We'll study differential forms more soon, but this suffices for now.

**Supergravity**

**Classical result:**

In D=4, the largest SUSY algebra is $N=8$ (i.e., 8 gravitinos).

An attempt to construct a larger algebra would give massless spin-3/2 fields, which are hardly ever part of a consistent theory.

This algebra contains $8 \times 4 = 32$ supercharges

⇒ SUSY can exist only in dimensions having space with $\leq 32$ components

⇒ D=11.

(NB use tori to connect to D=4.)
In $D=11$ $\mathcal{N}=1$ SUSY action,

$S = \frac{1}{2k^2} \int d^7x \sqrt{G} \left( R - \frac{1}{2} |F_4|^2 \right) - \frac{1}{12k^2} \int F_3 \wedge F_4 \wedge F_4 + \text{fermions}$

with $dF_3 = \nabla_4$

$d=11$ "maximal" $\mathcal{N}=1$ supergravity

But in $D=10$ $\mathcal{N}=2$ SUSY actions with

32 $\mathbb{C}$ supercharges (= number of components in SUSY variation and hence 2 gravitinos)

1D: $\Phi_x$ : $D=11$ Majorana spinor

$\{\Phi_x, \Phi_y\} = -2\eta_{xy} F^\mu_\mu$

10D: $\Phi_x \to \Phi_x^1, \Phi_x^2$

\[ \begin{array}{ccc} 1D M & 10D MW & 10D MW \\ 32 & 16 & 16' \end{array} \]
Two $\text{MW}$ spinors of opposite chirality

\[ \Phi_{x}^{(0)}, \Phi_{y}^{(0)} \]

\[ (16, 16') \]

Field content:
- gravitons $g_{ij}$
- antisymp tensor (two-form) $B_{ij}$
- scalar $\tilde{G}$
- one-form $C_{i}$
- three-form $C_{ijk}$

For the bosons:
- gravitinos $\tilde{G}_{i}$ of opposite
- $\tilde{G}_{i}$ chirality

Fermions
- spinors $\tilde{\psi}$

Action:

\[ S = \frac{1}{2\kappa_{0}^{2}} \int d^{10} \sqrt{G} \left( R + 4 \partial \Phi \partial \Phi - \frac{1}{2} \nabla \Phi \nabla \Phi \right) \]

\[ - \frac{1}{4 \kappa_{0}^{2}} \int d^{10} \sqrt{G} \left( \frac{1}{2} \Delta \Phi \Delta \Phi + \partial \Phi \partial \Phi \right) \]

\[ - \frac{1}{4 \kappa_{0}^{2}} \int d^{10} B_{2} \wedge F_{4} \wedge F_{4} \quad + \text{fermions} \]
with \( H_3 = dB \)

\( f_2 = dG \)

\( f_3 = dC_3 - G \wedge \star_3 \) 

As for type IIB:

Two M5 strings of (same) chirality:

\( \phi_1, \phi_2 \)  

16 16

Field content:

\( g_{ij} \)

\( B_{ij} \)

\( \phi^i \) of \( i \) \( \phi^i \) of

\( \phi^i \) of some chirality

\( C_{ij} \)

\( C^i \) \( C \) \( C \)

Action:

\[
S = \frac{1}{2 \kappa^2} \int d^8x \sqrt{-g} \left( R + 4 g_{ij} \phi^i \phi^j - \frac{1}{2} |H_4|^{2} \right) \\
- \frac{1}{4 \kappa^2} \int d^8x \left( |F_{ij}^{[8]}|^{2} + |F_{i}^{[8]}|^{2} + \frac{1}{2} |F_{i}^{[8]}|^{2} \right) \\
- \frac{1}{4 \kappa^2} \int d^4x \ W \wedge H_3 \wedge F_3
\]
With:  
\[ \tilde{F}_3 = F_3 - C_0 \wedge H_3 \]
\[ \tilde{F}_5 = F_5 - \frac{1}{2} C_0 \wedge H_3 + \frac{1}{2} B_2 \wedge F_3 \]

Aside: must impose \( * \tilde{F}_5 = \tilde{F}_5 \)

So from \((N, \phi) = (4,1)\), we get ...
depending on the GSO projection ... the two \( N=2 \) \( D=10 \) supergravity theories.

This is the massless spectrum of the string; of course, there are massive modes.

