We'll study differential forms more soon, but this suffices for now.

Supergravity

Classic result:

In $D=4$, the largest SUSY algebra is $\mathcal{N}=8$ (i.e, 8 gravitinos).
An attempt to construct a larger algebra would give massless spin >2 fields, which are hardly ever part of a consistent interacting theory.

This algebra contains $8\times4=32$ supercharges
$\Rightarrow$ SUSY can exist only in dimensions having spinors with ≤ 32 components
$\Rightarrow D\leq 11$.
(NB use tori to connect to $D=4$.)

In $D=11$ $\exists!$ SUSY action,
(Cremmer-Julia-Scherk)

$$S = \frac{1}{2\kappa^2}\int d^{11}x \sqrt{G}\left(R - \frac{1}{2}|F_4|^2\right)$$

$$- \frac{1}{12\kappa^2}\int A_3 \wedge F_4 \wedge F_4 \quad + \text{fermions}$$

with $dA_3 = F_4$

$d=11$ "maximal" $\mathcal{N}=1$ supergravity

But in $D=10$ $\exists$ 2 SUSY actions with 32 supercharges (= number of components in SUSY variation parameter ϵ) and hence 2 gravitinos.

11D: Q_α : $D=11$ Majorana spinor

$$\{Q_\alpha, \bar{Q}_\beta\} = -2P_\mu \Gamma^\mu_{\alpha\beta}$$

10D: $Q_\alpha \rightarrow Q^1_\alpha, \; Q^2_\alpha$

$$\underset{\substack{\text{11D M} \\ 32}}{Q_\alpha} \rightarrow \underset{\substack{\text{10D MW} \\ 16}}{Q^1_\alpha}, \; \underset{\substack{\text{10D MW} \\ 16'}}{Q^2_\alpha}$$

type IIA Two MW spinors of opposite chirality

$$\varphi^{(1)}_\alpha , \quad \varphi^{(2)}_\alpha$$
$$16 \qquad 16'$$

Field content:

graviton	g_{ij}
antisymm. tensor (two-form)	B_{ij}
scalar	Φ
one-form	C_i
three-form	C_{ijk}

for the bosons;

gravitinos	Ψ^i_s	of opposite
	$\hat{\Psi}^i_{s'}$	chirality
~~fermions~~ spinors (dilatinos)	λ_s	"
	$\hat{\lambda}_{s'}$	

Action:

$$S = \frac{1}{2\kappa_{10}^2}\int d^{10}x \sqrt{-G}\, e^{-2\Phi}\left(R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}|H_3|^2\right)$$
$$-\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-G}\left(|F_2|^2 + |\tilde{F}_4|^2\right)$$
$$-\frac{1}{4\kappa_{10}^2}\int B_2\wedge F_4\wedge F_4 \qquad + \text{fermions}$$

with $H_3 = dB_2$ (explain notation)

$$F_2 = dC_1$$

$$\tilde{F}_4 = dC_3 - C_1 \wedge H_3$$

As for type IIB:

Two MW spinors of same chirality:

$$Q^{(1)}_\alpha, \quad Q^{(2)}_\alpha$$

$$16 \quad 16$$

Field content:

g_{ij}	Ψ^i_s	
B_{ij}	Ψ^i_s	of
Φ	λ^i_s	same
C_0	$\hat{\lambda}^i_s$	chirality
C_2		
C_4		

Action:

$$S = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G}\, e^{-2\Phi} \left(R + 4 \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} |H_3|^2 \right)$$

$$- \frac{1}{4\kappa_{10}^2} \int d^{10}x \sqrt{-G} \left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2} |\tilde{F}_5|^2 \right)$$

$$- \frac{1}{4\kappa_{10}^2} \int C_4 \wedge H_3 \wedge F_3$$

With: $\tilde{F}_3 = F_3 - C_0 \wedge H_3$

$\tilde{F}_5 = F_5 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B_2 \wedge F_3$

Aside: must impose $* \tilde{F}_5 = \tilde{F}_5$

So from $(N, \tilde{N}) = (1,1)$ WS SCFT we get ... depending on the GSO projection ... the two $N=2$ $D=10$ supergravity theories.

