LECTURE 4

Piecewise matrix solutions to the linear trajectory equations

To understand strong focusing, we must consider what happens when K varies with s.

General linear trajectory equation

$$\frac{d^2z}{ds^2} + K(s)z = F(s)$$

A second order, inhomogeneous differential equation

General solution:

Let s_0 be the initial position. Define solutions to the homogeneous equation

$$C''(s,s_0) + K(s)C(s,s_0) = 0,\ \ C(s_0,s_0) = 1,\ \ C'(s_0,s_0) = 0$$

$$S''(s,s_0) + K(s)S(s,s_0) = 0,\ \ S(s_0,s_0) = 0,\ \ S'(s_0,s_0) = 1$$

$C(s,s_0)$ is called the "cosinelike" principal trajectory, and $S(s,s_0)$ is the "sinelike" principal trajectory.

Suppose K is constant and $K>0$: then we have

$$C(s,s_0) = \cos\left((s-s_0)\sqrt{K}\right),\quad S(s,s_0) = \frac{1}{\sqrt{K}}\sin\left((s-s_0)\sqrt{K}\right),$$

Hence the names.

Properties of C and S:

$$C'' + KC = 0 \quad \times\ S$$

$$\underline{S'' + KS = 0 \quad \times(-C)}$$

$$C''S - S''C = -\frac{d}{ds}(CS' - C'S) = 0$$

$$CS' - C'S = \text{constant} = 1$$

Back to the inhomogeneous equation:

$$\frac{d^2z}{ds^2} + K(s)z = F(s)$$

$$z'' + K(s)z = F(s) \times (-C)$$

$$\underline{C'' + K(s)C = 0} \quad \times z$$

Add:

$$C''z - Cz'' = \frac{d}{ds}(C'z - Cz') = -CF(s)$$

$$C'(s,s_0)z - C(s,s_0)z' + z'(s_0) = -\int_{s_0}^{s} C(t,s_0)F(t)dt$$

Same, but with S in place of C:

$$S'(s,s_0)z - S(s,s_0)z' - z(s_0) = -\int_{s_0}^{s} S(t,s_0)F(t)dt$$

Multiply first equation by -S, second by C, and add:

$$-C'Sz + Cz'S - z'(s_0)S = S\int_{s_0}^{s} C(t,s_0)F(t)dt$$

$$CS'z - CSz' - Cz(s_0) = -C\int_{s_0}^{s} S(t,s_0)F(t)dt$$

$$z(CS' - C'S) =$$

$$z(s,s_0) = C(s,s_0)z(s_0) + S(s,s_0)z'(s_0)$$

$$+S(s,s_0)\int_{s_0}^{s} C(t,s_0)F(t)dt - C(s,s_0)\int_{s_0}^{s} S(t,s_0)F(t)dt$$

This is the general solution. Applying it to our trajectory equations, we have

$$x(s,s_0) = C_x(s,s_0)x(s_0) + S_x(s,s_0)x'(s_0)$$

$$+\delta\left\{S_x(s,s_0)\int_{s_0}^{s}\frac{C_x(t,s_0)}{\rho(t)}dt - C_x(s,s_0)\int_{s_0}^{s}\frac{S_x(t,s_0)}{\rho(t)}dt\right\}$$

$$= C_x(s,s_0)x(s_0) + S_x(s,s_0)x'(s_0) + \delta D_x(s,s_0)$$

where

$$D_x(s,s_0) = S_x(s,s_0)\int_{s_0}^{s}\frac{C_x(t,s_0)}{\rho(t)}dt - C_x(s,s_0)\int_{s_0}^{s}\frac{S_x(t,s_0)}{\rho(t)}dt$$

is called the dispersion trajectory. In the other plane

$$y(s,s_0) = C_y(s,s_0)y(s_0) + S_y(s,s_0)y'(s_0)$$

Path-length equation:

$$\frac{dl}{ds} = 1 + \frac{x}{\rho}$$

$$\delta_l(s,s_0) = l(s) - l(s_0) - (s - s_0) = \int_{s_0}^{s}\frac{x(t,s_0)}{\rho(t)}dt =$$

$$x(s_0)\int_{s_0}^{s}\frac{C_x(t,s_0)}{\rho(t)}dt + x'(s_0)\int_{s_0}^{s}\frac{S_x(t,s_0)}{\rho(t)}dt + \delta\int_{s_0}^{s}\frac{D_x(t,s_0)}{\rho(t)}dt$$