Next step: search for supersymmetric string vacua that are four-dimensional.
Supersymmetry

An extension of the Poincaré algebra by the anticommuting symmetries:

\[ \{ \Phi_\alpha, \Phi_\beta \} = -2P^\mu \Gamma^{\mu}_{\alpha \beta} \]

\[ [P^\mu, \Phi_\alpha] = 0 \]

\[ \Phi_\alpha : \text{4}\tau \text{ Majorana spinor} \quad (\bar{\Phi} = \Phi^T \Gamma^0 = \Phi^* \Gamma^0) \]

"supercharges"

\[ N=1 \quad (4 \text{ real supercharges}) \]

Easy to show that supersymmetric states (1|\Psi\rangle \leftrightarrow |\Phi\rangle) are zero-energy states of even (basically, \( H = \Phi^+ \Phi \))

An action invariant under SUSY is in particular, invariant under infinitesimal transformations mixing

fermions + bosons,

\[ \text{eg. 2D} \quad S = \frac{1}{4m \hbar} \int d^2x \left( \partial^\mu \Phi \partial^\nu \Phi + \bar{\Phi} \partial^\mu \Gamma^{\mu}_{\alpha \beta} \partial^\nu \Phi \right) \]

invariant under:

\[ \delta \Phi^\mu = \int d^2x \epsilon^\mu, \quad \delta \bar{\Phi}^\mu = \int d^2x \bar{\epsilon}^\mu \]

\( \Sigma \) constant infinitesimal Majorana spinor

(Gravit...
Supersymmetric configurations obey the e.o.m. So we will seek a \textit{supersymmetric} vacuum. Suppose \(|\Sigma\rangle\) is such a vacuum. Then

\[(i) \quad \langle \Sigma | \{q_k, \theta\} | \Sigma \rangle = 0 \quad \text{for all supercharges} \quad q_k\]

Now this is equivalent to

\[(ii) \quad \langle \Sigma | \{q_k, \theta\} | \Sigma \rangle = 0 \quad \text{A} \quad \theta\]

\[
\begin{align*}
\langle \Sigma | q_k | \Sigma \rangle = 0 & \iff (ii) \\
(i) \Rightarrow (ii) & \text{ trivial} \\
(ii) \Rightarrow (i) & \text{ consider } \theta = 1.
\end{align*}
\]

However, fermion vevs vanish in a classical vacuum. So for \(\theta \text{ bosonic}\), \(\{q_k, \theta\} \text{ Fermionic} \) (ii) is guaranteed.

But for \(\theta \text{ fermionic}\),

\[
\{q_k, \theta\} = \delta_{q_k} \theta
\]

"variation of \(\theta\)."

\[\Rightarrow \text{ to find a classical supersymmetric vacuum (i.e. SUSY)}
\]

we must find a configuration of bosonic fields such that \(\delta_{q_k} \theta = 0 \quad \text{A} \quad \theta\).
Can do a bit of algebra + guessing to determine variations that are invariances of $S$.

The full transformations are somewhat complicated but simplify if all field strengths are taken to vanish:

\[
\begin{align*}
&H_3 = F_i = F_3 = F_8 = 0 & \text{II B} \\
&H_3 = F_8 = F_4 = 0 & \text{II A}
\end{align*}
\]

Recall the fermions are $\lambda$ "dilatino" $\Phi^M$ "gravitino".

The variation under a susy generated by a constant (Grassmann) parameter $\bar{\varepsilon}$ (Majordona-Weyl) is:

\[
\begin{align*}
\delta \lambda &= \frac{1}{2\Delta_2} \nabla_M \Phi^M \Gamma^3 \varepsilon \\
\delta \Phi^M &= \nabla_M \varepsilon
\end{align*}
\]

Now one has to be careful to distinguish II A + II B in general, but for us focusing on one MW dilatino gravitino, our will suffice.
8 \Phi = 0 \quad \text{trivial: set } \Phi = \text{const.} \\
8 \Phi^M = 0 \iff \exists \Xi \text{ s.t. } \nabla^M \Xi = 0 \\
\iff \exists \text{ "covariantly constant spinor"} \\

Now we're trying to solve for the classical bg. 

Easy! 10D flat space; \( G_{\mu \nu} = 0 \).

Then, \( \Xi = \text{constant} \) is covariantly constant.

Less trivially, seek

\[ M_{10} = M_4 \times K \]

\( (g_{MN}, g_{\mu \nu}, g_{ij}) \)

and assume: \( M_4 \) maximally symmetric (Mink, dS, AdS) 
\( K \) compact.
Now $\nabla_M \Sigma = 0$

$\Rightarrow [\nabla_M, \nabla_N] \Sigma = 0$

$\Rightarrow \frac{1}{4} R_{MNPQ} \Gamma^{PQ} \Sigma = 0$

But maximal symmetry of $M_4 \Rightarrow R_{MNPQ} = R_{MNPQ}^{\Sigma}$

$R_{MNPQ} = \left( \frac{r}{16} \right) \left( \sigma_p \sigma_q - \sigma_q \sigma_p \right)$

$r = 4D$ Ricci scalar.

$\Rightarrow 0 = \left( \frac{r}{16} \right) \left( \Gamma_M - \Gamma_M \right) \Sigma = \frac{r}{16} \Gamma_M \Sigma \Rightarrow r = 0$

$\Rightarrow$ Minkowski.