This is the massless spectrum of the string; of course there are massive modes.

Next step: search for supersymmetric string vacua that are <u>four-dimensional</u>!

Supersymmetry

An extension of the Poincaré algebra by the anticommuting symmetries:

4D:
$$\{Q_\alpha, \bar{Q}_\beta\} = -2P_\mu \Gamma^\mu_{\alpha\beta}$$
$$[P^\mu, Q_\alpha] = 0$$

Q_α: 4_R Majorana spinor "supercharges" $\left(\bar{Q} = Q^T C = Q^\dagger \Gamma^0 \right)$

$\mathcal{N}=1$ (4 real supercharges)

also true in SUGRA, but proof trivial for global SUSY

Easy to show that supersymmetric states ($|\chi\rangle$ st $Q|\chi\rangle = 0$) are zero-energy solns of eom (basically, $H = Q^\dagger Q$.)

An action invariant under SUSY is, in particular, invariant under infinitesimal transformations mixing fermions + bosons,

eg. 2D $S = \frac{1}{2\pi}\int d^2\sigma \left(\partial_\alpha X^\mu \partial^\alpha X_\mu + \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu\right)$

invt under: $\delta X^\mu = \bar{\varepsilon}\psi^\mu$

$\delta\psi^\mu = \rho^\alpha \partial_\alpha X^\mu \varepsilon$

ε constant infinitesimal Majorana spinor (Grassmann)

Supersymmetric configurations obey the eom. So we will seek a supersymmetric vacuum. $|\Omega\rangle$

Suppose $|\Omega\rangle$ is such a vacuum. Then

(i) $Q_\alpha|\Omega\rangle = 0$ for all the unbroken supercharges Q_α

Now this is equivalent to

(ii) $\langle\Omega|\{Q_\alpha, \mathcal{O}\}|\Omega\rangle = 0 \quad \forall\, \mathcal{O}$

$$\left(\begin{array}{l} Q_\alpha|\Omega\rangle = 0 \Leftrightarrow \text{(ii)} \\ \text{(i)} \Rightarrow \text{(ii)} \quad \text{trivial} \\ \text{(ii)} \Rightarrow \text{(i)} \quad \text{consider } \mathcal{O} = \mathbb{1}. \end{array} \right)$$

However, fermion vevs vanish in a classical vacuum. So for $\mathcal{O}$ bosonic, $\Rightarrow \{Q_\alpha, \mathcal{O}\}$ fermionic, (ii) is guaranteed.

But for $\mathcal{O}$ fermionic,

$$\{Q_\alpha, \mathcal{O}\} = \delta_{Q_\alpha}\mathcal{O} \qquad \text{"variation of } \mathcal{O}\text{"}$$

$\Rightarrow$ to find a classical supersymmetric vacuum (i.e. SUSY unbroken at tree level) we must find a configuration of bosonic fields such that $\delta_{Q_\alpha}\mathcal{O} = 0 \;\forall$ fermions $\mathcal{O}$.

Can do a bit of algebra + guessing to determine variations that are invariances of S.

The full transformations are somewhat complicated, but simplify if all <u>field strengths</u> are taken to vanish:

$$\begin{cases} H_3 = F_1 = F_3 = F_5 = 0 & \text{IIB} \\ H_3 = F_2 = F_4 \quad = 0 & \text{IIA} \end{cases}$$

Recall the fermions are λ "dilatino"

Ψ^M "gravitino"

The <u>variation</u> under a susy generated by a constant (Grassmann) parameter ε (Majorana-Weyl) is:

$$\begin{cases} \delta\lambda = \frac{1}{2\sqrt{2}} \nabla_M \Phi \, \Gamma^M \Gamma \varepsilon \\ \delta \Psi^M = \nabla^M \varepsilon \end{cases}$$

Now one has to be careful to distinguish IIA + IIB in general, but for us focusing on <u>one</u> MW dilatino/gravitino pair will suffice.

