The Five Superstring Theories

So far, we have found two superstring theories in ten dimensions: type IIA and type IIB string theory, which are related by T-duality. These nicely correspond to the two $\mathcal{N}=2$ supergravity theories in 10D, of the same names.

Are there more superstring theories? If so they should have corresponding supersymmetric low energy effective actions. Assuming 10D Lorentz invariance, what is left? Well, all $\mathcal{N}=1$ theories.

$\mathcal{N}=1$ Supergravity

For $\mathcal{N}=1_{10}$, there are 16 supercharges, and two massless multiplets. The supergravity multiplet contains the gravition $g_{\mu\nu}$, dilaton Φ, two-form $B_{\mu\nu}$, gravitino Ψ^{μ}_{α} and dilatino λ_{α}. In SO(8) notation, the bosonic sector is $[0]+[2]+(2)=\mathbf{1}+\mathbf{28}+\mathbf{35}$ and the fermionic sector is $\mathbf{56}+\mathbf{8}'$, for a total of 64 states in each sector. The gauge multiplet contains a gluon A_{μ} and gluino χ_{α}, i.e. $\mathbf{8}_v+\mathbf{8}'$ (an ultrashort multiplet). Thus, to specify the spectrum of the theory, we need only specify the "gluon" gauge group. This also suffices to determine the effective action, whose bosonic portion is:

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^{10}x\sqrt{-g}\left[R-\frac{1}{2}(\nabla\Phi)^2-\frac{1}{2}e^{-\Phi}|H|^2-\frac{\kappa^2 e^{-\Phi/2}}{2g^2}\mathrm{Tr}\left[F^2\right]\right]$$

where

$$\begin{aligned} F &= \mathrm{d}A - iA\wedge A \\ H &= \mathrm{d}B-\frac{\kappa^2}{g^2}\Omega_3 \\ \Omega_3 &= \mathrm{Tr}\left[A\wedge\mathrm{d}A-\frac{2i}{3}A\wedge A\wedge A\right] \end{aligned}$$

Note that the "Yang-Mills" coupling, g, is dimensionful, but the combination g^4/κ^3 is dimensionless. However, this combination can be shifted by shifting the dilaton, so in fact the action contains one dimensionful scale, and no adjustable parameters.

Classically, we can choose any gauge group. However, most gauge groups (including pure $\mathcal{N}=1$ supergravity) will be anomalous. It turns out that the only anomaly free gauge groups are the following:

$$\begin{array}{cc} \mathrm{Spin}(32)/\mathbb{Z}_2 & E_8\times E_8 \\ E_8\times U(1)^{248} & U(1)^{496} \end{array}$$

Here Spin(32)/$\mathbb{Z}_2$ is essentially SO(32) (they have the same Lie algebra, but different global properties), and E_8 is the rank 8, dimension 248 exceptional (simple) Lie group.

Note that all of these groups have dimension 496. In addition, the first two both have rank 16, and contain SO(16) $\times$ SO(16) as a subgroup. The later two have rank 256 and 496 respectively. It turns out that there *are* string theories corresponding to the Spin(32)/$\mathbb{Z}_2$ and $E_8\times E_8$ cases (in fact, two for Spin(32)/$\mathbb{Z}_2$), though not for the latter two groups.

How can we construct these string theories? So far we have considered only closed, *oriented* string theories. A straightforward generalization is to consider closed+open and/or unoriented theories (with the same worldsheet action as before).

Unoriented strings and orientifolds

What are unoriented strings? Define a world-sheet parity operator Ω by:

$$\Omega \;:\; \sigma \to -\sigma$$

This exchanges left and right movers. Thus, for instance, the NS-R and R-NS sectors are mapped into each other. In type IIB this is a global symmetry, since these sectors are identical. In type IIA, however, Ω must be combined with *spatial* parity in order to obtain a global symmetry, since these sectors have opposite parities.