As for $K$, want

$\frac{1}{4} R_{MNPQ} \Gamma^{PQ} \Sigma = 0$

(or more simply $\nabla_i \Sigma = 0$)

$K$ must admit a covariantly constant spinor.

$\Gamma_{kl}$ rotations must sit in subgroup that leaves spinor invariant.
Given a manifold $M$ of dimension $d$, consider the set of loops $lC_M$ based at a point $p$.

Transport a tangent vector around such a loop. It will come back rotated (in general).

Now (local) rotations of $TM \subset SO(d)$.

[Technically, consider parallel transport in a vector bundle $V$ with a connection and holonomy as structure group of $V$.]

Example.
These relations form a group $h_p \subset \text{SO}(d)$.

The holonomy at $p$. Now considering $h_p \times \text{p} \subset \text{M}$, we define the holonomy group of $\text{M}$ as the group of all (local) holonomies associated with any $\text{p} \subset \text{M}$.

This is a very basic property of a manifold.

Comment: $h \subset \text{O}(d)$ for unorientable manifolds. I've assumed orientable.

Comment: $h = \text{SO}(d) \iff$ not-very-symmetric $\text{M}$

$h = \mathbb{1} \iff$ very symmetric $\text{M}$ (torus, flat space).

Now our parameter $\varepsilon$ is a MW' spinor in 10D.

$\text{SO}(9,1) \to \text{SO}(3,1) \times \text{SO}(6)$

$\varepsilon(x) = \varepsilon^1 + \varepsilon^2 \mp \varepsilon^3 + \varepsilon^4 + \varepsilon^5 + \varepsilon^6 + \varepsilon^7 + \varepsilon^8 + \varepsilon^9 + \varepsilon^{10}$

$16 \to (\mathbb{1}, 4) \oplus (\mathbb{1}', 4)$

(10D, chirality $\mp 4, 6$ chirality correlated)

Now $2 \oplus 2' \leftrightarrow 4_R$ (Majorana)

$[(\mathbb{1})^* = 2']$
and so if we can find an invariant spinor \( \chi \) of \( So(6) \) then \((2,1) \oplus (2',1) \iff 4D \text{ Majorana spinor} \)

\[ (4) \]

i.e. one unbroken SUSY

4 D supercharges

Upshot: for each 10D supersymmetry

and each 6D covariantly constant spinor \( \eta \)

we get one unbroken SUSY in 4D.

10D \( N=2 \iff \exists ! \eta \) with \( \nabla \eta = 0 \), get 4D \( N=2 \)

We must seek out \( \eta \) with \( \nabla \eta = 0 \).
Since \( \text{SO}(6) \cong \text{SU}(4) \) \hspace{1cm} \text{(as Lie algebras)}

we can view \( h \subset \text{SU}(4) \).

Spinor of \( \text{SO}(6) \): \( 4 + \bar{4} \) of \( \text{SU}(4) \).

\[ \text{chirality:} \quad + - \]

If \( h = \text{SU}(4) \), \( \exists \eta \) with \( \nabla \eta = 0 \).

Maximal subgroup \( g \subset \text{SU}(4) \) leaving invariant spinor? 

\[ g = \text{SU}(3) \subset \text{SU}(4) \]

\[ \eta = \begin{pmatrix} 0 \\ 0 \\ \bar{0} \\ 0 \end{pmatrix} \]

If \( h(K) = \text{SU}(3) \), \( \exists \) spinor \( \eta \) with \( \nabla \eta = 0 \)

i.e. \( 4 \) \text{ singlet} \hspace{1cm} \overline{4} \text{ singlet}^* 

Really, \( \exists \) positive-chirality \( \frac{1}{2} \eta \), with \( \nabla \eta = 0 \)
and \( \exists \) negative \( \text{chirality} \quad \frac{1}{2} \bar{\eta} \) with \( \nabla \bar{\eta} = 0 \)

\( \bar{\eta} = \text{complex conjugate of } \eta \).
If $\mathfrak{h}(K) < \text{SU(3)}$, more supersymmetry is present

\[ \mathfrak{h}(K) = 1 \] (76)

\[
\begin{align*}
16 & \rightarrow (3,4) \oplus (\bar{3}, \bar{4}) \rightarrow 4 \times (N=1) \\
\{ 16' \rightarrow (\bar{2},4) \oplus (2,\bar{4}) \rightarrow 4 \times (N=1) \\
\{ 16 \rightarrow (2,4) \oplus (\bar{2},4) \rightarrow 4 \times (N=1) \\
\end{align*}
\]

\[ \Rightarrow 4D \ N=8 \ (32 \ \text{supercharges}). \]

So let's seek $K$ with $\mathfrak{h}(K) = \text{SU(3)}$.

