# LECTURE 6

Emittance in multi-particle beams
Lattice functions in non-periodic systems
Adiabatic damping
Momentum dispersion
Momentum compaction



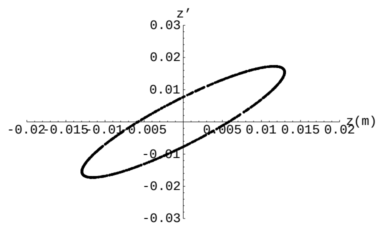
## Emittance in multi-particle beams

Up until now, we have been discussing single particles, and their trajectories. Let us now consider many particles in an accelerator, for which the trajectory of the ith particle has the form

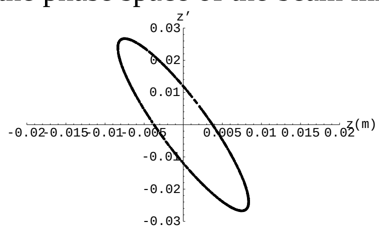
$$z_i(s) = \sqrt{\varepsilon\beta(s)}\cos\left(\Phi(s) + \delta_i\right)$$

The particles all have the same value of the emittance $\varepsilon$ but are randomly distributed in the phase $\delta_i$. The phase space of the *multi-particle beam* , at a particular point in the machine, at a particular time, might look like



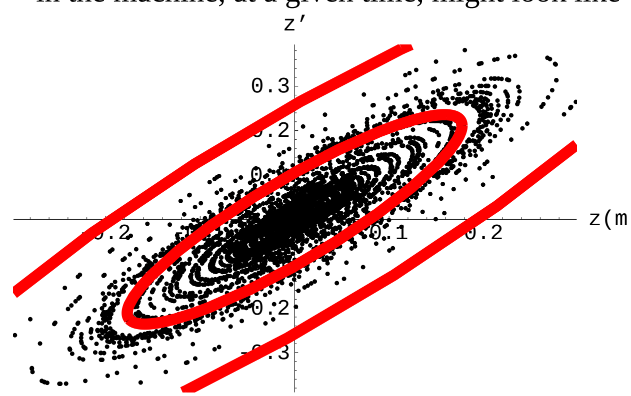



The emittance of this beam is ε. At a different point in the accelerator, the phase space of the beam might look like






The area is the same: the phase space area of the beam is constant. Now let the beam particles also have different values of the emittance $\varepsilon$. A phase space plot of the whole beam at a given point in the machine, at a given time, might look like



The points represent the beam particles. The *rms emittance* of the beam is defined as the area (divided by π) of the (matched) ellipse containing 39% of the particles. This is the smaller heavy (red, if you have color) ellipse beam.

The larger heavy (red) ellipse contains 95% of the particles, and has an area (divided by π) defined to be the *95% emittance* of the beam.

IF the distribution of the beam particles in phase space is *Gaussian*, then

$$\sqrt{\langle z^2 \rangle} = \sqrt{\beta \varepsilon_{rms}} \text{ and}$$

$$\sqrt{6\langle z^2 \rangle} = \sqrt{\beta \varepsilon_{95\%}}$$



Beware! The word "emittance" is often used to mean one or another of these, without specifying which one. Rms emittance is typical used at electron machines, 95% emittance at proton machines.

Also: often emittance is defined to be the **area** of the ellipse in phase space (as in Syphers and Edwards), not the area/π. In this case, the "emittance" is usually written in the form επ, where ε is a number: e.g., an emittance of "10π mm-mrad" would correspond to ε=10 mm-mrad.



The phase space area enclosed by all particles with a given emittance is constant as they move around the machine. Since the number of particles is also constant, *the local phase space density is constant.* This statement is called "Liouville's theorem".

This theorem does not hold in the presence of acceleration, particle losses, dissipative processes (like scattering), or damping processes (like radiation damping or cooling)

The emittance is a property of a trajectory (or a collection of trajectories: a beam). The *admittance* or *acceptance* of a beam transport system, or an accelerator, is the largest value of the emittance which the system will transport without loss.