$$= x(s_0)M_{51}(s,s_0) + x'(s_0)M_{52}(s,s_0) + \delta M_{56}(s,s_0)$$

These results are all combined in the following <u>matrix equation</u>:

$$
\begin{pmatrix} x(s,s_0) \\ x'(s,s_0) \\ y(s,s_0) \\ y'(s,s_0) \\ \delta_l(s,s_0) \\ \delta \end{pmatrix} = \vec{z}(s,s_0) =
$$

$$
\begin{pmatrix}
C_x(s,s_0) & S_x(s,s_0) & 0 & 0 & 0 & D_x(s,s_0) \\
C'_x(s,s_0) & S'_x(s,s_0) & 0 & 0 & 0 & D'_x(s,s_0) \\
0 & 0 & C_y(s,s_0) & S_y(s,s_0) & 0 & 0 \\
0 & 0 & C'_y(s,s_0) & S'_y(s,s_0) & 0 & 0 \\
M_{51}(s,s_0) & M_{52}(s,s_0) & 0 & 0 & 1 & M_{56}(s,s_0) \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix} x(s_0) \\ x'(s_0) \\ y(s_0) \\ y'(s_0) \\ \delta_l(s_0) \\ \delta \end{pmatrix}
$$

This matrix is sometimes called the transfer matrix It depends only on *K(s)* between *s* and s_0. It can also accommodate coupling, using the off-diagonal terms.

If we're only interested in, for example, x motion, and $\delta = 0$, we have

$$
x(s,s_0) = x(s_0)C_x(s,s_0) + x'(s_0)S_x(s,s_0)
$$

$$
x'(s,s_0) = x(s_0)C'_x(s,s_0) + x'(s_0)S'_x(s,s_0)
$$

$$
\vec{x}(s,s_0) = \begin{pmatrix} x(s,s_0) \\ x'(s,s_0) \end{pmatrix}
$$

$$
\Rightarrow = \begin{pmatrix} C_x(s,s_0) & S_x(s,s_0) \\ C'_x(s,s_0) & S'_x(s,s_0) \end{pmatrix} \vec{x}(s_0) = \mathbf{M}(s,s_0)\vec{x}(s_0)
$$

Notes:

1. Focusing: $\dfrac{dx'}{dx_0} = C' = \mathbf{M}_{21} \begin{cases} < 0 \Rightarrow \text{focusing} \\ > 0 \Rightarrow \text{defocusing} \end{cases}.$

Focal length $f = -\dfrac{dx_0}{dx'} = -\dfrac{1}{C'} = -\dfrac{1}{\mathbf{M}_{21}}$

2. $CS' - C'S = \begin{vmatrix} C & S \\ C' & S' \end{vmatrix} = \text{Det } \mathbf{M} = 1$

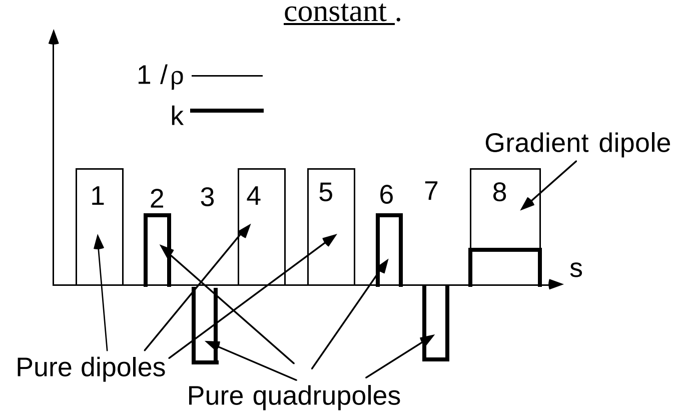
3. If $K(s)$ is constant, then, for any Δ,

$$\mathbf{M}(s+\Delta, s_0+\Delta) = \mathbf{M}(s,s_0)$$

4. If $K(s)$ is periodic with period L (i.e., $K(s+L) = K(s)$), then $\mathbf{M}(s+nL, s_0+nL) = \mathbf{M}(s,s_0)$, n=1,2,3…

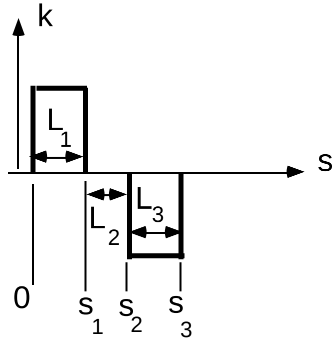
<u>Evaluation of C and S for idealized fields</u>

For the idealized fields, although K varies with s, it is <u>piecewise constant</u> .