In sum:

\[ \text{Ansatz: } M_4 \times K \text{ for supersymmetric classical vacuum} \]

leads to

\[ M_4 = \text{Minkowski space} \]

\[ K = \text{special holonomy manifold} \]

\[ \mathfrak{h}(K) \leq \text{SU(3)} < \text{SU(4)} \]

\[ \text{Key case, } \mathfrak{h}(K) = \text{SU(3)} \Leftrightarrow K \text{ is a Calabi-Yau manifold}. \]
Theorem: (Berger–Simons)

Let \((M,g)\) be a Riemannian manifold that is simply connected. Then either

(i) \((M,g)\) is a symmetric space \(G/H\) and \(h(M)=H\)

or

(ii) \(g\) has irreducible holonomy, and

<table>
<thead>
<tr>
<th>(h(M))</th>
<th>(\dim_{\text{tr}} M)</th>
<th>(\text{remark})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(SO(d))</td>
<td>(d)</td>
<td>(\text{no})</td>
</tr>
<tr>
<td>(O(d))</td>
<td>(2d)</td>
<td>(\text{yes})</td>
</tr>
<tr>
<td>(SU(d))</td>
<td>(2d)</td>
<td>(\text{yes})</td>
</tr>
<tr>
<td>(Sp(1)Sp(n))</td>
<td>(4d)</td>
<td>(\text{no})</td>
</tr>
<tr>
<td>(Sp(n))</td>
<td>(4d)</td>
<td>(\text{yes})</td>
</tr>
<tr>
<td>(Spin(7))</td>
<td>(8)</td>
<td>(\text{no})</td>
</tr>
<tr>
<td>(G_2)</td>
<td>(7)</td>
<td>(\text{no})</td>
</tr>
</tbody>
</table>

or

(iii) \(g\) has reducible holonomy and \((M,g)\) is a product, each factor obeying (i) or (ii).

Corollary: Let \(N_+ = \#\) positive chirality spinors even dim  
\[(\text{coinciding constant})\]  
\(N = \#\) spinors odd dim
Then:  $SU(2)$:  \(N^+=2, N^-=0\)
$SU(3)$:  \(N^+=1, N^-=-1\)
$SU(4)$:  \(N^+=2, N^-=0\)
$G_2$:  \(N=1\)
$Sp(1)$:  \(N^+=1, N^-=0\)

are the interesting cases with \(N_{\text{tot}} > 0\).

i.e. a metric admits a covariantly constant spinor \(\iff\) its holonomy is one of the Ricci-flat holonomies.

\[ G_2: \text{rank 2, obviously} \]
\[ \dim 14 \]
\[ \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \]

The study of supersymmetric string vacua is the study of manifolds with special holonomy.
Cohomology

Recall $d$ is independent of metric.

Given $\omega_p \in \Omega^p(M)$,

$\omega_p$ is closed $\iff$ $d\omega_p = 0$

exact $\iff$ $\omega_p = dx_{p-1}$ for $\omega_{p-1}$ defined globally

Poincaré lemma: all $\omega_p$ are locally exact, i.e.

$\forall \omega_p \in \Omega^p$, defined st. $d\omega_p = d\omega_{p-1}$ locally

(cf. $dA = F$)

Now $\Omega^p$ is a vector space.

Define $C_p$: space of closed $\omega_p$

$E_p$: "exact"

and $H^p(M, \mathbb{R}) \equiv C_p / E_p$

(two closed $\omega_p, \omega'_p$ open $\iff$ $\omega_p - \omega'_p = d\omega_{p-1}$)

$\dim(H^p(M, \mathbb{R})) \equiv$ Betti number $b_p$

de Rham cohomology
Homology

Let \( \Sigma_p \) be a closed \( p \)-dim submanifold (w/o \( \partial \)).

Then if \( \Sigma_p \) closed,

\[
\int_{\Sigma_p} \omega_p \text{ depends only on cohomology class of } \omega_p :
\]

\[
\int_{\Sigma_p} \omega_p - \int_{\Sigma_p'} \omega_p' = \int_{\partial \Sigma_p} \omega_{p-1} = \int_{\partial \Sigma_p} 0 = 0
\]

Used Stokes' thm \( \int_{\Sigma} d\omega = \int_{\partial \Sigma} \omega \).

And if

\[
\begin{array}{c}
\Sigma_p \\
\uparrow
\end{array}
\begin{array}{c}
B_{p+1}
\end{array}
\begin{array}{c}
\Sigma_p'
\end{array}
\]

Then \( \int_{\Sigma_p} \omega_p - \int_{\Sigma_p'} \omega_p' = \int_{\partial \Sigma_p} \omega_{p-1} - \int_{\partial B_{p+1}} (d\omega)_{p-1} = 0 \)

\( \Rightarrow \) \( \int \) of forms depends only on cycles