Superstrings: GSO Projection + Spacetime Spectrum

we started from the action

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma \left(\partial_\alpha X^\mu \partial^\alpha X_\mu + \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu \right)$$

and, decomposing to $1_\mathbb{R}$ MW spinors,

$$\mathcal{L}_F = (d^2w)\left[\psi^\mu \bar{\partial} \psi_\mu + \tilde{\psi}^\mu \partial \tilde{\psi}_\mu\right]$$

$$\bar{\psi} = \psi^+ i\rho^0$$
$$\{\rho^\alpha, \rho^\beta\} = 2\eta^{\alpha\beta}$$

Canonical quantization gave

$$[\alpha^\mu_m, \alpha^\nu_n] = m\delta_{m+n}\,\eta^{\mu\nu} = [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n]$$
$$\{\psi^\mu_m, \psi^\nu_n\} = \delta_{m+n}\,\eta^{\mu\nu} = \{\tilde{\psi}^\mu_m, \tilde{\psi}^\nu_n\}$$

so $\exists$ wrong-sign (anti) [,] corresp. to $\mu, \nu = 0$.

Conformal symmetry allows us to set

$$X^+(\sigma,\tau) = x^+ + p^+\tau \qquad \alpha^+ \text{ oscillators removed}$$

then in LCG we could solve for

$X^-(\sigma,\tau)$ in terms of transverse modes $X^i(\sigma,\tau)$.

with $[\alpha^i_m, \alpha^j_n] = m\delta_{m+n}\delta^{ij}$

Similarly, SC symmetry allows us to set

$$\psi^+(\sigma,\tau) = 0$$

and we can solve for ψ^- in terms of ψ^i.

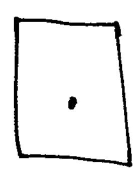
with $\{\psi_m^i, \psi_n^j\} = \delta_{m+n}\delta^{ij}$.

Moreover, we considered two distinct b.c.:

$$\psi^\mu(w+2\pi) = \pm\psi(w) \quad \begin{pmatrix} R \\ NS \end{pmatrix}, \;)$$

$$\tilde\psi^\mu(w+2\pi) = \pm\tilde\psi(w) \quad (\;, \begin{pmatrix} R \\ NS \end{pmatrix})$$

+ : R
− : NS

finally, mapping $w \to z$, 

$$\psi^\mu(z) = \sum_{r\in\mathbb{Z}+\nu} \frac{\psi_r^\mu}{z^{r+1/2}} \qquad \tilde\psi^\mu(\bar z) = \sum_{r\in\mathbb{Z}+\tilde\nu} \frac{\tilde\psi_r^\mu}{\bar z^{r+1/2}}$$

$(\nu,\tilde\nu) =$	$(0,0)$	RR
	$(0,\frac12)$	RNS
	$(\frac12,0)$	NSR
	$(\frac12,\frac12)$	NSNS

so, $\overset{(-)}{z}{}^{\mathbb{Z}/2}$ R

$\overset{(-)}{z}{}^{\mathbb{Z}}$ NS

for the vacuum:

may take $|0\rangle_{NS}$ to obey

$$\psi^\mu_r |0\rangle_{NS} = 0 \quad r>0 \quad (r = \tfrac{1}{2}, \tfrac{3}{2}, \ldots)$$

then $\psi^\mu_{-\frac{1}{2}}|0\rangle_{NS}$ is 1[st] excited state.
Lorentz ⇒ massless.

with general mass formula $\alpha' M^2 = \sum_{n=1}^{\infty} \alpha^i_{-n}\alpha^i_n + \sum_{r=\frac{1}{2}}^{\infty} r\,\psi^i_{-r}\psi^i_r - \frac{1}{2}$.

In R sector,

$$\psi^i_r|0\rangle_R = 0 \quad r>0$$

but $\psi^i_0|0\rangle_R \neq 0$

because $\{\psi^i_0, \psi^j_0\} = \delta^{ij}$

So the R ground states in LCG furnish a spinor rep'n of SO(8),

since $\{\sqrt{2}\psi_0^i, \sqrt{2}\psi_0^j\} = 2\delta^{ij}$ is the spin(8) Clifford algebra

Now one can build the corresponding spinor by writing down Γ matrices for SO(8) (or spin(8)).

