LECTURE 5

<u>Periodic systems</u>
<u>Twiss parameters and stability</u>
<u>Hill's equation and its solution</u>
<u>Courant-Snyder invariant and emittance</u>

<u>Periodic systems</u>

Everything we have done up to this point can be applied to beam transfer lines, linacs, or circular accelerators. We now specialize to circular accelerators, which are <u>periodic systems</u> with period C, where C=circumference=length of the (closed) reference orbit.

Consider only (x,x') motion for the moment (or only (y, y'))- just two dimensions. If I start at the point s_0 on the reference orbit, then, after one turn,

$$\vec{x}(s_0+C,s_0)=\mathbf{M}(s_0+C,s_0)\vec{x}(s_0)$$

where C is the circumference (length of the closed reference orbit).

The matrix $\mathbf{M}(s+C,s)$ is called the <u>one-turn transfer matrix</u> at the point s.

Important properties of this matrix:

1. $\mathbf{M}$ is periodic in s with period C:

$$\mathbf{M}(s+C,s)=\mathbf{M}(s,s-C)$$

2. Det $\mathbf{M}(s+C,s)=1$

3. Trace $\mathbf{M}(s+C,s)=\mathbf{M}_{11}+\mathbf{M}_{22}$ is independent of s.

Why?

<u>Theorem</u>: The trace of a matrix product is invariant under a permutation of the matrices.

$$\mathbf{M}(s+C,s)=$$
$$\mathbf{M}(s+C,s+C-\delta s)\mathbf{M}(s+C-\delta s,s+C-2\delta s)\times$$
$$...\mathbf{M}(s+2\delta s,s+\delta s)\mathbf{M}(s+\delta s,s)$$

$$\mathbf{M}(s+C-\delta s,s-\delta s)=$$
$$\mathbf{M}(s+C-\delta s,s+C-2\delta s)...\mathbf{M}(s+\delta s,s)\mathbf{M}(s,s-\delta s)$$
$$=\mathbf{M}(s+C-\delta s,s+C-2\delta s)...\mathbf{M}(s+\delta s,s)\mathbf{M}(s+C,s+C-\delta s)$$

$\mathbf{M}(s+C-\delta s,s-\delta s)$ is related to $\mathbf{M}(s+C,s)$ by a permutation of the matrices in the matrix product.

So the trace of $\mathbf{M}$ is independent of s.

Taking advantage of these properties, we write the 2x2 total one-turn matrix **M(**s+C,s**)** as the sum of a constant matrix plus a traceless matrix, periodic in s with period C.

$$\mathbf{M}(s+C,s) = A\mathbf{I} + B\mathbf{J}(s)$$

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \mathbf{J}(s) = \begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix}$$

Here *A* and *B* are constants, and $\alpha(s)$, $\beta(s)$, and $\gamma(s)$ are functions of *s*, periodic in *s* with period *C*. Since Det **M**=1, we have

$$A^2 + B^2\left(-\alpha(s)^2 + \beta(s)\gamma(s)\right) = 1$$

The *s*-dependent coefficient of B^2 must be a constant. Since $\alpha(s)$, $\beta(s)$, and $\gamma(s)$ are arbitrary functions of s, we can choose them so that

$$-\alpha(s)^2 + \beta(s)\gamma(s) = \text{constant} = 1$$

Then

$$A^2 + B^2 = 1$$

Let $A = \cos\mu, \quad B = \sin\mu$: (note that μ could be imaginary), we have

$$\mathbf{M}(s+C,s) = \mathbf{I}\cos\mu + \mathbf{J}(s)\sin\mu =$$

$$\begin{pmatrix} \cos\mu + \alpha(s)\sin\mu & \beta(s)\sin\mu \\ -\gamma(s)\sin\mu & \cos\mu - \alpha(s)\sin\mu \end{pmatrix}$$

When **M** is written in this form, it is called the "Twiss matrix"; The Twiss parameters α, β, and γ are periodic functions of s, with period C, related by $-\alpha^2 + \beta\gamma = 1$. Because these functions completely describe the properties of the magnetic lattice, they are also called *lattice functions*.