To obtain unoriented strings, we *gauge* Ω (or Ω + parity). In the type IIB case, it is easy to see that in the NS-NS sector, this removes $B_{\mu\nu}$ and leaves $g_{\mu\nu}$ and Φ. The NS-R and R-NS sectors are identified, leaving the same fermion content as the $\mathcal{N}=1$ SUGRA multiplet. Thus, we must have a total of 64 bosonic states. The only way to achieve this is if $C_{\mu\nu}$ is projected in an C_0 and $C_{\mu\nu\rho\sigma}$ are projected out (one can verify this in a more fancy manner...)

Now consider the IIA case. We require a $\mathbb{Z}_2$ involution $\Sigma\colon x \to x'$ which reverses spatial parity. For concreteness, consider $\Sigma: x^9 \to -x^9$. Thus, away from the fixed plane, have ordinary type IIA. At the fixed plane, we find that $B_{\mu\nu}$ and $g_{\mu 9}$ are projected out, whereas g_{99}, $g_{\mu\nu}$, $B_{\mu 9}$, and Φ are projected in. The NS-R and R-NS sectors are identified as before, so we still need a total of 64 bosonic states. We have at present $[0]+[0]+[1]+(2) = \mathbf{1}+\mathbf{1}+\mathbf{7}+\mathbf{27}$ in reps of SO(7). The rest of the multiplet is logically filled out by $[1]+[3] = \mathbf{7}+\mathbf{21}$, which can be explained if C_μ and $C_{\mu\nu 9}$ are projected in, whereas C_9 and $C_{\mu\nu\rho}$ are projected out.

In general this construction (gauging a combination of a $\mathbb{Z}_2$ involution and world-sheet parity) is called an *orientifold*. For type IIB, the involution must preserve spatial orientation, and for IIA it must reverse spatial orientation. The simplest orientifold is the trivial involution for IIB (the only one which results in a Lorentz invariant theory), but in general the involution can be complicated. The fixed planes of the involution are called orientifold fixed planes, or Op planes (e.g. O9 planes for the spacetime filling case in IIB).

However, the orientifold of IIB is anomalous... will need to add a gauge multiplet...

Open Strings

What about open strings? Adding open strings to the spectrum is equivalent to adding any number/configuration of Dp branes. For instance, the simplest case is to add a single D9 brane (Neumann BCs). What if we add multiple D9 branes? This leads to “Chan-Paton” factors, i.e. labels carried at each end of an open string which indicate which of the coincident D9 branes the string ends on. If we compute e.g. scattering amplitudes, one finds that the open string vector A_μ, which carries two Chan-Paton labels (one for each end) now lives in a *gauge* group $U(N)$ (where N is the number of D9 branes).

Why $U(N)$? The massless level states are:

$$\sum_{i,j} \mathcal{A}_{\mu,ij} \left(\psi^{\mu}_{-1/2} |p,i,j\rangle \right)$$

Choose as basis N^2 Hermitean $N \times N$ matrices:

$$\mathcal{A}_{\mu,ij} \;=\; \sum_a A^a_\mu \lambda^a_{ij}$$

The λ^a_{ij} are the generators of the $U(N)$ Lie algebra, where $\mathrm{Tr}(\lambda^a\lambda^b) = \delta^{ab}$... so we have gauge group $U(N)$.

What about unoriented open strings? Ω must exchange i and j. Turns out there is a sign reversal too:

$$\Omega\left(\psi^{\mu}_{-1/2}|p,i,j\rangle\right) = -\left(\psi^{\mu}_{-1/2}|p,j,i\rangle\right)$$

In this case, we must choose the λ^a to be antisymmetric, in which case we obtain the gauge group SO(N).

More generally, can combine Ω with a $U(N)$ rotation γ_{ij} on the Chan-Paton factors:

$$\Omega\left(\psi^{\mu}_{-1/2}|p,i,j\rangle\right) = \epsilon\gamma_{il}\left(\psi^{\mu}_{-1/2}|p,k,l\rangle\right)\left(\gamma^{-1}\right)_{kj}$$

where ϵ is a phase. Since Ω must square to the identity,

$$\begin{aligned}\Omega^2\left(\psi^{\mu}_{-1/2}|p,i,j\rangle\right) &= \epsilon\gamma_{il}\left(\epsilon\gamma_{kn}\left(\psi^{\mu}_{-1/2}|p,m,n\rangle\right)\left(\gamma^{-1}\right)_{ml}\right)\left(\gamma^{-1}\right)_{kj}\\ &= \epsilon^2\left(\gamma\,(\gamma^T)^{-1}\right)_{im}\left(\psi^{\mu}_{-1/2}|p,m,n\rangle\right)\left(\gamma^T\gamma^{-1}\right)_{nj}\end{aligned}$$