One finds 16-component spinors:

$$\Gamma^1, \ldots \Gamma^8 \text{ are } 16 \times 16$$

and $\Gamma^9 \equiv \Gamma^1 \times \cdots \times \Gamma^8$ obeys $\{\Gamma^9, \Gamma^i\} = 0$

so $\underline{16} \to \underline{8} + \underline{8}'$

$\hookrightarrow \Gamma^9 \underline{8} = +\underline{8}$

$\Gamma^9 \underline{8}' = -\underline{8}'$.

But SO(8) has a third $\underline{8}$, $\underline{8}_v$. (the obvious one).

We may take $|0\rangle_R$ to be on $\underline{8}$ or on $\underline{8'}$ of SO(8).

One finds the mass formula

$$\alpha' M^2 = \sum_{n \geq 1}^{\infty} \alpha^i_{-n} \alpha^i_n + \sum_{r=1}^{\infty} r \psi^i_{-r} \psi^i_r$$

in the R sector.

(Here I've set the :: constant to zero, as one can establish using SC symmetry.)

Upshot:

$\|0\rangle_{NS}$	$\alpha' M^2 = -\frac{1}{2}$	tachyon
$\psi^i_{-1/2}\|0\rangle_{NS}$	$\alpha' M^2 = 0$	$\underline{8}_v$ massless vector
$\|0\rangle^+_R$	$\alpha' M^2 = 0$	$\underline{8}$ massless +-chirality spinor
$\|0\rangle^-_R$	$\alpha' M^2 = 0$	$\underline{8'}$ " − " ".

Now, we'll define a consistent projection that gives a theory with spacetime SUSY.

GSO Projection

Problem: do not like $|0\rangle_{NS}$ tachyon.

Idea: define operator ~~$(-1)^F$~~ Θ

with
$$\begin{cases} \Theta^2 = 1 \\ \Theta|0\rangle_{NS} = -1 \\ \Theta\psi^i_{-1/2}|0\rangle_{NS} = +1 \end{cases}$$

and project onto Θ's $+$-eigenspace.

We'll define

$$(-1)^F \equiv (-1)^{\sum_{r=1/2}^{\infty} \psi^i_{-r}\psi^i_r + 1} \quad \text{in NS sector}$$

$$\equiv \Gamma^9 (-1)^{\sum_{r=1}^{\infty} \psi^i_{-r}\psi^i_r} \quad \text{in R sector}$$

Then, $\{(-1)^F, \psi^i_{-r}\} = 0$

since any fermion raising op. ψ^i_{-r} changes F.

F = WS fermion number mod 2.

(NB. defined up to sign; we've chosen conventions for sign of $|0\rangle_{NS}$, $|0\rangle_R$.)

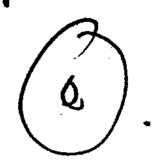
Then
$$(-1)^F |0\rangle_{NS} = -|0\rangle_{NS}$$
$$(-1)^F \psi^i_{-1/2} |0\rangle_{NS} = + \psi^i_{-1/2} |0\rangle_{NS}$$
$$(-1)^F |0\rangle^{\pm}_R = \pm |0\rangle^{\pm}_R$$

[Remark: we'll see that (the naive), unprojected spectrum contains a massless spin-3/2 field (massless gravitino). Such a Thy. is consistent only if it is supersymmetric.]

Next, we again require consistency at the quantum level, specifically invariance of $\int D[\,]$ measure under modular transformations.

'modular invariance'.

Namely, compute vacuum amplitude on T^2,

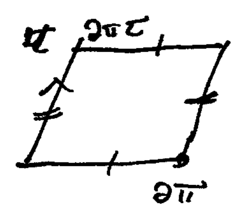


T^2 has a complex structure, the 'modular parameter' τ.