An additional restriction on the constant μ comes from the requirement of *stability*.

After *n* turns in the accelerator, we have

$$\vec{x}(s+nC,s) =$$

$$\mathbf{M}(s_0+nC, s_0+(n-1)C)\ldots\mathbf{M}(s_0+2C, s_0+C)\mathbf{M}(s_0+C, s_0)\vec{x}(s_0)$$

$$= \left[\mathbf{M}(s_0+C,s_0)\right]^n \vec{x}(s_0)$$

Stability requires that all the elements of $\left[\mathbf{M}(s_0+C,s_0)\right]^n$ remain finite as $n \to \infty$.

Note that

$$\mathbf{J}^2(s) = \begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix}\begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathbf{I}$$

J is the matrix equivalent of $i = \sqrt{-1}$.

Hence we can write

$$\mathbf{M}(s+C,s) = \mathbf{I}\cos\mu + \mathbf{J}(s)\sin\mu = \exp[\mathbf{J}(s)\mu]$$

So

$$[\mathbf{M}(s+C,s)]^n = \exp[\mathbf{J}(s)n\mu] = \mathbf{I}\cos n\mu + \mathbf{J}\sin n\mu$$

For the matrix elements to be finite as $n \to \infty$ requires that μ is real.

This also implies that

$$|\text{Trace } \mathbf{M}(s+C,s)| = |\mathbf{M}_{11} + \mathbf{M}_{22}| = |2\cos\mu| < 2$$

The condition

$$|\text{Trace } \mathbf{M}(s+C,s)| < 2$$

is a general condition for the stability of trajectories in any periodic system.

Example 3:

Suppose we make a circular accelerator out of a collection of *m* identical symmetric FODO cells. The one-turn matrix is the product of *m* identical matrices, each of the form

$$\mathbf{M}_c(L,0) = \begin{pmatrix} 1 - \frac{L^2}{8f^2} & L + \frac{L^2}{4f} \\ -\frac{L}{4f^2}\left(1 - \frac{L}{4f}\right) & 1 - \frac{L^2}{8f^2} \end{pmatrix}$$

The one-turn matrix for *m* FODO cells of length *L* is

$$\mathbf{M}(C+s,s) = \mathbf{M}_c(C+s,C)\mathbf{M}_c(C,C-L)...\mathbf{M}_c(2L,L)\mathbf{M}_c(L,s)$$

$$= \mathbf{M}_c(s,0)[\mathbf{M}_c(L,0)]^{m-1}\mathbf{M}_c(L,s)$$

where $C=mL$.

This system is periodic in s with period$L$.

For *n* turns, the stability argument applies to the FODO cell matrix and leads to the requirement that

$$|\text{Trace}(\mathbf{M}_c(L,0))| = \left|2 - \frac{L^2}{4f^2}\right| < 2 \Rightarrow \left|\frac{L}{4f}\right| < 1$$

The values of the Twiss parameters at the beginning of the FODO cell (at the F quad) can be found from

$$\mathbf{M}_c(L,0) = \begin{pmatrix} 1 - \frac{L^2}{8f^2} & L + \frac{L^2}{4f} \\ -\frac{L}{4f^2}\left(1 - \frac{L}{4f}\right) & 1 - \frac{L^2}{8f^2} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\mu + \alpha(0)\sin\mu & \beta(0)\sin\mu \\ -\gamma(0)\sin\mu & \cos\mu - \alpha(0)\sin\mu \end{pmatrix}$$

from which we find

$$\sin\frac{\mu}{2} = \frac{L}{4f}; \quad \beta(0) = \frac{L\left(1 + \sin\frac{\mu}{2}\right)}{\sin\mu};$$