Thus, γ must be either symmetric or antisymmetric, and $\epsilon^2=1$ so ϵ is a sign. The generators λ, will have to satisfy:

$$\lambda = \epsilon\gamma\lambda^T\gamma^{-1}$$

Commutator of two λ's:

$$\begin{aligned}[\lambda_1,\lambda_2] &= \epsilon^2\,\gamma\left[\lambda_1^T,\lambda_2^T\right]\gamma^{-1}\\ &= -\,\gamma\,[\lambda_1,\lambda_2]^T\gamma^{-1}\end{aligned}$$

Thus, to obtain a Lie algebra (closed under commutation) must choose $\epsilon = -1$. What does the corresponding Lie group look like (require λ's to be Hermitean)? Well, we have $\lambda\gamma = -\gamma\lambda^T$. Thus,

$$\begin{aligned}e^{i\lambda}\gamma e^{i\lambda^T} &= e^{i\lambda}e^{-i\lambda}\gamma\\ &= \gamma\end{aligned}$$

so the Lie group preserves the symmetric (antisymmetric) nondegenerate matrix γ. This leads to the Lie groups SO(N) and USp(N) respectively (both simple Lie groupds, and subgroups of $U(N)$). Call the O plane which produces SO(N) an O9$^-$ plane, and that for USp(N) an O9$^+$ plane. Note that for the USp(N) case, N will have to be even.

Type I superstring theory

Now consider adding open strings to the spectrum on either type II string theory. We will need to do a GSO projection in the open string sector. There are two choices:

$$\begin{aligned}\mathrm{NS}+, R+ &: \mathbf{8}_v+\mathbf{8}\\ \mathrm{NS}+, R- &: \mathbf{8}_v+\mathbf{8}'\end{aligned}$$

These are both multiplets of $\mathcal{N}=1$ SUGRA but *not* of $\mathcal{N}=2$ SUGRA, so we must *combine* the open and unoriented theories. Thus, take the IIB orientifold and add an open string sector (must be the NS $+$, $R+$ GSO projection for consistency). We can add any number of D-branes/Chan-Paton factors, and use either an O9$^+$ or an O9$^-$, obtaining either a symplectic or orthogonal gauge group. However, the only nonanomalous case is 32 D9's + O9$^-$, with gauge group SO(32). This is Type I superstring theory, and gives the low energy effective theory $\mathcal{N}=1$, Spin(32)/$\mathbb{Z}_2$ SUGRA.

D branes

The existence of the type I theory tells us a lot about D-branes in superstring theories via T-duality. In particular, consider the T-dual theory to type I compactified on a circle (known as type I′). Recall that T-duality replaces:

$$X^9 = X_L^9 + X_R^9$$

with

$$\tilde{X}^9 = X_L^9 - X_R^9$$

The world-sheet parity operator Ω exchanges left and right movers, so:

$$\Omega : X_L \leftrightarrow X_R$$

Thus

$$\Omega : \tilde{X}^9 \to -\tilde{X}^9$$

Thus, on the T-dual side, we find an *orientifold* of type IIA with a nontrivial $\mathbb{Z}_2$ involution $\tilde{X}^9 \to -\tilde{X}^9$. Since $\tilde{X}^9$ is periodically identified, $\tilde{X}^9 \sim \tilde{X}^9 + 2\pi\tilde{R}$, there are two fixed planes at $\tilde{X}^9 = 0$ and $\tilde{X}^9 = \pi\tilde{R}$. There are also 32 D8 branes which come in $\tilde{X}^9 \to -\tilde{X}^9$ image pairs, so there are only 16 D8 branes on the target space. When all 16 coincide with one orientifold plane, the gauge group is SO(32) as before. Adjusting Wilson lines in the type I theory, we can break the gauge group (without reducing the rank). On the far side, this corresponds to letting some of the D8 branes move off the orientifold plane. A stack of N of them will then give a $U(N)$ gauge group, etc. As a simple example, moving half of them onto the other O-plane will give SO(16) $\times$ SO(16). None of this will break supersymmetry (example of a moduli space).