Can write T^2 with metric

$$ds^2 = dw\, d\bar{w}$$

and $\left\{ \begin{array}{l} w \cong w + 2\pi \\ w \cong w + 2\pi\tau \end{array} \right\}$



Now under $\tau' = \dfrac{a\tau + b}{c\tau + d}$ with $ad - bc = 1$,

we find an equivalent torus.
The above is a diff. of T^2, a 'large coordinate transform'.

Group: $PSL(2, \mathbb{Z})$ since $a,b,c,d \mapsto -a,-b,-c,-d$

Define periodicity + modular parameter by

$$0 \le \sigma_1 \le 2\pi \qquad \sigma_1 \cong \sigma_1 + 2\pi$$
$$0 \le \sigma_2 \le 2\pi \qquad \sigma_2 \cong \sigma_2 + 2\pi$$

$$w = \sigma^1 + \sigma^2 \tau \qquad \tau \in \mathbb{C}$$
$$ds^2 = dw\, d\bar{w}$$

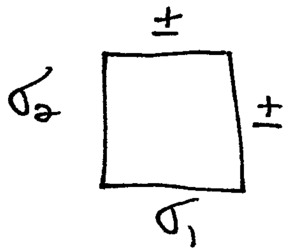
Under $\begin{pmatrix} \sigma^{1\prime} \\ \sigma_2' \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \begin{pmatrix} \sigma^1 \\ \sigma^2 \end{pmatrix}$

and $\tau' = \dfrac{a\tau + b}{c\tau + d}$

The metric has the same form, but with $\tau \to \tau'$!

(use) $w = \sigma_1 + \sigma_2 \tau$
$= \sigma_1'(d + c\tau) + \sigma_2'(b + a\tau)$
$= (c\tau + d)\left[\sigma_1' + \sigma_2' \tau'\right]$

Under such modular transformations, periodicites on T^2 change.



eg. take $a=d=0$ $b=1$ $c=-1$

$$\begin{pmatrix}\sigma_1'\\ \sigma_2'\end{pmatrix} = \begin{pmatrix}-\sigma_2\\ \sigma_1\end{pmatrix}$$

so $(\alpha,\beta) \Leftrightarrow (\beta,\alpha)$ eg $(+,-) \leftrightarrow (-,+)$.

[This is $\tau' = -\frac{1}{\tau}$]

eg. $a=b=d=1$, $c=0$ [This is $\tau' = \tau+1$]

$$\begin{pmatrix}\sigma_1'\\ \sigma_2'\end{pmatrix} = \begin{pmatrix}\sigma_1+\sigma_2\\ \sigma_2\end{pmatrix}$$

Then if get $-$ on σ_1
$-$ on σ_2

on $\sigma_1+\sigma_2$, get $+$.

so if $\alpha, \beta \in \pm 1$,

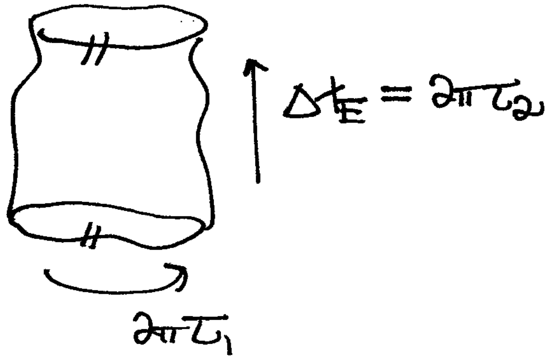
$$(\alpha,\beta) \to (\alpha\beta, \beta)$$

eg $(-,-) \to (+,-)$.

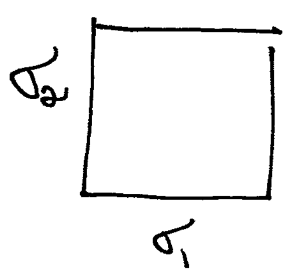
So far this was for a Euclidean T^2.

What does this have to do with the consistency of string theory?

$$\langle 1 \rangle_{T^2(\tau)} = Z(\tau)$$



In quantum stat mech, fermions have antiperiodic bc in Euclidean time.