## Lattice functions in non-periodic systems

The Twiss parameters are uniquely defined only for circular accelerators. Nevertheless, the language is used also to describe beam optics in linacs and beam transfer lines. The approach is the following:

1. Establish values for the lattice functions at the start of the transfer line or linac.
2. Calculate the cosinelike and sinelike trajectories for the line from the fields in the magnet lattice.
3. Propagate the lattice functions from the staring point through the line using the relations discussed in lecture 5.

*This is straightforward except for step 1.*

If the starting point of the transfer line is the exit of a circular accelerator, then the lattice functions at this point are well-defined: they are those of the circular accelerator at the point of extraction.

If the starting point is a particle source, then it's trickier.

*In this case, we use the standard relation between the lattice functions and the shape of the beam in phase space*

$$\gamma z^2 + 2\alpha z z' + \beta z'^2 = \varepsilon$$

*to <u>define</u> the starting values for the lattice functions and the emittance.*



The output of a particle source will have some distribution in $\{z, z'\}$ phase space.
We overlay an ellipse, whose general equation is

$$az^2 + bzz' + cz'^2 = d$$

on the source output phase space, adjusting the ellipse axes and tilt to find the smallest ellipse which contains, for example, 39% of the phase space points.
Then we identify

$$a = \gamma_0, b = 2\alpha_0, c = \beta_0, d = \varepsilon_{rms},$$

$$\text{with } 1 + \alpha_0^2 = \beta_0 \gamma_0$$

and propagate the lattice functions forward from here.



This procedure is undesirable in that it makes the lattice functions depend not only on the magnet lattice but also on the input beam distribution. However, it does preserve the general relations between lattice functions, beam envelopes, and beam phase space distributions, which are true for circular machines.

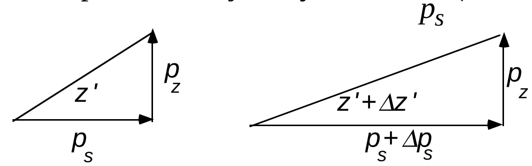
## <u>Adiabatic damping</u>

The Courant-Snyder invariant emittance ε decreases if we the accelerate the particle. This is called "adiabatic damping" (a misnomer: there is no damping process involved).
Suppose we have a particle of momentum $p_0$

$$p_0^2 = p_{s0}^2 + p_{x0}^2 + p_{y0}^2$$



The slope of the trajectory is $z'' = \dfrac{p_z}{p_s}$ ($z$=$x$ or $y$)




Accelerate the particle: $p_s$ increases to $p_s+\Delta p_s$, but $p_z$ doesn't change=> slope changes. The new value of the trajectory slope is

$$z' + \Delta z' = \frac{p_z}{p_s + \Delta p_s} = \frac{p_z}{p_s\left(1 + \dfrac{\Delta p_s}{p_s}\right)} \approx z'\left(1 - \frac{\Delta p}{p}\right) \Rightarrow$$

$$\Delta z' = -z' \frac{\Delta p}{p}$$

What happens in $\{z, z'\}$ phase space?

Let us consider a beam of particles, all with the same emittance ε, but with random phases. For particle *i*, at a point where α=0, we have

$$z_i = \sqrt{\varepsilon\beta}\cos(\Phi+\delta_i), \quad z_i' = -\sqrt{\frac{\varepsilon}{\beta}}\sin(\Phi+\delta_i)$$

The emittance is

$$\varepsilon = \beta z_i'^2 + \gamma z_i^2$$

If we change $z'$, the resulting emittance change is

$$\Delta\varepsilon = 2\beta z_i' \Delta z_i' = -2\beta z_i'^2 \frac{\Delta p}{p} = -2\varepsilon \sin^2(\Phi+\delta_i)\frac{\Delta p}{p}$$



Averaging over all the particles to get the emittance of the beam, we have

$$\langle\Delta\varepsilon\rangle = -\varepsilon\frac{\Delta p}{p} \Rightarrow \frac{d\varepsilon}{\varepsilon} = -\frac{dp}{p},$$

$$\varepsilon(p) = \varepsilon_0 \frac{p_0}{p}$$

The “invariant” emittance is thus a decreasing function of the momentum. To keep track of this, the “normalized” emittance is defined as

$$\varepsilon_n = \varepsilon\beta\gamma$$

in which $\beta = \frac{v}{c}$ and $\gamma^2 = \frac{1}{1-\beta^2}$

Ideally, the normalized emittance does not change during acceleration.