$$\alpha(0) = 0$$

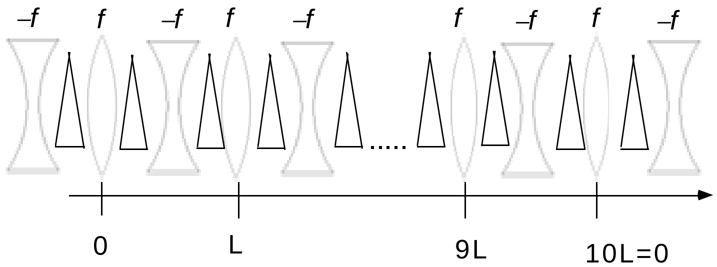
We can find the Twiss parameters at a general point s within the cell from

$$\mathbf{M}_c(L+s,s) = \mathbf{M}_c(L+s,L)\mathbf{M}_c(L,s)$$

$$= \mathbf{M}_c(s,0)\mathbf{M}_c(L,0)\mathbf{M}_c^{-1}(s,0)$$

$$== \begin{pmatrix} \cos\mu + \alpha(s)\sin\mu & \beta(s)\sin\mu \\ -\gamma(s)\sin\mu & \cos\mu - \alpha(s)\sin\mu \end{pmatrix}$$

Numerical results:
Bend example 1 around into a circle with bending magnets in FODO drifts.



Circular symmetric FODO lattice C=10L.
L=1 m. Lens focal length f=0.45 m.

For this FODO cell we have

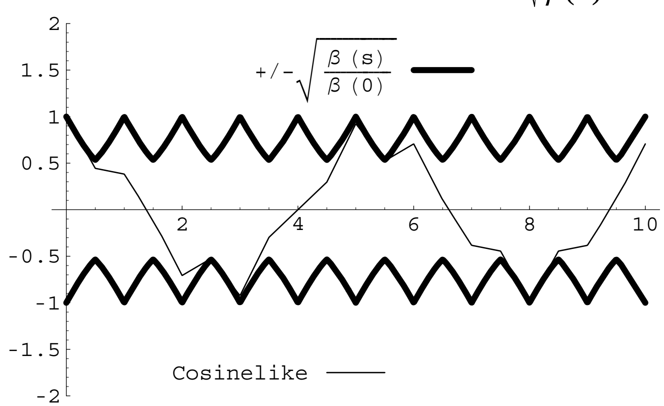
$$\sin\frac{\mu}{2}=\frac{L}{4f}=\frac{1}{4\times 0.45}=0.5555$$

$$\mu=1.178\ (67.5^o)$$

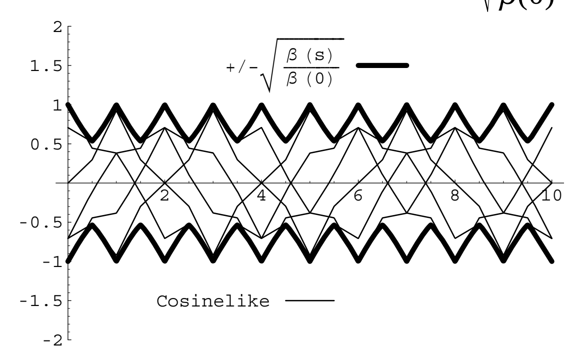
$$\beta(0)=\frac{L\left(1+\sin\frac{\mu}{2}\right)}{\sin\mu}=\frac{1(1+0.555)}{0.924}=1.68\ \text{ m}$$

We can calculate the trajectories and Twiss parameter β through the accelerator
In the following figures, the sinelike trajectories and β are in m.