Now consider a point along the $\tilde{X}^9$ which is separated from the O-planes and D-branes. The local physics here is just type IIA string theory, as only closed strings can propagate in the bulk. We can decompactify, keeping some of the D8 branes at a finite distance, which shows that D8 branes can occur in type IIA. T-dualizing further, we obtain in general Dp branes, where for even p they live in a type IIA background, and for odd p they live in a type IIB background. If we also turn on gauge field backgrounds on the D-branes, then we can obtain D-branes at angles to each other, and D-branes of different dimensions at the same time.

The upshot seems to be that type IIA, type IIB, and type I string theories are all states of the same theory, where we can add general D brane content (even or odd p depending on which T-dual frame we are in), and orientifold if desired.

There is another way to see that D-branes must be present in the type IIA and type IIB spectra:

BPS states

A BPS (Bogolmol'nyi-Prasad-Sommerfeld) state is a state in a supersymmetric theory which is invariant under some fraction of the supercharges. A supermultiplet can be generated by acting on a state with all available supercharges. Thus, a BPS state lives in a *short* multiplet. Moreover, BPS states will saturate an appropriate BPS bound.

For $0+1$ dimensional objects, (particles), the BPS bound is a bound on the mass. Roughly:

$$\text{mass} \geqslant \text{charge}$$

in appropriate units. For $p+1$ dimensional objects, (p-branes), the BPS bound is a bound on the p-brane *tension*. Roughly:

$$\text{tension} \geqslant \text{charge}$$

A state is BPS iff it saturates the appropriate BPS bound; thus mass = charge or tension = charge.

Because BPS states live in short multiplets, their mass relation is exact, even accounting for nonperturbative effects. No corrections can increase their mass, since they do not furnish a full (*long*) representation of the SUSY algebra.[1] As such, they are important tools for obtaining nonperturbative information about supersymmetric theories.

For those familiar with classical GR, the BPS bound looks something like the *cosmic censorship principle* and BPS states look something like *extremal black holes*. This is not a coincidence. In supergravity theories, BPS states will correspond to semiclassical extremal charged black holes, and most extremal charged black holes will be BPS.

This provides an easy way to find the BPS states of superstring theories. All we need to do is consider the low energy effective theory, and construct black p-branes carrying various charges in the extremal limit. If these are supersymmetric states, they are BPS. The no-hair theorem ensures that there is a unique BPS state for a given set of charges.

For instance, consider type IIA string theory. We have the bosonic fields $g_{\mu\nu}$, Φ, $B_{\mu\nu}$, C_1 and C_3. In general, if A_{p+1} is a $p+1$ form potential, it naturally couples to p branes via:

$$S \supset \mu_p \int_\Sigma A_{p+1}$$

(This is an electric coupling). The field strenth, $F_{p+2} = \mathrm{d}A_{p+1}$ has dual field strength $F_{d-p-2} = \mathrm{d}A_{(d-3)-p}$. This dual potential naturally couples electrically to $(d-4)-p$ branes, i.e. $6-p$ branes in $d = 10$. This is a "magnetic" coupling.

Now consider what branes couple to $B_{\mu\nu}$. $B_{\mu\nu}$ couples electrically to 1-branes, i.e. *strings*, via the term:

$$S_{\text{int}} \propto \int_\Sigma B_2$$

In fact, the fundamental string has such a term in its worldsheet action. We reach the following conclusion: in type IIA (and IIB), F-strings are BPS branes which are minimally coupled to $B_{\mu\nu}$. What about C_1 and C_3? These couple to 0-branes and 2-branes respectively. It turns out that these are just the D0 and D2 branes we argued must be present above. Furthermore, C_1 and C_3 couple magnetically to 6 and 4-branes respectively. These are just the D6 and D4 branes.