The trajectory of a particle through this array of magnets can be calculated by finding the matrix $\mathbf{M}_i$ for each of the i magnets, and multiplying the matrices together.

Example:



$$\vec{z}_1 = \mathbf{M}_1(s_1,0)\vec{z}_0$$
$$\vec{z}_2 = \mathbf{M}_2(s_2,s_1)\vec{z}_1 = \mathbf{M}_2(s_2,s_1)\mathbf{M}_1(s_1,0)\vec{z}_0$$
$$= \mathbf{M}_2(s_2 - s_1,0)\mathbf{M}_1(s_1,0)\vec{z}_0 = \mathbf{M}_2(L_2,0)\mathbf{M}_1(L_1,0)\vec{z}_0$$
$$\vec{z}_3 = \mathbf{M}_3(s_3,s_2)\vec{z}_2 = \mathbf{M}_3(L_3,0)\mathbf{M}_2(L_2,0)\mathbf{M}_1(L_1,0)\vec{z}_0 = \mathbf{M}(s_3,0)\vec{z}_0$$
$$\mathbf{M}(s_3,0) = \mathbf{M}_3(L_3,0)\mathbf{M}_2(L_2,0)\mathbf{M}_1(L_1,0)$$

So we just need to find the **M** matrices for the individual magnets. We really just need the gradient dipole, since a pure dipole and a pure quadrupole are special cases of a gradient dipole.

In a gradient dipole of length L:

Let $K_x = k + \dfrac{1}{\rho^2}$

$$\begin{matrix} C_x'' + K_x C_x = 0 \\ S_x'' + K_x S_x = 0 \end{matrix};\quad \begin{matrix} C_x(0,0) = 1;\ C_x'(0,0) = 0; \\ S_x(0,0) = 0;\ S_x'(0,0) = 1; \end{matrix} \quad =>$$

$$C_x(L,0) = \cos\left(L\sqrt{K_x}\right),\ S_x(L,0) = \frac{1}{\sqrt{K_x}}\sin\left(L\sqrt{K_x}\right),$$

$$\begin{matrix} C_y'' - kC_y = 0 \\ S_y'' - kS_y = 0 \end{matrix};\quad \begin{matrix} C_y(0,0) = 1;\ C_y'(0,0) = 0; \\ S_y(0,0) = 0;\ S_y'(0,0) = 1; \end{matrix} \quad =>$$

$$C_y(L,0) = \cosh\left(L\sqrt{k}\right),\ S_y(L,0) = \frac{1}{\sqrt{k}}\sinh\left(L\sqrt{k}\right)$$

Then

$$D_x(L,0) = \frac{1-\cos\left(L\sqrt{K_x}\right)}{\rho K_x};\quad D_x'(L,0) = \frac{\sin\left(L\sqrt{K_x}\right)}{\rho\sqrt{K_x}}$$
$$M_{51}(L,0) = \frac{\sin\left(L\sqrt{K_x}\right)}{\rho\sqrt{K_x}};\quad M_{52}(L,0) = \frac{1-\cos\left(L\sqrt{K_x}\right)}{\rho K_x};$$
$$M_{56}(L,0) = \frac{1}{\rho^2 K_x}\left[L - \frac{\sin\left(L\sqrt{K_x}\right)}{\sqrt{K_x}}\right]$$

Transfer matrices for drift spaces, pure dipoles and quadrupoles:

Simplest case: Gradient dipole with 1/ρ–>0, k–>0 =>

Drift space, length L:

$$\mathbf{M}(L,0)=\begin{pmatrix}1 & L & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & L & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}$$

Gradient dipole with k->0=>

pure "sector" dipole, length L_D:

Let $\phi = \dfrac{L_D}{\rho}$ = dipole bend angle

Transfer matrix:

$$\mathbf{M}(L_D,0)=\begin{pmatrix}\cos\phi & \rho\sin\phi & 0 & 0 & 0 & \rho(1-\cos\phi)\\ -\frac{1}{\rho}\sin\phi & \cos\phi & 0 & 0 & 0 & \sin\phi\\ 0 & 0 & 1 & \rho\phi & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ \sin\phi & \rho(1-\cos\phi) & 0 & 0 & 1 & \rho(\phi-\sin\phi)\\ 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}$$

Focal length in x is $f=\dfrac{\rho}{\sin\phi}$. If, as is usual for high-energy accelerators, $\phi \ll 1$, $\sin\phi \approx \phi = \dfrac{L_D}{\rho}$. This corresponds to a focal length $f \approx \dfrac{\rho^2}{L_D} \gg L_D$.

In this approximation, the transfer matrix is

$$\mathbf{M}(L_D,0)=\begin{pmatrix}1 & L_D & 0 & 0 & 0 & \frac{L_D}{2}\sqrt{\frac{L_D}{f}}\\ -\frac{1}{f} & 1 & 0 & 0 & 0 & \sqrt{\frac{L_D}{f}}\\ 0 & 0 & 1 & L_D & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ \sqrt{\frac{L_D}{f}} & \frac{L_D}{2}\sqrt{\frac{L_D}{f}} & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}$$

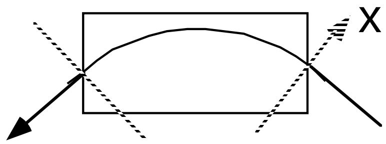
Except for very weak focusing in x, the sector dipole looks like a drift section of length L_D. The lowest order dispersive effects are a change in the trajectory's angle by the bend angle times times the relative momentum deviation:

$$\Delta x' \approx \delta\sqrt{\frac{L_D}{f}} = \delta\frac{L_D}{\rho} = \delta\phi$$

and a change in the trajectory's position by the bend angle, time the relative momentum deviation, times half the magnet length:

$$\Delta x \approx \delta\frac{L_D}{2}\sqrt{\frac{L_D}{f}} = \delta\frac{L_D}{2}\phi$$

A rectangular dipole



has an effective extra field length at the ends for negative x, absence of field for positive x, compared to a wedge dipole. The

net effect of this is to make it weakly focusing in y, neutral in x.

Gradient dipole with 1/ρ->0=>

pure quadrupole, length L_Q.

Let $\phi = L_Q\sqrt{k}$, for k>0

Transfer matrix

$$\mathbf{M}(L_Q,0) = \begin{pmatrix} \cos\phi & \frac{1}{\sqrt{k}}\sin\phi & 0 & 0 & 0 & 0 \\ -\sqrt{k}\sin\phi & \cos\phi & 0 & 0 & 0 & 0 \\ 0 & 0 & \cosh\phi & \frac{1}{\sqrt{k}}\sinh\phi & 0 & 0 \\ 0 & 0 & \sqrt{k}\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Focal length in x is $f = \frac{1}{\sqrt{k}\sin\phi}$

For $k<0$: interchange x and y submatrices, replace k->$|k|$

Thin lens approximation:

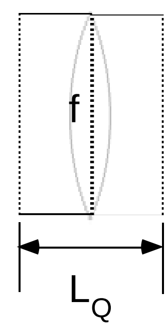
If $\phi << 1$, $\sin\phi \approx \phi = L_Q\sqrt{k}$

This corresponds to a focal length $f \approx \frac{1}{kL_Q} >> L_Q$

Transport matrix (x-y part only) in the limit L->0 while kL_q remains finite:

$$\mathbf{M}_{thin\ \ lens} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{f} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{f} & 1 \end{pmatrix}, \quad f = \frac{1}{kL_Q}$$

A physical lens of length L_Q is formed by sandwiching the zero-length thin-lens between two drift spaces of length $L_Q/2$.