Single-pass cosinelike trajectory and $\sqrt{\frac{\beta(s)}{\beta(0)}}$

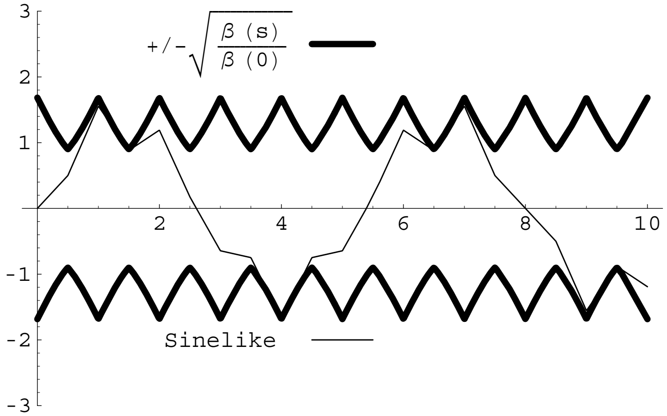


Multi-pass cosinelike trajectory and $\sqrt{\frac{\beta(s)}{\beta(0)}}$

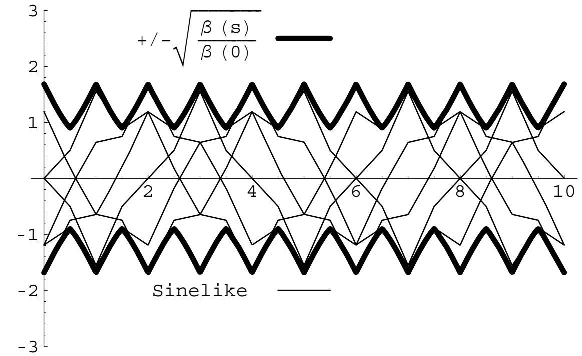


$\sqrt{\frac{\beta(s)}{\beta(0)}}$ forms the envelope of the cosinelike trajectory

Single-pass sinelike trajectory and $\sqrt{\beta(s)\beta(0)}$



Multi-pass sinelike trajectory and $\sqrt{\beta(s)\beta(0)}$



$\sqrt{\beta(s)\beta(0)}$ forms the envelope of the sinelike trajectory

Hill's equation and its solution

We see from the above example that there is a very direct relation between the particle trajectories and the Twiss parameters.

To explore this relation further, we return to the general homogeneous differential equation for the trajectories:

$$\frac{d^2 z}{ds^2} + K(s)z = 0$$

where *z*=*x* or *y*, and for a circular accelerator of circumference *C*, *K* is periodic in *s* with period *C*.

This type of differential equation is called *Hill's equation.*

Floquet's theorem (a result from the 19th century) states that the solution can be written in the form

$$z(s) = af(s)\cos(\Phi(s)+\delta)$$

in which *a* and δ are arbitrary constants, *f(s)* is a function which has the same periodicity as that of *K* (i.e, periodic in s with period *C*).

For a trajectory in an accelerator, $\cos(\Phi(s)+\delta)$ (and z(s)) should be *non-periodic*.

Differential equations for f and Φ can be obtained by requiring that z satisfy Hill's equation:

$$z' = a\left(\cos(\Phi+\delta)f' - f\sin(\Phi+\delta)\Phi'\right)$$

$$z'' + Kz = \cos(\Phi+\delta)\left(fK - f\Phi'^2 + f''\right) + \sin(\Phi+\delta)(-2f'\Phi' - f\Phi'') = 0$$

Coefficients of sine and cosine must be separately zero, since δ is arbitrary

From sine coefficient (times f): integral equation for Φ

$$f(2f'\Phi' + f\Phi'') = 2ff'\Phi' + f^2\Phi'' = \left(f^2\Phi'\right)' = 0$$

$$f^2(s)\Phi'(s) = f^2(s_0)\Phi'(s_0) + k_1 = k_2$$

in which k_1 and k_2 are constants. Absorb the constant k_2 into the arbitrary constant a which multiplies f:

$$\Phi' = \frac{1}{f^2} \Rightarrow \Phi(s) = \Phi(s_0) + \int_{s_0}^{s} \frac{dt}{f(t)^2} = \int_{s_0}^{s} \frac{dt}{f(t)^2}$$

where the last equation follows if we absorb $\Phi(s_0)$ into the arbitrary constant δ, which is added to Φ. Then s_0 is the location from which we measure the *phase advance* $\Phi(s)$.

