LECTURE 8

Linear deviations from an ideal lattice:
Dipole errors and closed orbit deformations (continued)
Quadrupole errors and tune shifts
Chromaticity
Sextupole Compensation of Chromaticity

Example: Our 500 m accelerator has an F quad misaligned in the x-direction by 1 mm at s_k =50 m. What is the resulting orbit deformation?

The focal length of the quad is f=4.5 m. The quad strength is

$$k \approx \frac{1}{fL_Q} = \frac{B'}{(B_0\rho)}$$

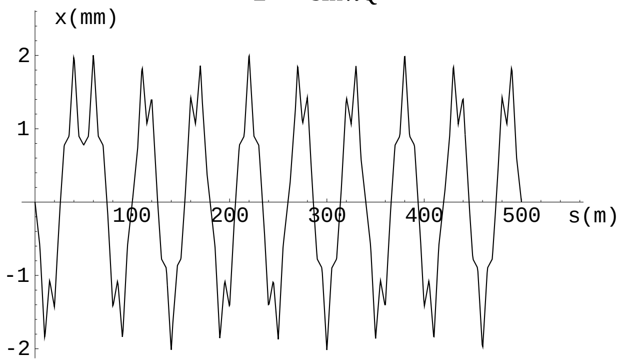
If the misalignment is Δx, the kick angle due to the misalignment is

$$\theta = \frac{\Delta(BL)}{(B_0\rho)} = \frac{B'\Delta x L_Q}{(B_0\rho)} = \frac{\Delta x}{f} = \frac{10^{-3}}{4.5} = 2.22\times10^{-4}$$

The kick occurs at β_k = 16.8 m, and Q=9.3747. The orbit displacement at the kick is

$$z(s_k) = \frac{\beta(s_k)}{2}\theta\cot\pi Q = \frac{16.83\times2.222\times10^{-4}}{2\tan 9.3747\pi} = 0.78\ \text{mm}$$

Plot of the "closed orbit" $z(s) = \frac{\theta}{2}\frac{\sqrt{\beta(s)}\sqrt{\beta(s_k)}}{\sin\pi Q}\cos\left(\left|\Phi(s)-\Phi(s_k)\right| - \pi Q\right)$



For a given kick angle, the orbit deviation is much greater, the closer Q is to an integer. For Q=integer, the orbit diverges. So Q's close to an integer must be avoided.

If I have not just one error, but N, then the orbit distortions add linearly and we have

$$z(s) = \frac{\sqrt{\beta(s)}}{2\sin\pi Q}\sum_{k=1}^{N}\theta_k\sqrt{\beta(s_k)}\cos\left(\left|\Phi(s)-\Phi(s_k)\right| - \pi Q\right)$$

If the N errors are uncorrelated and randomly distributed in phase, then

$$\sqrt{\left\langle z(s)^2\right\rangle} = \frac{\sqrt{\beta(s)}}{2\sqrt{2}\sin\pi Q}\sqrt{\sum_{k=1}^{N}\theta_k^2\beta(s_k)}$$

$$\approx \frac{\sqrt{N}\sqrt{\beta(s)\langle\beta\rangle}}{2\sqrt{2}\sin\pi Q}\sqrt{\left\langle\theta^2\right\rangle}$$

Note dependence on $\sqrt{N}$: Very large machines typically have very tight error tolerances on individual components.

In our example, if all quads were misaligned with an rms error of 1 mm, we would expect a maximum orbit distortion with rms

$$\sqrt{\langle z^2 \rangle} \approx 2 \times \sqrt{50} \ \text{mm} \ = \ 1.4 \ \text{cm}$$

Typical size of a high-energy beam: take ε_{rms}=10^{-6} m-rad and δ_{rms}=10^{-3}. Then

$$z_{max} \approx \pm 2 z_{rms} = \pm 2\left(\sqrt{\varepsilon_{rms}\beta} + \delta_{rms}\eta\right)$$
$$= \pm 2\left(\sqrt{10^{-6} \times 16.83} + 10^{-3} \times 1.3\right) \ \text{m}$$
$$= \pm 10.8 \ \text{mm}$$

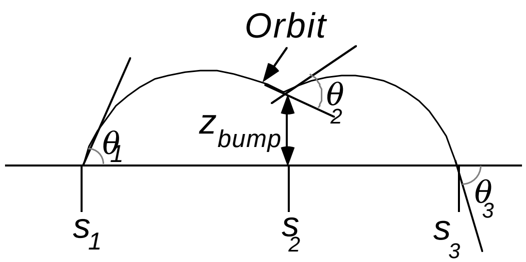
An orbit distortion 1.4 times the beam size generally can not be tolerated.

