear and coupled.

To start the general study of the trajectory equations:
Take the linear terms only, with $\Delta\vec{B}$=0.

$$x'' + x\left(k + \frac{1}{\rho^2}\right) = \frac{\delta}{\rho(s)}$$

$$y'' - yk = 0$$

These both have the general form

$\frac{d^2z}{ds^2} + K(s)z = F(s)$ with z=x or y, and

with $K_x(s) = k + \frac{1}{\rho^2}, \quad K_y(s) = -k, \; F_x(s) = \frac{\delta}{\rho(s)}, \; F_y(s) = 0$

In general, k and ρ depend on s. If K were constant, (as in a classical cyclotron) this is the equation for simple harmonic motion.

Solution:

$$z(s) = \begin{cases} z_0 \cos\left(s\sqrt{K}\right) + \frac{z_0'}{\sqrt{K}} \sin\left(s\sqrt{K}\right) & if \quad K > 0 \\ z_0 \cosh\left(s\sqrt{|K|}\right) + \frac{z_0'}{\sqrt{|K|}} \sinh\left(s\sqrt{|K|}\right) & if \quad K < 0 \end{cases}$$

Oscillatory (stable) if K>0; unstable if K<0.

For $K>0$, magnitude of K measures the strength of the focusing.

In the x-direction:

The $1/\rho^2$ term is the radial "weak focusing" provided by a uniform dipole field. In terms of the field index n, the quadrupole strength provided by a non-uniform dipole field

$$B_y(x) = B_0\left(\frac{\rho}{\rho + x}\right)^n \text{ is}$$

$$k = \frac{1}{B_0\rho}\frac{\partial B_y}{\partial x}\bigg|_{x=0} = \frac{-n}{\rho^2}$$

$$\text{So } K = -\frac{n}{\rho^2} + \frac{1}{\rho^2} > 0 \Rightarrow n < 1$$

$$\text{For y, } K = \frac{n}{\rho^2} > 0 \Rightarrow n > 0$$

Stability in both planes requires $0 < n < 1$ (weak focusing)

Some typical numbers for high-energy accelerators:

CESR: B_0 = 0.18 T; p_0 = 5.2 GeV; => ρ = 96.4 m; $B\rho$=17.3 T-m

Weak focusing strength $1/\rho^2 = 10^{-4}$ m^{-2}.

Typical CESR quadrupole:

$$B' = 5 \ \frac{\text{T}}{\text{m}}, \ L = 0.5 \ \text{m}; \ k = \frac{B'}{(B\rho)} = 0.289 \ \text{m}^{-2}$$

Tevatron: B_0 = 4.4 T; p_0 = 1000 GeV; =>ρ = 758 m; $B\rho$=3335 T-m

Weak focusing strength $1/\rho^2 = 1.7\text{x}10^{-6}$ m^{-2}.

Typical Tevatron quadrupole:

$$B' = 76 \ \frac{\text{T}}{\text{m}}, \ L = 1.7 \ \text{m}; \ k = \frac{B'}{(B\rho)} = 0.0228 \ \text{m}^{-2}$$