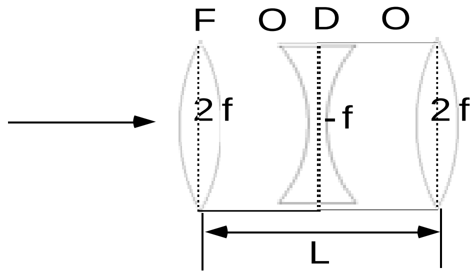


The transport matrix for a physical lens of length L_Q, in the thin-lens approximation, is the matrix product:

$$\mathbf{M}(L_Q,0)=\begin{pmatrix}1 & \frac{L_Q}{2} & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & \frac{L_Q}{2}\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\\ -\frac{1}{f} & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & \frac{1}{f} & 1\end{pmatrix}\begin{pmatrix}1 & \frac{L_Q}{2} & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & \frac{L_Q}{2}\\ 0 & 0 & 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}1-\frac{L_Q}{2f} & L_Q-\frac{L_Q^2}{4f} & 0 & 0\\ -\frac{1}{f} & 1-\frac{L_Q}{2f} & 0 & 0\\ 0 & 0 & 1+\frac{L_Q}{2f} & L_Q+\frac{L_Q^2}{4f}\\ 0 & 0 & \frac{1}{f} & 1+\frac{L_Q}{2f}\end{pmatrix}\approx\begin{pmatrix}1 & L_Q & 0 & 0\\ -\frac{1}{f} & 1 & 0 & 0\\ 0 & 0 & 1 & L_Q\\ 0 & 0 & \frac{1}{f} & 1\end{pmatrix}$$

A symmetric *FODO cell* is formed from a two half-strength focusing lenses and a full-strength defocusing lens, separated by a drift space of length L/2:



L is called the cell length. Transfer matrix:

$$\mathbf{M}(L,0)=\begin{pmatrix}1 & 0 & 0 & 0\\ -\frac{1}{2f} & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & \frac{1}{2f} & 1\end{pmatrix}\begin{pmatrix}1 & L/2 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & L/2\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\\ \frac{1}{f} & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & -\frac{1}{f} & 1\end{pmatrix}$$

$$\times\begin{pmatrix}1 & L/2 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & L/2\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\\ -\frac{1}{2f} & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & \frac{1}{2f} & 1\end{pmatrix}$$

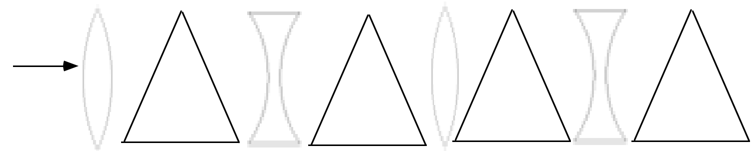
$$\mathbf{M}(L,0)=\begin{pmatrix}1-\frac{L^2}{8f^2} & L+\frac{L^2}{4f} & 0 & 0\\ -\frac{L}{4f^2}\left(1-\frac{L}{4f}\right) & 1-\frac{L^2}{8f^2} & 0 & 0\\ 0 & 0 & 1-\frac{L^2}{8f^2} & L-\frac{L^2}{4f}\\ 0 & 0 & -\frac{L}{4f^2}\left(1+\frac{L}{4f}\right) & 1-\frac{L^2}{8f^2}\end{pmatrix}$$

Net focal length in both planes $\sim\frac{4f^2}{L}$: **alternating gradient focusing**

The FODO cell is the basic unit used to provide strong focusing, or alternating gradient focusing, in most modern accelerators.

The arrangement of idealized fields in an accelerator is called the *magnetic lattice*. A FODO lattice is simply a collection of FODO cells.

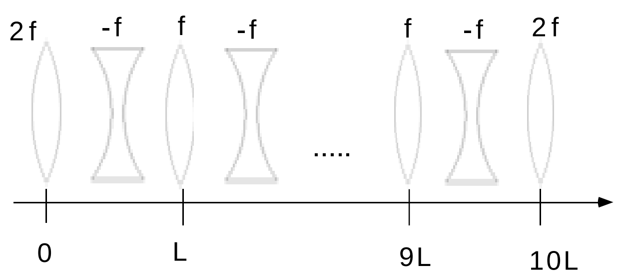
In a circular accelerator, the O's (drift spaces) in the FODO cells are usually occupied by bending magnets (which have very little focusing strength, and are almost equivalent to drift spaces).



Linear particle transport through each element corresponds to multiplication by the appropriate **M** matrix. The alternating gradient provides strong focusing in both planes.

Passage through the whole lattice once corresponds to multiplication by the product matrix $\mathbf{M} = \prod_{i=1}^{n} \mathbf{M}_i$, where the product is over all n magnetic elements in the machine.