D8 branes are slightly more subtle. It turns out that one can add a "mass parameter" to type IIA SUGRA. This is a 0-form field strength, which is nonpropagating, and thus does not show up in the spectrum. Its dual 10-form field strength has a 9-form potential which couples naturally to D8 branes. In fact, the mass parameter must be a constant, at least locally, and the D8 branes act as domain walls between regions with different mass parameters.

What about branes which couple magnetically to $B_{\mu\nu}$? These are 5-branes, known as NS5 branes. Unlike D-branes, open strings do not end on NS5 branes. Instead, D2 branes can end on them...! They appear as solitons in the weak coupling theory, but become lighter at strong coupling (a hint about M-theory.) Instead of a world-volume one-form, they possess a world-volume two-form which is self-dual (further hints).

1. Technically, two or more BPS states can combine to fill out a long multiplet, and then obtain a mass. Otherwise, this statement is rigorous.

Now what about type IIB? The story concerning D-branes is much the same as before, except that now we have C_0, C_2, and C_4. C_0 couples electrically to D(-1) branes (i.e. D-instantons) and magnetically to D7 branes. C_2 couples electrically to D1 branes (i.e. D-strings) and magnetically to D5 branes. Finally C_4 couples both electrically and magnetically to D3 branes (like C_4, D3 branes are "self-dual" in some sense).

What about $B_{\mu\nu}$? As before, $B_{\mu\nu}$ couples electrically to fundamental strings and magnetically to NS5 branes. However, type IIB NS5 branes are somewhat different; they possess a world-volume one-form, and D-strings can end on them (can see this via S-duality/T-duality).

Now what about type I? In this case, we have the closed-string fields $g_{\mu\nu}$, Φ as well as C_2 (and C_{10}). As in type IIB, C_2 couples electrically to D-strings, and magnetically to D5 branes. The other D-branes and the NS5 brane, which were BPS in type IIB, are no longer charged, and therefore not BPS. In fact, the fundamental string is no longer BPS! It is unstable, and can decay (by fragmentation), though the lifetime goes to infinity at weak coupling.

Now that we have learned some basic facts about D-branes/BPS states, we are ready to describe our first strong-coupling duality.

S-duality

Notice that in type IIB, there are two two-forms B_2 and C_2, and correspondingly two types of strings: F-strings and D-strings. This leads to the question: are D-strings related to F-strings in anyway?

For technical reasons, we will want the *Einstein*-frame tensions of F-strings, Dp branes, and NS5 branes. The F-string has tension $1/2\pi\alpha' = 1/2\pi l_s^2$ in string frame. In Einstein frame, this becomes:

$$T_{F1} = \frac{g_s^{1/2}}{2\pi l_s^2}$$

By contrast, Dp branes have tensions $(2\pi)^{-p} g_s^{-1} l_s^{-(p+1)}$ in string frame, and thus

$$T_{Dp} = \frac{g_s^{(p-3)/4}}{l_s (2\pi l_s)^p}$$

in Einstein frame. NS5 branes have tension $(2\pi)^{-5} l_s^{-6} g_s^{-2}$ in string frame. Therefore, its Einstein-frame tension is:

$$T_{\text{NS5}} = \frac{g_s^{-1/2}}{l_s (2\pi l_s)^5}$$

Now return to the D-string/F-string comparison. They have different tensions:

$$T_{F1} = \frac{g_s^{1/2}}{2\pi\alpha'}$$

$$T_{D1} = \frac{g_s^{-1/2}}{2\pi\alpha'}$$

so that $T_{F1}/T_{D1} = g_s = e^{\Phi}$. Thus, at weak coupling, the F-string is much "lighter" than the D-string. However, at strong coupling, the D-string becomes lighter, and for $g_s > 1$, it is lighter than the F-string (recall that these tension computations are exact by the BPS property), and is (in some sense) more fundamental.

Thus, we conjecture that type IIB is self-dual under the transformation $g_s \to 1/g_s$ and where the D-string an F-string exchange roles (a strong-weak duality is generically referred to as an "S-duality").