From cosine coefficient: Differential equation for f:

$$fK - f\Phi'^2 + f'' = -\frac{1}{f^3} + fK + f'' = 0$$

Relation to the Twiss parameters:

From the Twiss matrix

$$\begin{pmatrix} z(s+C) \\ z'(s+C) \end{pmatrix} = \mathbf{M}(s+C,s)\begin{pmatrix} z(s) \\ z'(s) \end{pmatrix}$$

$$\mathbf{M}(s+C,s) = \begin{pmatrix} \cos\mu + \alpha(s)\sin\mu & \beta(s)\sin\mu \\ -\gamma(s)\sin\mu & \cos\mu - \alpha(s)\sin\mu \end{pmatrix}$$

so

$$z(s+C) = z(s)\left[\cos\mu + \alpha(s)\sin\mu\right] + z'(s)\beta(s)\sin\mu$$

Compare with solution to Hill's equation:

$$z(s+C) = af(s+C)\cos\left(\Phi(s+C) + \delta\right)$$

Now

$$\Phi(s+C) - \Phi(s) = \int_{s}^{s+C} \frac{dt}{f(t)^2} = \oint_C \frac{dt}{f(t)^2} = 2\pi Q$$

is a constant since $\Phi' = \frac{1}{f^2}$ is periodic in s with period C.

So

$$\begin{aligned} z(s+C) &= af(s)\cos\left(\Phi(s) + 2\pi Q + \delta\right) \\ &= af(s)\cos\left(\Phi(s)+\delta\right)\cos 2\pi Q - af(s)\sin\left(\Phi(s)+\delta\right)\sin 2\pi Q \\ &= z(s)\cos 2\pi Q - af(s)\sin\left(\Phi(s)+\delta\right)\sin 2\pi Q \end{aligned}$$

Use

$$z'(s) = -\frac{a}{f}\sin\left(\Phi(s)+\delta\right) + z(s)\frac{f'}{f}$$

to obtain

$$z(s+C) = z(s)[\cos 2\pi Q - f(s) f'(s) \sin 2\pi Q] + z'(s) f(s)^2 \sin 2\pi Q$$

Compare with

$$z(s+C) = z(s)[\cos \mu + \alpha(s) \sin \mu] + z'(s)\beta(s) \sin \mu$$

thus

$$\beta(s) = f(s)^2$$

$$\alpha(s) = -f(s) f'(s) = -\frac{\beta'(s)}{2}$$

$$\mu = 2\pi Q = \oint_C \frac{ds}{\beta(s)}$$

Interpretation:
The trajectory

$$z(s) = a\sqrt{\beta(s)} \cos(\Phi(s) + \delta)$$

is called a betatron oscillation. The amplitude is determined both by a (initial conditions) and by β (magnet lattice). As we saw earlier, the trajectory envelope varies like $\sqrt{\beta}$. But β also determines the wavelength of the trajectory's oscillation. For constant λ, the phase is $\Phi = \frac{2\pi s}{\lambda}$. For variable wavelength, this generalizes to $\Phi = 2\pi \int \frac{ds}{\lambda(s)} = \int \frac{ds}{\beta(s)}, \;\; \lambda(s) = 2\pi\beta(s)$

The function $\Phi = \int \frac{ds}{\beta(s)}$ is called the phase advance (for obvious reasons). The phase of the oscillating trajectory advances at different rates around the machine, changing fast when β is small, and slowly when β is large.

The total number of oscillations per turn is

$$\oint_C \frac{ds}{\lambda(s)} = \frac{1}{2\pi} \oint_C \frac{ds}{\beta(s)} = Q$$

which is called the tune of the accelerator. Note that, we have, roughly

$$Q = \frac{1}{2\pi} \oint_C \frac{ds}{\beta(s)} \approx \frac{C}{2\pi\langle\beta\rangle} = \frac{\langle R\rangle}{\langle\beta\rangle}$$

where $\langle R\rangle = \frac{C}{2\pi}$ mean radius of the accelerator.

Back to the numerical example: circular accelerator, 10 FODO cells, cell length L=1 m, focal length f=0.45 m.