Short <u>*correction dipoles*</u> are placed into the lattice at high-beta locations (next to F quads in *x*, and next to D quads in *y*). These dipoles can be tuned to introduce kicks that compensate for the field errors.

The orbit correction is done using position information from *beam position monitors*, which have to be carefully aligned onto the reference orbit. The orbit can typically be corrected to a level of a few tenths of a mm.

Frequently one wishes purposefully to deform the closed orbit from the reference orbit in a local region of the machine. The most common purpose is to facilitate injection or extraction; other purposes might be for beam collimation, to accommodate an asymmetric physical aperture, or for diagnostic purposes.

This local orbit deformation is called a “bump”. Bumps are created using combinations of (usually three or four) dipole correctors.



We use three corrector dipoles, at s_1, s_2, and s_3, which deliver kick angles θ_1, θ_2, and θ_3. The phases at each of these points are $\Phi(s_1) = \Phi_1;\ \Phi(s_2) = \Phi_2;\ \Phi(s_3) = \Phi_3$, and the beta functions are

$\beta(s_1) = \beta_1;\ \beta(s_2) = \beta_2; \beta(s_3) = \beta_3$. The relation between the kick angles, which is required to make the bump *local* (that is, only non-zero between s_1 and s_3), is

$$\frac{\theta_2}{\theta_1} = -\sqrt{\frac{\beta_1}{\beta_2}} \frac{\sin(\Phi_3 - \Phi_1)}{\sin(\Phi_3 - \Phi_2)}$$
$$\frac{\theta_3}{\theta_1} = \sqrt{\frac{\beta_1}{\beta_3}} \frac{\sin(\Phi_2 - \Phi_1)}{\sin(\Phi_3 - \Phi_2)}$$

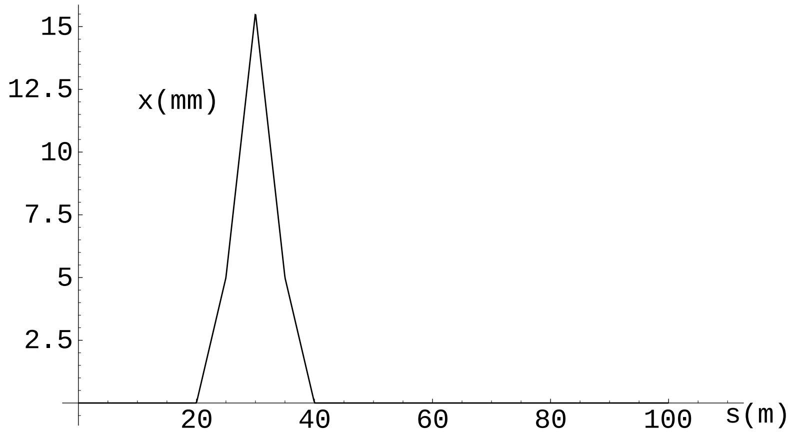
The bump amplitude at s_2 is

$$z_{bump} = \theta_1 \sqrt{\beta_1 \beta_2} \sin(\Phi_2 - \Phi_1)$$

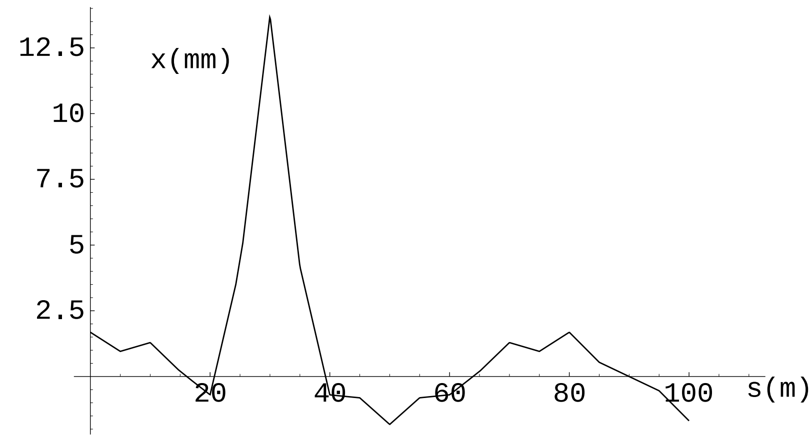
Exercise: derive these relations.