## Momentum dispersion

Review: we solved the linear trajectory equation

$$\frac{d^2x}{ds^2} + K(s)x = \frac{\delta}{\rho(s)}$$

in terms of cosinelike, sinelike, and dispersion trajectories:

$$x(s,s_0) = C_x(s,s_0)x(s_0) + S_x(s,s_0)x'(s_0) + \delta D_x(s,s_0)$$

The momentum dependence of the solution was determined by the dispersion trajectory $D_x(s,s_0)$



When we started talking about circular accelerators (periodic systems) and the Twiss matrix, however, we took δ=0. Now we return and look at momentum dependence in periodic systems.

The general trajectory is written as a betatron oscillation plus a momentum-dependent piece described by a new lattice function, the *dispersion function η(s):*

$$z(s) = \sqrt{\varepsilon\beta(s)}\cos(\Phi(s)+\varphi) + \delta\eta(s)$$

where $\delta = \frac{p-p_0}{p_0}$ is the relative momentum deviation from the reference momentum $p_0$. The one-turn transfer matrix is expanded, as before, to accommodate momentum deviation :

$$\mathbf{M}(s+C,s)=\begin{pmatrix}\cos\mu+\alpha\sin\mu & \beta\sin\mu & D_x(s+C,s)\\ -\gamma\sin\mu & \cos\mu-\alpha\sin\mu & D_x{}'(s+C,s)\\ 0 & 0 & 1\end{pmatrix}$$

$$\begin{pmatrix}z\\ z'\\ \delta\end{pmatrix}_{s+C}=\mathbf{M}(s+C,s)\begin{pmatrix}z\\ z'\\ \delta\end{pmatrix}_s$$

To calculate the dispersion function, we take $\varepsilon$=0 in the solution for $z(s)$ and substitute in the above matrix equation:

$$\begin{pmatrix}\eta(s+C)\\ \eta'(s+C)\\ 1\end{pmatrix}=\mathbf{M}(s+C,s)\begin{pmatrix}\eta(s)\\ \eta'(s)\\ 1\end{pmatrix}$$

in which the $\delta$'s on both sides have been cancelled.



Now the dispersion function must be periodic in $s$ with period $C$:

$$\eta(s+C)=\eta(s)$$

Because it is proportional to the reference orbit at a different momentum, and so must be closed. Hence we have

$$\begin{pmatrix}\eta(s)\\ \eta'(s)\\ 1\end{pmatrix}=\mathbf{M}(s+C,s)\begin{pmatrix}\eta(s)\\ \eta'(s)\\ 1\end{pmatrix}$$

$$\eta(s)=\eta(s)\left(\cos\mu+\alpha(s)\sin\mu\right)+\eta'(s)\beta(s)\sin\mu+D_x(s+C,s)$$

$$\eta'(s)=-\eta(s)\gamma(s)\sin\mu+\eta'(s)\left(\cos\mu-\alpha(s)\sin\mu\right)+D_x{}'(s+C,s)$$

These equations are solved simultaneously for $\eta$ and $\eta'$. The result is



$$\eta=\frac{2D_x\sin^2\frac{\mu}{2}+\left(\alpha D_x+\beta D_x'\right)\sin\mu}{4\sin^2\frac{\mu}{2}}$$

$$\eta'=\frac{2D_x'\sin^2\frac{\mu}{2}-\left(\alpha D_x'+\gamma D_x\right)\sin\mu}{4\sin^2\frac{\mu}{2}}$$

These expressions are divergent if $\mu=2\pi n$, where n is an integer. For the one-turn matrix, $\mu=2\pi Q$, so integral tune leads to a divergent dispersion function.