For a single FODO cell, we found $\mu_C = \oint_C \frac{ds}{\beta(s)} = 1.178.$

For the whole machine

$$\mu = 10\mu_C = 11.78 = 2\pi Q$$

$$Q = 1.8748$$

And

$$\langle R\rangle = \frac{C}{2\pi} = \frac{10}{2\pi} \text{ m } = 1.59 \text{ m}$$

$$\langle\beta\rangle \approx \frac{\langle R\rangle}{Q} = \frac{1.59}{1.8748} \text{ m} = 0.85 \text{ m}$$

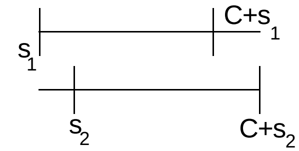
Exercises: 1.Using $$z = a\sqrt{\beta}\cos(\Phi+\delta)$$ $$z' = -\frac{a}{\sqrt{\beta}}(\alpha\cos(\Phi+\delta)+\sin(\Phi+\delta)),$$
write the trajectory equations for the cosinelike and sinelike trajectories in terms of α, β and Φ. Show that the transfer matrix from one point in the accelerator, s_0, to another, s, can be written in terms of β, α, and Φ, as

$$\mathbf{M}(s,s_0) = \begin{pmatrix} C(s,s_0) & S(s,s_0) \\ C'(s,s_0) & S'(s,s_0) \end{pmatrix} =$$

$$\begin{pmatrix} \sqrt{\frac{\beta(s)}{\beta(s_0)}}[\cos\Delta\Phi + \alpha(s_0)\sin\Delta\Phi] & \sqrt{\beta(s)\beta(s_0)}\sin\Delta\Phi \\ \sqrt{\frac{1}{\beta(s_0)\beta(s)}}\begin{bmatrix} (\alpha(s_0)-\alpha(s))\cos\Delta\Phi \\ -(1+\alpha(s)\alpha(s_0))\sin\Delta\Phi \end{bmatrix} & \sqrt{\frac{\beta(s_0)}{\beta(s)}}[\cos\Delta\Phi - \alpha(s)\sin\Delta\Phi] \end{pmatrix}$$

where $\Delta\Phi = \Phi(s) - \Phi(s_0)$

2. The one-turn matrix $\mathbf{M}(s_1+C,s_1) = \mathbf{I}\cos 2\pi Q + \mathbf{J}(s_1)\sin 2\pi Q$



propagates a particle around one turn starting at s_1. $\mathbf{M}(s_2+C,s_2)$ does the same starting at s_2. Let $\mathbf{M}(s_2,s_1) = \begin{pmatrix} C(s_2,s_1) & S(s_2,s_1) \\ C'(s_2,s_1) & S'(s_2,s_1) \end{pmatrix}$ be the transfer matrix from s_1 to s_2. Then we have

$$\mathbf{M}(s_2,s_1)\mathbf{M}(s_1+C,s_1) = \mathbf{M}(s_2+C,s_2)\mathbf{M}(s_2,s_1)$$

$$\mathbf{M}(s_2+C,s_2) = \mathbf{M}(s_2,s_1)\mathbf{M}(s_1+C,s_1)\mathbf{M}(s_2,s_1)^{-1}$$

So $\mathbf{J}(s_2) = \mathbf{M}(s_2,s_1)\mathbf{J}(s_1)\mathbf{M}(s_2,s_1)^{-1}$.

Use this matrix equation to show that

$$\begin{pmatrix} \beta(s_2) \\ \alpha(s_2) \\ \gamma(s_2) \end{pmatrix} =$$

$$\begin{pmatrix} [C(s_2,s_1)]^2 & -2C(s_2,s_1)S(s_2,s_1) & [S(s_2,s_1)]^2 \\ -C(s_2,s_1)C'(s_2,s_1) & C'(s_2,s_1)S(s_2,s_1)+S'(s_2,s_1)C(s_2,s_1) & -S(s_2,s_1)S'(s_2,s_1) \\ [C'(s_2,s_1)]^2 & -2C'(s_2,s_1)S'(s_2,s_1) & [S'(s_2,s_1)]^2 \end{pmatrix} \begin{pmatrix} \beta(s_1) \\ \alpha(s_1) \\ \gamma(s_1) \end{pmatrix}$$

This allows us to propagate the lattice functions from one point to another.