$\delta\lambda = 0$ trivial: set $\Phi = \text{const.}$

$\delta \Psi^M = 0 \Leftrightarrow \exists\, \varepsilon$ st $\nabla^M \varepsilon = 0$

$\Leftrightarrow \exists$ "covariantly constant spinor"

Now we're trying to solve for the classical bg.

Easy! 10D flat space, $G_{\mu\nu} = \eta_{\mu\nu}$.

then $\varepsilon =$ constant is covariantly constant.

Less trivially, seek

$M_{10} = M_4 \times K$

$(G_{MN} \quad g_{\mu\nu} \quad g_{ij})$

and assume: M_4 maximally symmetric (Mink, dS, AdS)

K compact.

Now $\nabla_M \varepsilon = 0$

$$\Rightarrow [\nabla_M, \nabla_N]\varepsilon = 0$$

$$\Rightarrow \frac{1}{4} R_{MNPQ}\Gamma^{PQ}\varepsilon = 0$$

= exercise 9.6 of BBS

recall: $\nabla_\mu\nabla_\nu\omega_\rho - \nabla_\nu\nabla_\mu\omega_\rho = R_{\mu\nu\rho}{}^{\sigma}\omega_\sigma$

But maximal symmetry of M_4 $\Rightarrow$

$$R_{\mu\nu\rho\sigma} = \left(\frac{r}{12}\right)\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right)$$

r = 4D Ricci Scalar.

$$\Rightarrow 0 = \left(\frac{r}{12}\right)\left(\Gamma_{\mu\nu} - \Gamma_{\nu\mu}\right)\varepsilon = \frac{r}{6}\Gamma_{\mu\nu}\varepsilon \Rightarrow r = 0$$

$\Rightarrow$ Minkowski.

As for K, want

$$\frac{1}{4} R_{ijkl}\Gamma^{kl}\varepsilon = 0$$

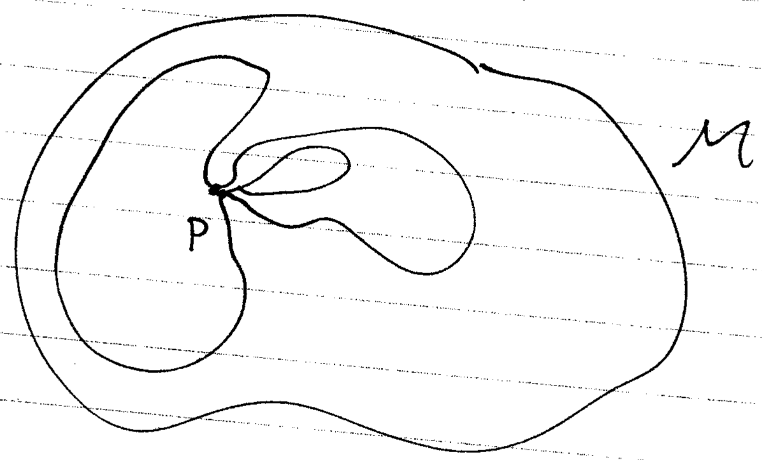
(or more simply $\nabla_i\varepsilon = 0$)

K must admit a covariantly constant spinor.

(Γ^{kl} rotations must sit in subgroup that leaves spinor invariant.)

Holonomy

Given a manifold M of dimension $d_{\mathbb{R}}$, consider the set of loops $\ell \subset M$ based at a point p.

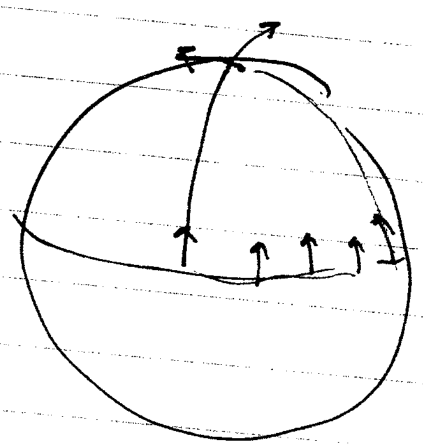


Transport a tangent vector around such a loop. It will come back rotated (in general).