Example calculation of dispersion:

<u>500</u> m accelerator with <u>50</u> FODO cells, each of length $L$=10 m. We said that there were bending magnets in the FODO cell drift spaces. We now calculate the dispersion function due to these bending magnets.

We take all the 100 bends to be of equal bend radius $\rho$ and length $L/2$: then we have $50\frac{L}{\rho}=2\pi,\ \rho=\frac{50L}{2\pi}=\frac{500}{2\pi}\ \mathrm{m}=79.58\ \mathrm{m}$. The bend angle is $\phi=\frac{2\pi}{100}=0.0628$

We expand the FODO cell transfer matrix to include the dispersion trajectory by using for the dipole the matrix

$$\mathbf{M}_D\left(\frac{L}{2},0\right)=\begin{pmatrix} 1 & \frac{L}{2} & \frac{L\phi}{4} \\ 0 & 1 & \phi \\ 0 & 0 & 1 \end{pmatrix}$$

where the weak focusing of the dipole has been ignored. Then the FODO transfer matrix is



$$\mathbf{M}_c(L,0)=$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{L}{2} & \frac{L\phi}{4} \\ 0 & 1 & \phi \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{L}{2} & \frac{L\phi}{4} \\ 0 & 1 & \phi \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{M}_c(L,0)=\begin{pmatrix} 1-\frac{L^2}{8f^2} & L+\frac{L^2}{4f} & \frac{L\phi}{2}\left(1+\frac{L}{8f}\right) \\ -\frac{L}{4f^2}\left(1-\frac{L}{4f}\right) & 1-\frac{L^2}{8f^2} & 2\phi\left(1-\frac{L}{8f}-\frac{L^2}{32f^2}\right) \\ 0 & 0 & 1 \end{pmatrix}$$



Then the dispersion at the beginning of the FODO cell is found using

$$\sin\frac{\mu}{2}=\frac{L}{4f};\quad \beta(0)=\frac{L\left(1+\sin\frac{\mu}{2}\right)}{\sin\mu};\ \ \alpha(0)=0;\ \ \gamma(0)=\frac{\sin\mu}{L\left(1+\sin\frac{\mu}{2}\right)};$$

$$D_x(L,0)=\frac{L\phi}{2}\left(1+\frac{L}{8f}\right)=\frac{L\phi}{2}\left(1+\frac{1}{2}\sin\frac{\mu}{2}\right);$$

$$D_x'(L,0)=2\phi\left(1-\frac{L}{8f}-\frac{L^2}{32f^2}\right)=2\phi\left(1-\frac{1}{2}\sin\frac{\mu}{2}-\frac{1}{2}\sin^2\frac{\mu}{2}\right)$$

So we get



$$\eta(0)=\frac{L\phi\left(1+\frac{1}{2}\sin\frac{\mu}{2}\right)\sin^2\frac{\mu}{2}+\left(\frac{L\left(1+\sin\frac{\mu}{2}\right)}{\sin\mu}2\phi\left(1-\frac{1}{2}\sin\frac{\mu}{2}-\frac{1}{2}\sin^2\frac{\mu}{2}\right)\right)\sin\mu}{4\sin^2\frac{\mu}{2}}=\frac{L\phi\left(1+\frac{1}{2}\sin\frac{\mu}{2}\right)}{2\sin^2\frac{\mu}{2}}$$

$$\eta'(0)=\frac{4\phi\left(1-\frac{1}{2}\sin\frac{\mu}{2}-\frac{1}{2}\sin^2\frac{\mu}{2}\right)\sin^2\frac{\mu}{2}-\left(\frac{\sin\mu}{L\left(1+\sin\frac{\mu}{2}\right)}\frac{L\phi}{2}\left(1+\frac{1}{2}\sin\frac{\mu}{2}\right)\right)\sin\mu}{4\sin^2\frac{\mu}{2}}=0$$

Note that for fixed energy and field (=>fixed $\rho$), $\phi=L/(2\rho)$, and the dispersion varies like the square of the cell length.