Example 1:



Symmetric FODO lattice, no bends. 10 cells L=1 m. Lens focal length *f*. Let $\mathbf{M}_c(f,s,0)$ = transfer matrix from 0 to *s* for symmetric FODO cell of focal length *f*.

Then: overall transfer matrix at $s=nL+\Delta s$

$$\mathbf{M}(s,0) = \mathbf{M}(nL+\Delta s,0)$$
$$\mathbf{M}_c(f,nL+\Delta s,nL)\mathbf{M}_c(f,nL,(n-1)L)\ldots\mathbf{M}_c(f,2L,L)\mathbf{M}_c(f,L,0)$$
$$= \mathbf{M}_c(f,\Delta s,0)\left[\mathbf{M}_c(f,L,0)\right]^n$$

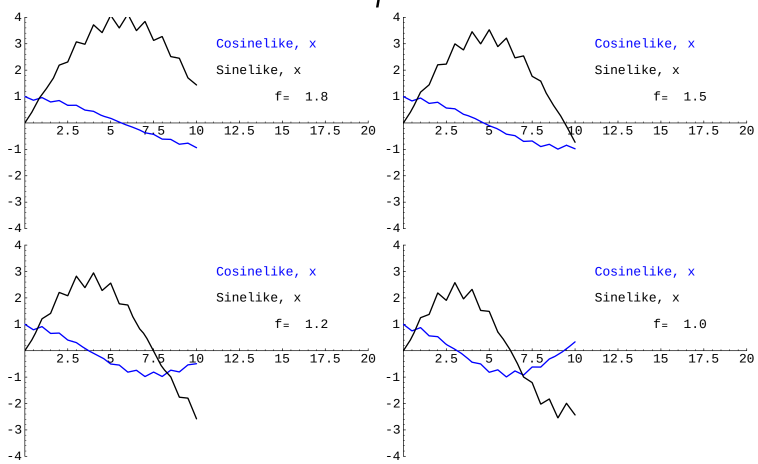
because of the periodicity of $\mathbf{M_c}$

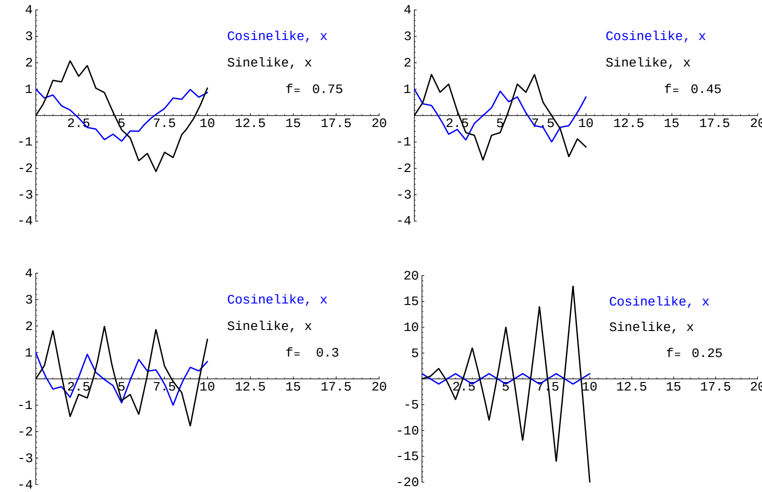
The principal trajectories are

$$C_x(s,0) = \mathbf{M}_{11}(s,0);\ S_x(s,0) = \mathbf{M}_{12}(s,0);$$
$$C_y(s,0) = \mathbf{M}_{33}(s,0);\ S_y(s,0) = \mathbf{M}_{34}(s,0)$$