For example, in a drift space of length s, $C = 1,\ S = s,\ C' = 0, S' = 1$

$$\begin{pmatrix} \beta(s) \\ \alpha(s) \\ \gamma(s) \end{pmatrix} = \begin{pmatrix} 1 & -2s & s^2 \\ 0 & 1 & -s \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix} \Rightarrow \begin{matrix} \beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2 \\ \alpha(s) = \alpha_0 - \gamma_0 s \\ \gamma(s) = \gamma_0 \end{matrix}$$

If $\alpha_0 = 0,\ \beta(s) = \beta_0 + \frac{s^2}{\beta_0},\ \ \alpha(s) = -\frac{s}{\beta_0}$

In a thin lens $C=1$, $S=0$, $C'=\mp\frac{1}{f}, S'=1$

$$\begin{pmatrix}\beta_1\\ \alpha_1\\ \gamma_1\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ \pm\frac{1}{f} & 1 & 0\\ \frac{1}{f^2} & \pm\frac{2}{f} & 1\end{pmatrix}\begin{pmatrix}\beta_0\\ \alpha_0\\ \gamma_0\end{pmatrix}\Rightarrow \begin{array}{c}\beta_1=\beta_0\\ \alpha_1=\alpha_0\pm\frac{\beta_0}{f}\\ \gamma_1=\gamma_0+\frac{\beta_0}{f^2}\pm\frac{2\alpha_0}{f}\end{array}$$

Courant-Snyder invariant and emittance

Back to the trajectory solutions:

$$z=a\sqrt{\beta}\cos\theta$$

$$z'=-\frac{a}{\sqrt{\beta}}(\alpha\cos\theta+\sin\theta)$$

$$\theta=\Phi+\delta$$

Form the combination

$$\begin{aligned}&\gamma z^2+2\alpha zz'+\beta z'^2\\ &=a^2\beta\gamma\cos^2\theta-2\alpha a^2\cos\theta(\alpha\cos\theta+\sin\theta)+a^2(\alpha\cos\theta+\sin\theta)^2\\ &=a^2\left[\cos^2\theta\left(\beta\gamma-\alpha^2\right)+\cos\theta\sin\theta(-2\alpha+2\alpha)+\sin^2\theta\right]\\ &=a^2\end{aligned}$$

So

$$\gamma(s)z(s)^2+2\alpha(s)z(s)z'(s)+\beta(s)z'(s)^2=a^2$$

(the Courant-Snyder invariant)

-it is constant along a particular particle trajectory.

Let us pick a particular position in the ring s_0. Let the values of z and z' at this position, on turn n, be $z_n(s_0)$ and $z_n'(s_0)$.

$$\begin{aligned}z_n(s_0)&=z(s_0+nC)=a\sqrt{\beta(s_0)}\cos(\Phi(s_0+nC)+\delta)\\ &=a\sqrt{\beta(s_0)}\cos(2\pi nQ+\Phi(s_0)+\delta)=a\sqrt{\beta(s_0)}\cos[\phi(n)]\end{aligned}$$

$$z_n'(s_0)=z'(s_0+nC)=-\frac{a}{\sqrt{\beta(s_0)}}(\alpha(s_0)\cos[\phi(n)]+\sin[\phi(n)])$$