For our numerical example: we'll take f=4.5 m, so we get the same cell advance as in our previous, smaller ring, example:

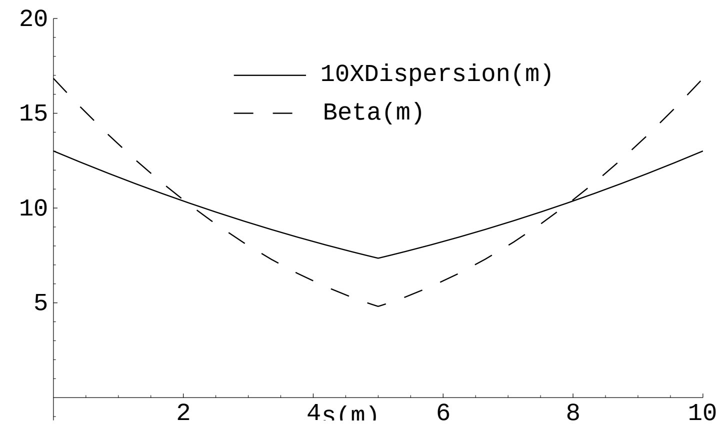
$$\sin\frac{\mu}{2} = 0.5555; \quad L = 10 \;\; \text{m}$$

$$\beta(0) = \frac{L\left(1+\sin\frac{\mu}{2}\right)}{\sin\mu} = \frac{10(1+0.555)}{0.924} = 16.83 \;\; \text{m}$$

$$\eta(0) = \frac{L\phi\left(1+\frac{1}{2}\sin\frac{\mu}{2}\right)}{2\sin^2\frac{\mu}{2}} = \frac{10\times 0.0628\times\left(1+\frac{1}{2}\times 0.5555\right)}{2(0.5555)^2} \;\; \text{m} = 1.30 \;\; \text{m}$$



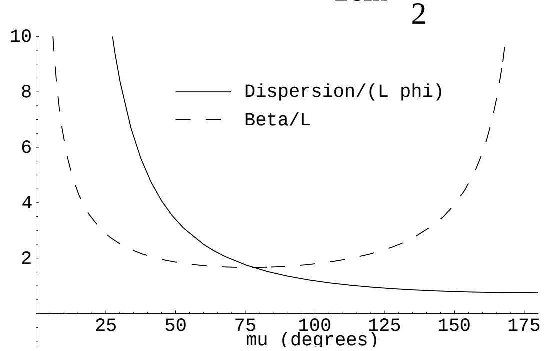
Dispersion and beta function vs. *s* in the cell






Dependence of the disperison and beta function on cell phase advance μ

$$\beta(0) = \frac{L\left(1+\sin\frac{\mu}{2}\right)}{\sin\mu}; \quad \eta(0) = \frac{L\phi\left(1+\frac{1}{2}\sin\frac{\mu}{2}\right)}{2\sin^2\frac{\mu}{2}}$$






Cell phases advances are typically in the range of 60-120 degrees

The dispersion function can also be calculated from the following expression

$$\eta(s) = \frac{\sqrt{\beta(s)}}{2\sin\pi Q}\oint_C dt\,\frac{\sqrt{\beta(t)}}{\rho(t)}\cos\left(\left|\Phi(t)-\Phi(s)\right|-\pi Q\right)$$

For a derivation, see Ref. 2, p 72. Typically, dispersion is calculated from transfer matrices, rather than this result.

## Momentum compaction

The path length of a closed off-momentum trajectory will differ from the length of the reference trajectory (which is defined to be the circumference, $C$):

$$\frac{dl}{ds} = 1 + \frac{x(s)}{\rho(s)}; \quad x(s) = \delta\eta(s) \Rightarrow$$

$$dl = ds + \delta\frac{\eta(s)}{\rho(s)}ds$$

Integrate around the circumference:



$$\oint_c dl = C + \Delta C = C + \delta\oint_C \frac{\eta(s)}{\rho(s)}ds$$

$$\Delta C = \delta\oint_C \frac{\eta(s)}{\rho(s)}ds = \delta C\alpha_C$$

in which $\alpha_C = \frac{1}{C}\oint_C \frac{\eta(s)}{\rho(s)}ds$ is called the momentum compaction It measures the relative change in circumference per unit relative momentum offset.

$$\frac{\Delta C}{C} = \alpha_C\delta$$

$\alpha_C$ measures how 'closely packed' orbits with different momenta are.