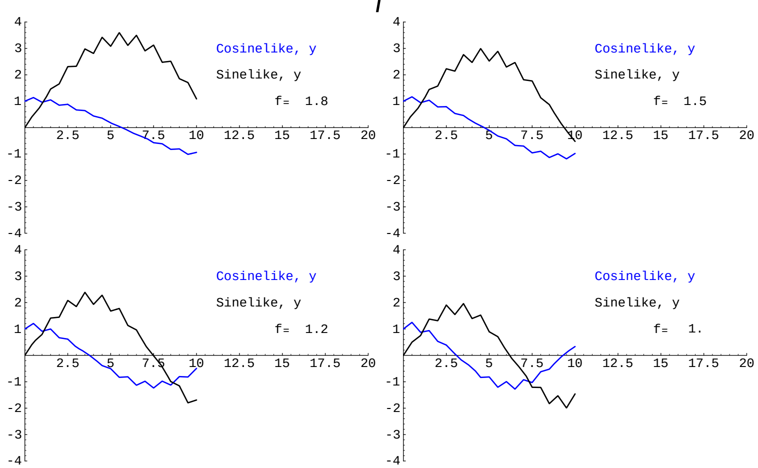
In the following figures, *f* is in m, and the sinelike trajectories are in m.

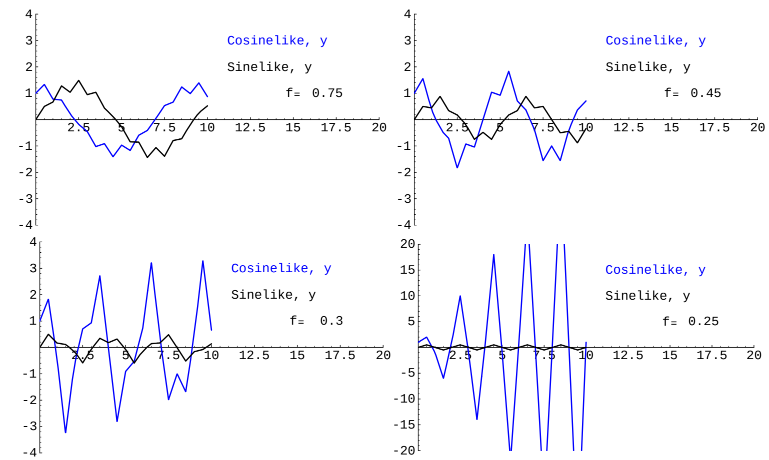
Cosinelike and sinelike *x* trajectories vs *s*(m), for various values of *f*





Cosinelike and sinelike *y* trajectories vs *s*(m), for various values of *f*





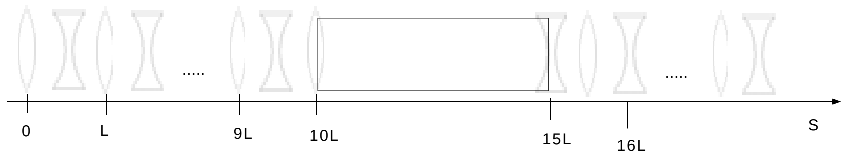
Note:

1. Changing f changes the wavelength of the trajectory oscillation, and also changes the amplitude.

2. Decreasing f makes the wavelength shorter

3. Decreasing f makes the amplitude smaller, until we get to very small f, when the amplitude blows up (trajectory increases exponentially)

Example 2:



10 cell Symmetric FODO lattice focal length f, 5 m bend with ϕ=0.1, 10 cell symmetric FODO lattice, focal length $-f$. L=1 m.

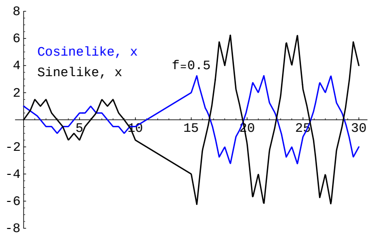
Then: overall transfer matrix is again obtained by taking appropriate matrix products.

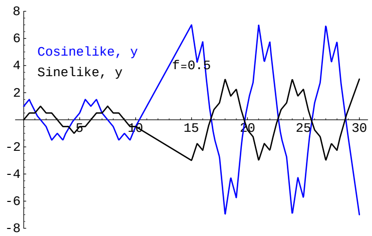
The principal trajectories are

$$C_x(s,0) = \mathbf{M}_{11}(s,0);\ S_x(s,0) = \mathbf{M}_{12}(s,0);$$

$$C_y(s,0) = \mathbf{M}_{33}(s,0);\ S_y(s,0) = \mathbf{M}_{34}(s,0)$$

Trajectories vs. s(m), for f=0.5 m; sinelike trajectory in m.





The dispersive trajectory is

$$D_x(s,0) = \mathbf{M}_{16}(s,0)$$

Dispersive trajectory (m). vs s(m)