Now (local) rotations of $T_pM \subset SO(d)$.

[technically: considering parallel transport in a vector bundle V with a connection, and holonomy $\subset$ structure group of V.]

Example.



These rotations form a group $h_p \subset SO(d)$.
the holonomy at p.

Now considering h_p $\forall\, p \in M$, we define the holonomy group, h, of M as the group of all (local) holomies associated with any $\ell \subset M$.

This is a crucial + basic property of a manifold.

Comment. $h \subset O(d)$ for unorientable manifolds. I've assumed orientable.

Comment. $h = SO(d) \Leftrightarrow$ not-very-symmetric M
$h = \mathbb{1} \Leftrightarrow$ very symmetric M (torus, flat space).

Now our parameter ε is a MW spinor in 10D.

$$SO(9,1) \to SO(3,1) \times SO(6)$$

$$\varepsilon(x,y) = \chi(x) \otimes \eta(y) = \chi_+ \otimes \eta_+ + \chi_- \otimes \eta_-$$

$$16 \to (2,4) \oplus (2',\bar{4})$$

(10D, chirality $\Rightarrow$ 4D, 6D chirality correlated.)

now $2 \oplus 2' \Leftrightarrow 4_R$ (Majorana)
$[(2)^* = 2']$

and so if we can find an invariant spinor η $1 \subset 4$ of $SO(6)$

then $(2,1) \oplus (2',1) \Leftrightarrow$ 4D Majorana spinor (4)

i.e one unbroken SUSY
4 " Supercharges

Upshot: for each 10D supersymmetry
and each 6D covariantly constant spinor η
we get one unbroken SUSY in 4D.

10D $\mathcal{N}=2 \Rightarrow$ if $\exists! \, \eta$ with $\nabla_i \eta = 0$, get 4D $\mathcal{N}=2$

We must seek out η with $\nabla \eta = 0$.

Since $SO(6) \cong SU(4)$ (as Lie algebras)

can view $\mathfrak{h} \subset SU(4)$.

Spinor of $SO(6)$: 4 $\bar{4}$ of $SU(4)$.
chirality $+$ $-$

If $\mathfrak{h} = SU(4)$, $\nexists\ \eta$ with $\nabla_i \eta = 0$.

Maximal subgroup $g \subset SU(4)$ leaving invariant spinor?

$$g = SU(3) \subset SU(4)$$

$$\eta = \left.\begin{pmatrix} 0 \\ 0 \\ 0 \\ \eta_0 \end{pmatrix}\right. \begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} SU(3) \\ \\ \end{array} \Bigg\} SU(4)$$

If $\mathfrak{h}(K) = SU(3)$, $\exists!$ spinor η with $\nabla_i \eta = 0$

i.e $4 \supset$ singlet
$\bar{4} \supset$ singlet*

Really, $\exists!$ positive-chirality 4, η, with $\nabla \eta = 0$
and $\exists!$ negative " $\bar{4}$, $\bar{\eta}$, with $\nabla \bar{\eta} = 0$
$\bar{\eta}$ = complex conjugate of η.

If $h(K) \subset SU(3)$ (strictly), more supersymmetry is preserv

eg if $h(K) = \mathbb{1}$ (T^6),

$$16 \to (2,4) \oplus (2',\bar{4}) \to 4 \times \begin{pmatrix} 4D \\ \mathcal{N}=1 \end{pmatrix}$$

$$\left.\begin{cases} 16'_{IIA} \to (2',4) \oplus (2,\bar{4}) \\ 16_{IIB} \quad (2,4) \oplus (2',\bar{4}) \end{cases}\right\} \to 4 \times \begin{pmatrix} 4D \\ \mathcal{N}=1 \end{pmatrix}$$

$\Rightarrow$ 4D $\mathcal{N}=8$ (32 supercharges).

So let's seek K with $h(K) = SU(3)$ exactly.