where $\phi(n)=2\pi nQ+\text{constant}$

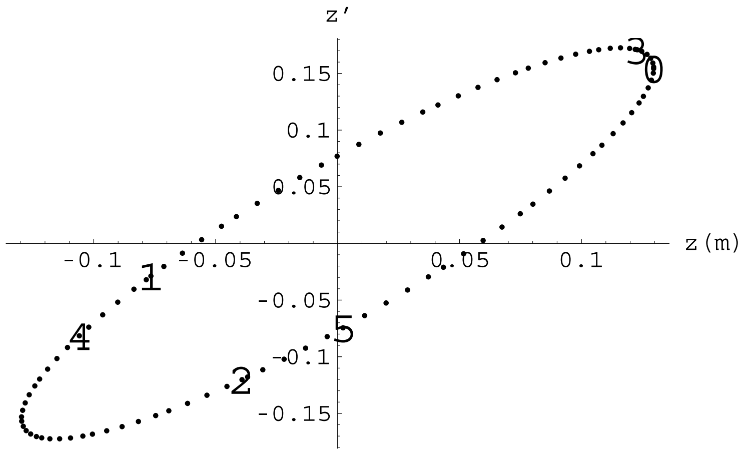
At this position, the Courant-Snyder invariant is

$$\gamma(s_0)z_n(s_0)^2+2\alpha(s_0)z_n(s_0)z_n'(s_0)+\beta(s_0)z_n'(s_0)^2=a^2$$

The two-dimensional space formed by $z_n(s_0)$ and $z_n'(s_0)$ is called phase space. The equation expressing the Courant-Snyder invariant is the equation of an ellipse in this phase space. From turn to turn, the phase space points $\{z_n(s_0),\ z_n'(s_0)\}$ map out this ellipse.

Example:

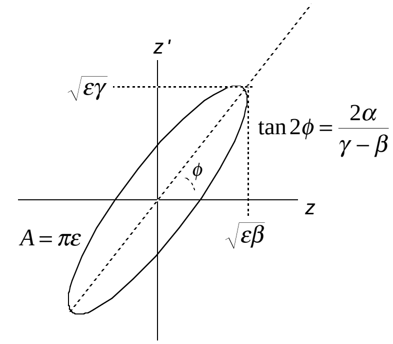
phase space plot (just upstream of the F quad) in the 10 m accelerator, with a^2 = 0.01 m. The first five turns are shown by the numbers.



a^2 is called the *emittance* of a particle which has this trajectory.

$$a^2 = \varepsilon = \frac{\text{Area of ellipse}}{\pi}$$

The parameters of the ellipse are determined by the lattice functions α, β, and γ, at the position s_0, and by the emittance:

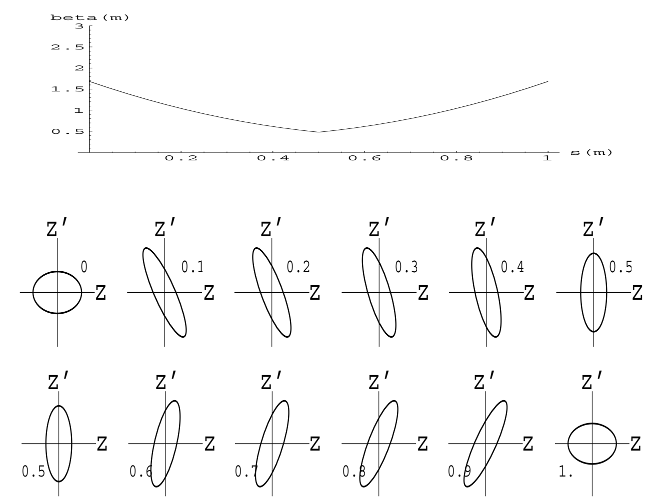


As we change the observation position s_0, the phase space ellipse changes its shape and orientation, but ε, proportional to the area, is constant:

Example; back to the 10 m accelerator with 10 FODO cells.

Plot β within one cell, and the phase space ellipse at several positions within the cell.

The number on the phase space ellipse plots indicates the value of *s* within the cell for which it is plotted.



The ellipse is upright when α=0. At these points

$$\beta = \frac{z_{\text{max}}}{z'_{\text{max}}}$$

If β is at a minimum, we call it a <u>waist</u>. In FODO cells, waists appear at the center of the D quads.