Examples:

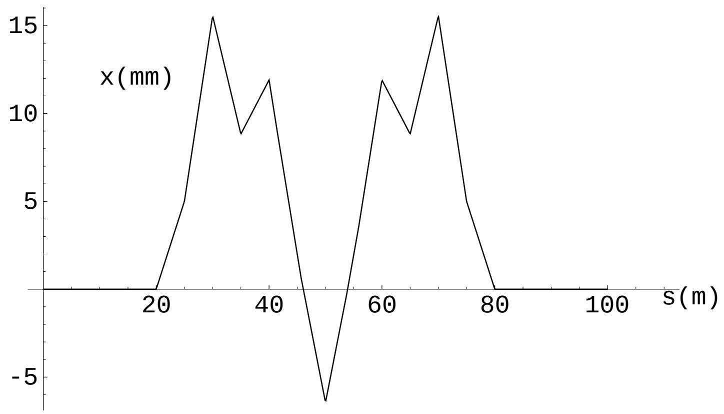
1. s_1=20 m, s_2=30 m, s_3=40 m, θ_1=1 mrad, θ_2=-0.76 mrad, θ_3=1 mrad.



2. s_1=20 m, s_2=30 m, s_3=40 m, θ_1=1 mrad, θ_2=-0.76 mrad, θ_3=0.8 mrad.



C. s_1=20 m, s_2=50 m, s_3=80 m, θ_1=1 mrad, θ_2=1.85mrad, θ_3=1 mrad.



Linear deviations from an ideal lattice: Quadrupole errors and tune shifts

After having explored some of the consequences of dipole field errors, we'll now take a look at the effects of quadrupole field errs.

Some quadrupole field error sources:

- Differences between the idealized quadrupole field and the true quadrupole field, due to fabrication errors in the magnets, and/or due to remnant field effects (this is usually the biggest source of error)
- Quadrupole fields due to errors in the dipole magnets

• Quadrupole fields due to sextupoles not being aligned on the reference orbit

• Stray fields on the reference orbit from other accelerator components

From Lecture 3, p 7: The trajectory equations, to lowest order in quadrupole field errors, are

$$x'' + x\left(k + \frac{1}{\rho^2}\right) = -\frac{\Delta B'(s)x}{B_0\rho}; \quad y'' - yk = \frac{\Delta B'(s)y}{B_0\rho}$$

Both of the form

$$z'' + \left(K(s) + \frac{\Delta B'(s)}{B_0\rho}\right)z = 0$$

A quadrupole field error produces a perturbation in the focusing function $K(s)$. The focusing function $K(s)$ determines the lattice functions β, and η, and quantities derived from them, such as Φ and Q. Thus, we expect all these quantities to change as a result of quadrupole field errors.

As in the case of dipole errors, we'll treat a single gradient error as localized at one point, and sum over these to treat a collection of gradient errors. Thus a single gradient error is treated as a thin lens, of focal length $\frac{1}{f} = \Delta kL = \frac{\Delta(B'L)}{B_0\rho}$, where L is the length of the gradient error along the reference orbit.

Suppose the gradient error is located at s_0. Then the one-turn matrix at this point becomes

$$\mathbf{M}(C+s_0, s_0) = \begin{pmatrix} \cos 2\pi Q + \alpha(s_0)\sin 2\pi Q & \beta(s_0)\sin 2\pi Q \\ -\gamma(s_0)\sin 2\pi Q & \cos 2\pi Q - \alpha(s_0)\sin 2\pi Q \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -\Delta(kL) & 1 \end{pmatrix} \begin{pmatrix} \cos 2\pi Q_0 + \alpha_0(s_0)\sin 2\pi Q_0 & \beta_0(s_0)\sin 2\pi Q_0 \\ -\gamma_0(s_0)\sin 2\pi Q_0 & \cos 2\pi Q_0 - \alpha_0(s_0)\sin 2\pi Q_0 \end{pmatrix}$$

in which the lattice functions with subscript 0 refer to the unperturbed lattice functions and tune.