Significance of the momentum compaction:

Consider a particle moving along the reference orbit with momentum $p_0$, velocity $v_0 = \beta_0 c$ , and energy $m_0c^2\gamma_0 = \frac{m_0c^2}{\sqrt{1-\beta_0^2}}$.

It moves distance $s$ along the reference orbit in time $t_0 = \frac{s}{v_0}$. (Take the distance $s$ to be much greater than one circumference). Another particle, with momentum $p = p_0(1+\delta)$, moves along a different trajectory; its projection on the reference orbit moves a distance $s$ in time $t = \frac{s+\Delta s}{v} = \frac{s+\alpha_C\delta s}{v}$

where $\Delta s$ is the extra path length required due to the momentum difference. The time difference between this particle and the reference particle is



$$\Delta t = t - t_0 = \frac{s+\alpha_C\delta s}{v} - \frac{s}{v_0} \approx \frac{s}{v_0}\left(\frac{v_0}{v} - 1 + \alpha_C\delta\right)$$

$$\frac{\Delta t}{t_0} \approx \frac{v_0}{v} - 1 + \alpha_C\delta$$

For small $\delta = \frac{p-p_0}{p_0}$, we have $v \approx v_0\left(1+\frac{\delta}{\gamma^2}\right)$

so

$$\frac{\Delta t}{t_0} \approx \left(\alpha_C - \frac{1}{\gamma^2}\right)\delta = \eta_C\delta$$

where $\eta_C = \left(\alpha_C - \frac{1}{\gamma^2}\right)$ is called the "slip factor": it measures how much off-momentum particles "slip" in time relative to the reference particle.

For some value of $\gamma$, the slip factor will be zero: this value is called the "transition gamma" $\gamma_t$. It is determined by the momentum compaction of the lattice.

$$\frac{1}{{\gamma_t}^2} = \alpha_C$$

For an accelerator operating at the transition gamma, there is no relative longitudinal motion: all particles take the same time to go around, irrespective of their momentum.

The implications of this will be discussed later.

A rough estimate of the transition gamma, for machines made entirely of simple FODO cells with phase advance $\mu$<<1, can be obtained as follows:



For such a machine,

$$\langle\beta\rangle \approx \frac{L\left(1+\sin\frac{\mu}{2}\right)}{\sin\mu} \approx \frac{L}{\mu}\;;\quad \langle\eta\rangle \approx \frac{L\phi\left(1+\frac{1}{2}\sin\frac{\mu}{2}\right)}{2\sin^2\frac{\mu}{2}} \approx \frac{2L\phi}{\mu^2} = \frac{L^2}{\mu^2\rho} \approx \frac{\langle\beta\rangle^2}{\rho}$$

$$Q = \frac{\langle R\rangle}{\langle\beta\rangle} \approx \frac{\rho}{\langle\beta\rangle} \Rightarrow \langle\eta\rangle \approx \frac{1}{\rho}\left(\frac{\rho}{Q}\right)^2 = \frac{\rho}{Q^2}$$

Then, we have

$$\alpha_C = \frac{1}{C}\oint_C \frac{\eta(s)}{\rho(s)}ds \approx \frac{1}{\rho}\langle\eta\rangle \approx \frac{1}{Q^2}$$

$$\gamma_t = \frac{1}{\sqrt{\alpha_C}} = Q$$



Remarks:

1. For high energy machines with large $\gamma$, the slip factor is dominated by $\alpha_C$. For a storage ring, if $\alpha_C$ is made very small, the ring will be "close to transition" at all times. Such a machine is called "quasi-isochronous", since all particles have almost exactly the same revolution frequency.

2. It is possible to design a lattice in which the momentum compaction is variable (without changing the tune much), and it can even be made negative. Such machines are said to have "flexible momentum compaction". If the momentum compaction is negative, the transition gamma is imaginary: there is no energy for which the longitudinal motion is isochronous.