$(\sigma_1 \quad \sigma_2)$

fermion: $(\quad -)$

NS
R
$\begin{pmatrix} + \\ - \end{pmatrix}$

We're going to study the role of $\psi(w)$ in $Z(\tau)$.
↑ fermion on WS!

Recall in QM

$$Z = \mathrm{Tr}(e^{-\beta H})$$

$$\left(= \sum_\alpha \langle \alpha | e^{-\beta H} | \alpha \rangle \right.$$

$$\left. = \sum_\alpha e^{-\beta E_\alpha} \right).$$

Can also compute 'twisted Z'

$$Z_F = \mathrm{Tr}\, (-1)^F e^{-\beta H}$$

$$= \sum_\alpha \langle \alpha | (-1)^F e^{-\beta H} | \alpha \rangle$$

or, $Z + Z_F = \mathrm{Tr}\left\{ [1 + (-1)^F] e^{-\beta H} \right\}$

which projects onto $(-1)^F = +1$, among states of the same energy.

Can think: $e^{-\beta H} | \phi, t_E = 0 \rangle = | \phi, t_E = 2\pi \tau_2 \rangle$

$|\alpha\rangle \leftrightarrow |\phi, t_E\rangle$ evolve with 'natural' bc

$(-1)^F e^{-\beta H} | \phi, t_E = 0 \rangle = -| \phi, t_E = 2\pi \tau_2 \rangle$

= time-evolution to $2\pi\tau_2$ with <u>opposite</u> bc.

Well, M.I. ⇒ NS → R → ?
(− −) (+ −) (− +)

(?) corresponds to NS sector with $\underline{\underline{+}}$ bc in t_E.

⇔ 'twisted' $Z(\tau)$,

$$Z_F^{NS} = \mathrm{Tr}\left[e^{-\beta H}(-1)^F\right] \quad \left(\begin{array}{l}\beta = 2\pi\, \mathrm{Im}\,\tau \\ 2\pi\tau = \alpha + i\beta\end{array}\right)$$

So: 1) if ∃ NS states, ∃ R states;
2) must impose projection $[1+(-1)^F]$ in NS sector.

<u>Two</u>-loop M.I. ⇒ also impose in R sector.

⇒ result is that we sum over all possible b.c.

(+ +)	R_F
(− −)	NS
(− +)	NS_F
(+ −)	R

i.e. we compute $\mathrm{Tr}\left\{\left[1+(-1)^F\right]e^{-\beta H}\right\}$

which projects onto $(-1)^F = +1$.

So, we need:

$$(--)\oplus(-+)$$
$$NS \oplus NS_F = \left[1+(-1)^F\right] NS$$

$$(+-)\oplus(++) = \left[1+(-1)^F\right] R$$
$$R + R_F$$

but, can choose overall sign of $(-1)^F$:

$$\begin{cases} NS+ \\ NS- \\ R+ \\ R- \end{cases}$$

$$(-1)^F = (\pm 1)(-1)^{\sum_{r=1/2}^{\infty} \psi^i_{-r}\psi^i_r + 1} \qquad \text{NS sector}$$

my choice

$$(\pm 1)\,\Gamma^9 (-1)^{\sum_{r=1}^{\infty} \psi^i_{-r}\psi^i_r} \qquad \text{R sector}$$

we'll take + for both to define $(-1)^F$,

but remember we're allowed to project onto + or −.

NS+ means

$$\left[\prod_{i=2}^{D-1}\prod_{n=1}^{\infty}\left(\alpha_{-n}^{i}\right)^{N_{in}}\right]\prod_{j=1}^{m}\psi_{-r_j}^{i_j}\,|0\rangle_{NS} \qquad m \text{ odd}$$

NS− $\qquad m$ even

R+ : $$\left[\prod_{i=2}^{D-1}\prod_{n=1}^{\infty}\left(\alpha_{-n}^{i}\right)^{N_{in}}\right]\prod_{j=1}^{m'}\psi_{-r_j}^{i_j}\,|0\rangle_{R}^{\pm} \qquad \begin{matrix}+,\ m' \text{ even}\\ -,\ m' \text{ odd}\end{matrix}$$

R− : " " $\qquad \begin{matrix}-,\ m' \text{ even}\\ +,\ m' \text{ odd}\end{matrix}$

lowest:

NS+ $\quad \psi_{-1/2}^{i}|0\rangle_{NS}$

NS− $\quad |0\rangle_{NS}$

R± $\quad |0\rangle_{R}^{\pm}$

This was for left-movers $\psi(z)$
What about right-movers $\tilde{\psi}(\bar{z})$?