Carrying out the matrix multiplication and equating the trace of the matrices on each side of the equation, we get

$$\cos 2\pi Q = \cos 2\pi Q_0 \cos 2\pi \Delta Q - \sin 2\pi Q_0 \sin 2\pi \Delta Q$$

$$= \cos 2\pi Q_0 - \frac{\beta_0(s_0)\Delta(kL)\sin 2\pi Q_0}{2}$$

in which the change in the tune is $\Delta Q = Q - Q_0$.

If ΔQ <<1, then we have

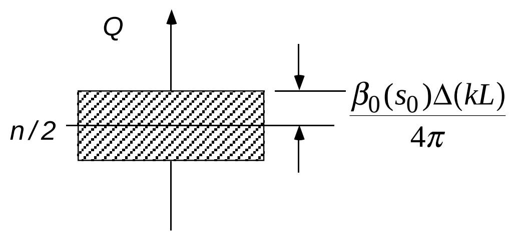
$$\Delta Q = \frac{1}{4\pi}\beta_0(s_0)\Delta(kL)$$

The lowest order effect of the quadrupole error is a change in the tune, proportional to the strength error, the error's length, and beta at the location of the error.

This result is only true to first order in Δk, since the lattice functions are also perturbed, and we have ignored this. Its accuracy also depends on the assumption that the pertubed motion is still stable. Stability requires that

$$|\cos 2\pi Q| = \left|\cos 2\pi Q_0 - \frac{\beta_0(s_0)\Delta(kL)\sin 2\pi Q_0}{2}\right| < 1$$

If the unperturbed tune Q_0 is close to *n/2*, where *n* is any integer, then $|\cos 2\pi Q_0|$ is close to 1, and the quadrupole perturbation could be large enough to violate the stability criterion. There is a range of tune values around Q_0=n/2 for which the motion is unstable. This range, which depends on *ΔkL*, is called the *half-integer stopband*.



So, in the presence of gradient errors, we must avoid a range of tunes around the half-integers.

Example: Take a quadrupole field error of 10%, in one of the F quads, in our 500 m accelerator.

Using *f*=4.5 m,

$$kL = \frac{1}{f} = 0.2222 \ \ \mathrm{m}^{-1}; \ \ \Delta(kL) = 0.02222 \ \ \mathrm{m}^{-1}$$

$$\Delta Q = \frac{1}{4\pi}\beta_0(s_0)\Delta(kL) = \frac{16.8\times 0.02222}{4\pi} = 0.03$$

This may seem small, but it should be compared with the fractional part of the tune (9.3747); it is about 10% of that.

The stopband width is twice this, or 0.06: so, with this gradient error, tunes from 0.47 to 0.53 must be avoided.

If we have many errors $\Delta(kL)_i$ at locations s_i, then the tune shift is

$$\Delta Q = \frac{1}{4\pi}\sum_{i=1}^{N}\beta_0(s_i)\Delta(kL)_i$$

In this case, the stopband width is no longer twice the tune shift, since the relative phases at the perturbations must be accounted for.

For a continuous distribution of errors, this generalizes to

$$\Delta Q = \frac{1}{4\pi}\oint_C ds\,\beta(s)\Delta k(s)$$

This result can be used for gradient errors due to any source: e.g., electric field gradients, space charge and beam-beam fields, etc.

Gradient errors cause a perturbation to the lattice functions everywhere in the machine. To calculate this, let the quadrupole error be at s_0 . The one-turn matrix at another point, *s*, is given by

$$\mathbf{M}(s+C,s) = \mathbf{M}_0(C+s,s_0)\begin{pmatrix} 1 & 0 \\ -\Delta(kL) & 1 \end{pmatrix}\mathbf{M}_0(s_0,s)$$

If we write the unperturbed transfer matrices $\mathbf{M}_0(C+s,s_0)$ and $\mathbf{M}_0(s_0,s)$ in terms of the unperturbed lattice functions β_0 and α_0 at the appropriate points, we can carry out the matrix multiplication on the right-hand side. Then, on the left-hand side, the matrix element

$$M_{12}(C+s,s) = \beta(s)\sin 2\pi Q = \left(\beta_0(s)+\Delta\beta(s)\right)\sin 2\pi\left(Q_0+\Delta Q\right)$$