<u>In sum:</u> ~~[illegible]~~

ansatz $M_4 \times K$ for supersymmetric classical vacuum

leads to

M_4 = Minkowski space

K = special holonomy manifold

$h(K) \subseteq SU(3) \subset SU(4)$.

Key case, $h(K) = SU(3) \Leftrightarrow K$ is a Calabi-Yau manifold.

Theorem: (Berger–Simons)

Let (M,g) be a Riemannian manifold that is simply connected. Then either

(i) (M,g) is a symmetric space G/H and $h(M)=H$

or

(ii) g has irreducible holonomy, and

$h(M)$	$\dim_{\mathbb{R}} M$	remark: Kähler	Ricci-flat
$SO(d)$	d	no	no
$U(d)$	$2d$	yes	no
$SU(d)$	$2d$	yes	yes
$Sp(1)Sp(n)$	$4d$	no	no
$Sp(n)$	$4d$	yes	yes
$Spin(7)$	8	no	yes
G_2	7	no	yes

or

(iii) g has reducible holonomy and (M,g) is a product, each factor obeying (i) or (ii).

Corollary: Let $N_\pm$ = # ~~positive~~ negative chirality covariantly constant spinors even dim

N = # spinors odd dim

Then: $SU(2)$: $N_+ = 2,\ N_- = 0$

$SU(3)$ $N_+ = 1,\ N_- = 1$

$SU(4)$ $N_+ = 2,\ N_- = 0$

G_2 : $N = 1$

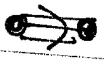
$Spin(7)$: $N_+ = 1,\ N_- = 0$

are the interesting cases with $N_{(\pm)} > 0$.

i.e. a metric admits a covariantly constant spinor
$\Leftrightarrow$ its holonomy is one of the Ricci-flat holonomies.

The study of supersymmetric string vacua is the study of manifolds with special holonomy.

[G_2: rank 2, obviously
dim 14



$$\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$
]

Cohomology

Recall d indep of metric.

Given $\omega_p \in \Omega_p(M)$,

ω_p is closed $\Leftrightarrow$ $d\omega_p = 0$

exact $\Leftrightarrow$ $\omega_p = d\alpha_{p-1}$ for α_{p-1} defined globally

Poincaré lemma: all ω_p are 'locally exact', i.e.
$\forall \omega_p \; \exists \; \alpha_{p-1}$ defined locally s.t. $\omega_p = d\alpha_{p-1}$

(cf. $dA = F$)

Now Ω_p is a vector space.

Define C_p: space of closed ω_p
E_p: " " exact

and $H^p(M, \mathbb{R}) \equiv C_p / E_p$.

(two closed ω_p, ω_p' equiv $\Leftrightarrow$ $\omega_p - \omega_p' = d\underbrace{\alpha_{p-1}}_{\text{global}}$)

$\dim(H^p(M,\mathbb{R})) \equiv$ Betti number b_p.

de Rham cohomology.

Homology

Let Σ_p be a closed p-dim submanifold (i.e. w/o ∂.)

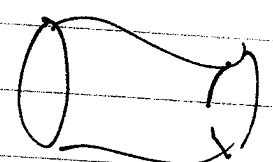
Then if ω_p closed,

$$\int_{\Sigma_p} \omega_p \quad \text{depends only on cohomology class of } \omega_p :$$

$$\int_{\Sigma_p} \omega_p - \int_{\Sigma_p} \omega_p' = \int_{\Sigma_p} d\alpha_{p-1} = \int_{\partial \Sigma_p} \alpha_{p-1} = 0$$

Used Stokes' thm $\displaystyle\int_{\Sigma} d\omega = \int_{\partial\Sigma} \omega$.

And if



$\Sigma_p \quad B_{p+1} \quad \Sigma_p'$

Then
$$\int_{\Sigma_p} \omega_p - \int_{\Sigma_p'} \omega_p = \int_{\partial B_{p+1}} \omega_p = \int_{B_{p+1}} (d\omega)_{p+1} = 0$$

$\Rightarrow \int$ of closed forms depends only on cycles/boundaries