Well, M. transforms act in same way on $\psi, \tilde{\psi}$ b.c.

Since we <u>do not like</u> $|0\rangle_{NS}$,

we'll choose to project onto NS+ (~~$|0\rangle_{NS}$~~)
for L, R.

Rule we've derived: for L and for R,
choose one projected NS sector
and one " R " .

	L	R
	NS+	NS+
and: i)	R+ ($|0\rangle_R^+$)	R+
or		
ii)	R−	R−
or		
iii)	R+	R−
or		
iv)	R−	R+

If we could endowe a tachyon $\exists$ two other M-I possibilities:

OA (NS+, NS+), (NS−, NS−), (R+, R−), (R−, R+)

OB (NS+, NS+), (NS−, NS−), (R+, R+), (R−, R−)

Clearly (i), (ii) are equivalent

and

(iii), (iv) are equivalent

But it does matter if we take

$R+, R-$

vs

$R+, R+$.

iii) : 'IIA'

i) : 'IIB'

More simply,

IIA:
$$\begin{pmatrix} \psi^i_{-1/2}|0\rangle_{NS} \\ \text{or} \\ |0\rangle^+_R \end{pmatrix} \otimes \begin{pmatrix} \tilde{\psi}^i_{-1/2}|0\rangle_{NS} \\ \text{or} \\ |0\rangle^-_R \end{pmatrix}$$

IIB
$$\begin{pmatrix} \psi^i_{-1/2}|0\rangle_{NS} \\ \text{or} \\ |0\rangle^+_R \end{pmatrix} \otimes \begin{pmatrix} \psi^i_{-1/2}|0\rangle_{NS} \\ \text{or} \\ |0\rangle^+_R \end{pmatrix}$$

OA, OB, IIA, IIB all consistent.

OA, OB have <u>tachyon</u> from $|0\rangle_{NS}$.
Also, ~~SUSY~~ in 10D.

IIA, IIB are supersymmetric + tachyon-free.

Sector

	NS NS	NS R	R NS	RR
IIA	$\psi^i_{-1/2}\|0\rangle_{NS} \otimes \tilde{\psi}^j_{-1/2}\|0\rangle_{NS}$	$\psi^i_{-1/2}\|0\rangle_{NS} \otimes \|0\rangle^-_R$	$\|0\rangle^+_R \otimes \tilde{\psi}^i_{-1/2}\|0\rangle_{NS}$	$\|0\rangle^+_R \otimes \|0\rangle^-_R$
IIB	"	$\psi^i_{-1/2}\|0\rangle_{NS} \otimes \|0\rangle^+_R$	$\|0\rangle^+_R \otimes \tilde{\psi}^i_{-1/2}\|0\rangle_{NS}$	$\|0\rangle^+_R \otimes \|0\rangle^+_R$

So we have

$$\left\{ \begin{array}{l} \left[\underline{8_v} \right] \quad \psi^i_{-\frac{1}{2}} |0\rangle_{NS} \quad \text{"NS+"} \\ \left\{ \begin{array}{l} \underline{8} \\ \underline{8}' \end{array} \right\} \begin{array}{l} |0\rangle_R^+ \\ |0\rangle_R^- \end{array} \end{array} \right.$$

Compute

$8_v \otimes 8_v$	NS NS	IIA, IIB
$8 \otimes 8$	RR	IIB
$8 \otimes 8'$	RR	IIA
$8_v \otimes 8$	NSR RNS	IIB IIA, IIB
$8_v \otimes 8'$	NSR	IIA