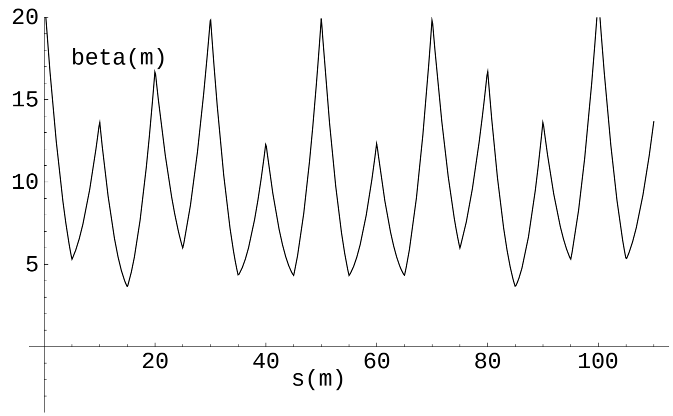
in which $\Delta\beta(s)$ is the perturbation β at *s*.

Use $\Delta Q = \frac{1}{4\pi}\beta_0(s_0)\Delta(kL)$, and then equate this to the corresponding matrix element on the right-hand side. Solve for $\Delta\beta(s)$. The result (for general s) is

$$\frac{\Delta\beta(s)}{\beta_0(s)} = -\frac{\Delta(kL)\beta_0(s_0)}{2\sin 2\pi Q_0}\cos\left[2\left(\left|\Phi_0(s)-\Phi_0(s_0)\right|-\pi Q\right)\right]$$

We again see the sensitivity to Q_0 near the half-integer: the beta function perturbation blows up at this point. The beta function perturbation oscillates twice as fast around the circumference as the closed orbit perturbation.

Example: $\Delta(kL) = 0.02222\ \ \mathrm{m}^{-1}$ at the F-quad at 50 m in our 500 m machine. The perturbed β function is shown below:



Chromaticity

Chromaticity refers to the dependence of the focusing function on momentum. Back to Lecture 3, p 7 again: Ignore field errors, but keep all terms linear in (x,y) or in the momentum deviation δ:

$$x'' + x\left(k(1-\delta) + \frac{1-2\delta}{\rho^2}\right) = \frac{\delta}{\rho}$$
$$y'' - yk(1-\delta) = 0$$

The constant $\frac{\delta}{\rho}$ is responsible for momentum dispersion, which we have already discussed. We'd now like to focus on the momentum dependence of the focusing terms. We'll neglect the

$\frac{1-2\delta}{\rho^2}$ term, which corresponds to the (weak) focusing in dipoles:

then

$$x'' + xk(1-\delta) = 0$$
$$y'' - yk(1-\delta) = 0$$

which is just equivalent to a gradient error of strength

$$\Delta k_{x,y} = \mp k\delta$$

This focusing error will produce a tune shift

$$\Delta Q_{x,y} = \frac{1}{4\pi}\oint_C ds\beta_{x,y}(s)\Delta k_{x,y}(s) = \mp\frac{\delta}{4\pi}\oint_C ds\beta_{x,y}(s)k(s)$$

The *chromaticity* ξ of the lattice is defined as the tune change per unit relative momentum change. Hence the chromaticity due to the dependence of quadrupole strength on momentum (called the *natural chromaticity*) is

$$\xi_{x,y} = \frac{\Delta Q_{x,y}}{\delta} = \mp\frac{1}{4\pi}\oint_C ds\beta_{x,y}(s)k(s)$$

For a strong focusing lattice, the natural chromaticity in <u>both</u> planes is always negative.