$8_v \otimes 8_v$: $c^{ij} \to s^{ij} + a^{ij}$

$$(s^{ij} - \tfrac{1}{8} S^i_i) + a^{ij} + \tfrac{1}{8} S^i_i$$

$$64 \to 35 \oplus 28 \oplus 1$$

$$\frac{8(8+1)}{2} - 1, \quad \frac{8(8-1)}{2}, \quad 1$$

$$\uparrow \qquad \uparrow \qquad \uparrow$$

graviton g_{ij} $\quad b_{ij} \quad \underline{\Phi}$.

$$8_v \otimes 8_v = (2) + [2] + [0]$$
$$= \underline{35} + \underline{28} + \underline{1}$$

$8_v \otimes 8$

8_v index i
8 index α
$8'$ index α'

given state $|\zeta\rangle_i^{\alpha}$

we construct $|\zeta\rangle_i^{\alpha}\, \Gamma^{i}_{\alpha\alpha'}$

which has free $8'$ index $\Rightarrow$ is on $8'$!

$$8_v \otimes 8 = 8' + 56$$

$\hookrightarrow$ vector-spinor irrep Ψ_i^{α}

$$8_v \otimes 8' = 8 + 56'$$

for RR, we recall that tensoring spinors gives tensors, with

$$\zeta \otimes \chi = \sum_{\text{suitable } m} \bar{\zeta}\, \Gamma^{[\mu_1} \cdots \Gamma^{\mu_m]} \chi$$

$\bar{\zeta}$ ↓ (secretly $\zeta^T C$)

Result: (cf. App. B of Joe II)

$$8 \otimes 8 = [0] + [2] + [4]_+$$
$$\underline{1} + \underline{28} + 35_+$$

here $\binom{8}{4} \cdot \frac{1}{2}$

$$8' \otimes 8' = [0] + [2] + [4]_-$$
$$\underline{1} + \underline{28} + \underline{35_-}$$

$$8 \otimes 8' = [1] + [3]$$
$$\underline{8_v} + 56_+$$

note 'ambiguity' given only the dimension:

$$\left\{ \begin{array}{l} 8_v,\ 8,\ 8' \\ 35,\ 35_+,\ 35_- \\ 56_+,\ 56,\ 56' \end{array} \right\}$$

In sum:

IIA: $\left(1 \oplus 28 \oplus 35\right) \oplus \left(8 + 56'\right) \oplus \left(8' + 56\right) \oplus \left(8_v + 56_+\right)$

NS NS — R NS — NS R — RR

$\Phi \quad B_{ij} \quad g_{ij} \qquad \lambda_s \quad \Psi_s^i \qquad \hat{\lambda}_s \quad \hat{\Psi}_s^i \qquad C_i \quad C_{ijk}$

$C_1 \quad C_3$

IIB $\left(1 \oplus 28 \oplus 35\right) \oplus \left(8' \oplus 56\right) \oplus \left(8' \oplus 56\right) \oplus \left(1 + 28 + 35_+\right)$

NS NS

$\Phi \quad B_{ij} \quad g_{ij} \qquad \lambda_s \quad \Psi_s^i \qquad \hat{\lambda}_s \quad \hat{\Psi}_s^i \qquad C_0 \quad C_2 \quad C_4$

IIA has opposite-chirality Ψ_s.
IIB " same " "

Universal NSNS sector:

$g_{ij}, B_{ij}, \Phi.$

NS-R R-NS

fermions: 2 gravitinos Ψ with $\begin{pmatrix} \text{same} \\ \text{opposite} \end{pmatrix} \begin{pmatrix} \text{IIB} \\ \text{IIA} \end{pmatrix}$ chirality.
2 dilatinos λ

NB Rarita-Schwinger eqn for $m=0$
$\epsilon^{\mu\nu\rho\sigma}\gamma^5\gamma_\nu\partial_\rho\psi_\sigma = 0$
$(\text{or} = -m\psi^\mu)$

RR:

antisymmetric p forms

$C_0 \quad C_2 \quad C_4$ IIB

$C_1 \quad C_3$ IIA

These spectra have the same # of bose + fermi states:
64 in each sector.
128 bosons, 128 fermions.