The natural chromaticity of a simple FODO lattice, in the thin lens approximation, is easy to calculate. For a single cell, we have

$$\xi_{c,x} = -\frac{1}{4\pi}\left(\frac{\beta_x(0)}{f} - \frac{\beta_x(\frac{L}{2})}{f}\right) = -\frac{1}{4\pi f}\left(\frac{L\left(1+\sin\frac{\mu}{2}\right)}{\sin\mu} - \frac{L\left(1-\sin\frac{\mu}{2}\right)}{\sin\mu}\right)$$

$$= -\frac{2L\sin\frac{\mu}{2}}{4\pi f\sin\mu} = -\frac{2\sin^2\frac{\mu}{2}}{\pi\sin\mu} = -\frac{1}{\pi}\tan\frac{\mu}{2}$$

For the whole machine, with N_c cells, we have:

$$\xi_x = N_c\xi_{c,x} = -\frac{N_c}{\pi}\tan\frac{\mu}{2}$$

For a design with μ<<1, we have

$$\xi_x = -\frac{N_c\mu}{2\pi} = -Q_x$$

The natural chromaticity is just the negative of the tune. For our 500 m accelerator example, with N_c = 50 and μ=1.178, we have

$$\xi_x = -\frac{50}{\pi}\tan\frac{1.178}{2} = -10.63$$

still not far from the negative of the tune. For this example, the natural *y*-chromaticity is the same as ξ_x.

For real machines with insertions, the chromaticity will of course be different. In particular, a machine with a low-β insertion can have a considerably larger natural chromaticity than that from the regular FODO lattice, because of the large value of β_{max} in the insertion, coupled with typically larger focusing strengths in the insertion matching quadrupoles.

Chromaticity is generally not desirable in a machine, for at least two reasons. Unfortunately, neither of these can be fully appreciated until further in the course.

1. If there is a spread in momentum δ in the beam(as there always will be), then there is spread in tune $\Delta Q = \delta\xi$. If large enough, this tune spread could put some of the beam dangerously

close to resonances. This is particularly important for large (high tune) machines. For example, if δ=10^{-3} and ξ=-100, then the chromatic tune spread will be ΔQ=-0.1, which is large compared to the typical spacing of high-order resonance lines.

2. The growth rate of a collective instability called the *head-tail instability* depends on the value of the chromaticity. Above transition, for positive chromaticity, this instability is very weak. Thus, machines are often operated with a small positive chromaticity above transition.

For a fixed lattice, how can we change the chromaticity? We need a field gradient which is a linear function of δ

Sextupole Compensation of Chromaticity

Recall from Lecture 3, p. 11: a sextupole has a field

$$B_y = \frac{B''}{2}(x^2 - y^2); \quad B_x = B''xy$$

and position dependent field gradients

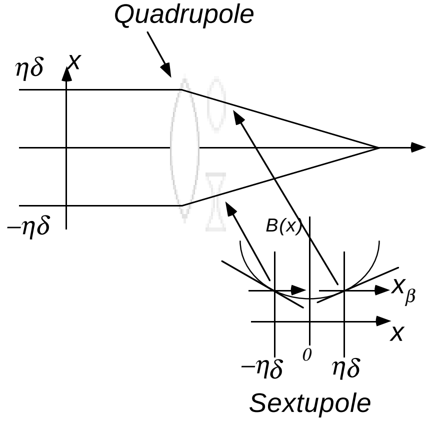
$$\frac{\partial B_y}{\partial x} = B''x = \frac{\partial B_x}{\partial y}$$

If we place a sextupole in a dispersive region, where $x = \eta\delta$, then the field gradients are momentum-dependent:

$$\frac{\partial B_y}{\partial x} = B''\eta\delta$$

This gives us what we want: a momentum dependent field gradient. By inserting sextupoles into the lattice with the

appropriate signs and strengths, we can cancel the natural chromaticity, or achieve any value of chromaticity that we want.



Equation of motion, including first-order chromatic terms, and sextupoles (neglecting dipole weak focusing term):

$$x'' + xk(1-\delta) + \frac{m}{2}\left[x^2 - y^2\right] = \frac{\delta}{\rho}$$

$$y'' - yk(1-\delta) - mxy = 0$$

where the sextupole strength is (see Lect. 3, p 11)

$$m = \frac{B''}{B_0\rho}, \quad m[\mathrm{m}^{-3}] = 0.2998\,\frac{B''\left[\mathrm{T/m}^2\right]}{p_0[\mathrm{GeV/c}]}$$

Let $x = x_\beta + \delta\eta; \quad y = y_\beta$, where x_β and y_β represent the betatron oscillations. Then, substituting, we have

$$x_\beta'' + \delta\eta'' + \left(x_\beta + \eta\delta\right)k(1-\delta) + \frac{m}{2}\left[\left(x_\beta + \eta\delta\right)^2 - y_\beta^2\right] = \frac{\delta}{\rho}$$

$$y_\beta'' - y_\beta k(1-\delta) - m\left(x_\beta + \eta\delta\right)y_\beta = 0$$

Expand. The dispersion function changes in the presence of the sextupoles, obeying the equation.

$$\eta'' + \eta k + \delta\eta\left(\frac{m}{2}\eta - k\right) = \frac{1}{\rho}$$

This differs by the term of order δ from the usual equation for the dispersion. The betatron motion equations are

$$x_\beta'' + x_\beta\left(k + \delta(m\eta - k)\right) + \frac{m}{2}\left[x_\beta^2 - y_\beta^2\right] = 0$$

$$y_\beta'' - y_\beta\left(k + \delta(m\eta - k)\right) - m x_\beta y_\beta = 0$$

The gradient error now has the form

$$\Delta k_{x,y} = \pm(m\eta - k)\delta$$

There are also nonlinear terms. We will return to these in future lectures.

The chromaticity becomes

$$\xi_{x,y} = \pm\frac{1}{4\pi}\oint_C ds\,\beta_{x,y}(s)\left(m(s)\eta(s) - k(s)\right)$$

and can be adjusted to a desired value with appropriate use of the sextupole strength *m(s)*.

In principle, only two sextupoles are required to compensate the chromaticity in both planes. For two thin lens sextupoles, m_1 located at s_1 and m_2 located at s_2, of length L_s,

$$\xi_x = \xi_{x,natural} + \left[\frac{\beta_x(s_1)\eta(s_1)m_1}{4\pi} + \frac{\beta_x(s_2)\eta(s_2)m_2}{4\pi}\right]L_s$$

$$\xi_y = \xi_{y,natural} - \left[\frac{\beta_y(s_1)\eta(s_1)m_1}{4\pi} + \frac{\beta_y(s_2)\eta(s_2)m_2}{4\pi}\right]L_s$$

Note that the sextupoles have opposite effects in the two planes.

If $\xi_{x,natural} = \xi_{y,natural} = \xi_{natural}$, to get zero total chromaticity, we need to have

$$m_1 L_s = \frac{4\pi\xi_{natural}\left(\beta_x(s_2) + \beta_y(s_2)\right)}{\eta(s_1)\left(\beta_x(s_2)\beta_y(s_1) - \beta_y(s_2)\beta_x(s_1)\right)}$$

$$m_2 L_s = -\frac{4\pi\xi_{natural}\left(\beta_x(s_1) + \beta_y(s_1)\right)}{\eta(s_2)\left(\beta_x(s_2)\beta_y(s_1) - \beta_y(s_2)\beta_x(s_1)\right)}$$

Typically, to minimize the required sextupole strength, we want $\eta(s_1)$ and $\eta(s_2)$ large, and also $\beta_y(s_1) >> \beta_x(s_1)$, and $\beta_x(s_2) >> \beta_y(s_2)$. Then

$$m_1 L_s \approx \frac{4\pi\xi_{natural}}{\eta(s_1)\beta_y(s_1)}; \quad m_2 L_s \approx -\frac{4\pi\xi_{natural}}{\eta(s_2)\beta_x(s_2)}$$

Example: FODO lattice of our 500 m model accelerator. Place m_1 at a D quad, m_2 at an F quad. Let L_s=0.1 m. Then

$$\beta_y(s_1)=\beta_x(s_2)=16.83\,\text{m};\quad \beta_x(s_1)=\beta_y(s_2)=4.8\,\text{m}$$

$$\eta(s_2)=1.30\,\text{m};\quad \eta(s_1)=0.735\,\text{m};\ \xi_{natural}=-10.35$$

$$m_1 \approx -\frac{4\pi\times 10.35}{0.735\times 16.83\,\text{m}^2\times 0.1\text{m}}=-105\ \text{m}^{-3}$$

$$m_2 \approx \frac{4\pi\times 10.35}{1.30\times 16.83\,\text{m}^2\times 0.1\text{m}}=59\ \text{m}^{-3}$$

In practice, the required sextupole strength is distributed around the circumference in at least two families of sextupoles. The inevitable nonlinear effects from a distributed sextupole system are less than for two strong sextupoles. More than two

families may also be used, with a local correction in the interaction region.