What happens to the other BPS states? Consider the D3, D5, and NS5 tensions:

$$T_{D3} = \frac{1}{l_s\,(2\pi l_s)^3}$$

and

$$\begin{aligned} T_{D5} &= \frac{g_s^{1/2}}{l_s\,(2\pi l_s)^5} \\ T_{\mathrm{NS5}} &= \frac{g_s^{-1/2}}{l_s\,(2\pi l_s)^5} \end{aligned}$$

This strongly suggests that the D3 brane is self-dual under S-duality, whereas the D5 and NS5 branes are exchanged, the former becoming heavier, and the latter becoming lighter.

What about the D7 brane? Its tension is

$$T_{D7} = \frac{g_s}{l_s\,(2\pi l_s)^7}$$

so it cannot be self-dual under S-duality. Thus, we are led to hypothesize the existence of an "NS7" brane, with Einstein-frame tension:

$$T_{\mathrm{NS7}} = \frac{g_s^{-1}}{l_s\,(2\pi l_s)^7}$$

or string frame tension $g_s^{-3}/(l_s\,(2\pi l_s)^7))$. In fact, such an object *does* exist. Since this object is very heavy in the $g_s \to 0$ limit, it won't be important in perturbative string theory, but *will* be essential when considering nonperturbative compactifications of type IIB (i.e. F-theory).

$\mathbf{SL(2,\mathbb{Z})}$

So far, we have consider the background $C_0 = 0$. Classically, the SUGRA action is invariant under arbitrary shifts $C_0 \to C_0 + k$ for k constant. However, some of the BPS tensions are *not* invariant under this shift. In particular, the D-string tension is:

$$\begin{aligned} T_{D1} &= \frac{(e^{\Phi}\,C_0^2 + e^{-\Phi})^{1/2}}{2\pi l_s^2} \\ &= \frac{g_s^{-1/2}\,\sqrt{1+(g_s\,C_0)^2}}{2\pi l_s^2} \end{aligned}$$

Thus, it seems that the axion shift symmetry is completely broken. However, this is not quite true. It turns out that there exist BPS bound states of p F-strings and q D-strings for any p, q coprime. For instance, the F-D bound state is easy to understand (draw nice picture). The tension of these $(p,\ q)$ strings is:

$$T_{(p,q)} = \frac{\sqrt{e^{\Phi}\,(p+q\,C_0)^2 + q^2\,e^{-\Phi}}}{2\pi l_s^2}$$

Thus, we see that starting with a D-string, i.e. a $(0,1)$ string, and shifting $C_0 \rightarrow C_0 + k$, we obtain a $(k,1)$ string. Therefore, k must be an integer, i.e. the axion shift symmetry is broken by quantum effects to a discrete $\mathbb{Z}$ symmetry.

Another way to see the effect on D-strings is to consider that under $C_0 \rightarrow C_0 + k$:

$$C_2 \;\rightarrow\; C_2 + k B_2$$

Thus, D-strings pick up F-string charge.

In fact the $\mathbb{Z}_2$ described above and the $\mathbb{Z}$ derived here do not commute. This is because in general for $C_0 \neq 0$ the $\mathbb{Z}_2$ symmetry is not $\Phi \rightarrow -\Phi$, but the more complicated expression:

$$\tau \;\rightarrow\; -\frac{1}{\tau}$$

where

$$\tau \;\equiv\; C_0 + i e^{-\Phi}$$

The axion shift symmetry is generated by:

$$\tau \;\rightarrow\; \tau + 1$$

Together, these generate the modular group, PSL(2, $\mathbb{Z}$). However, note that B_2 and C_2 transform under the full SL$(2,\mathbb{Z})$, since the $-1 \in \mathrm{SL}(2,\mathbb{Z})$ will take $F_3 \rightarrow -F_3$, $H_3 \rightarrow -H_3$, but $\tau \rightarrow \tau$.

What does a general transformation look like? It acts on τ like:

$$\tau' \;=\; \frac{a\tau + b}{c\tau + d}$$

on C_2 and B_2:

$$\begin{pmatrix} C_2' \\ B_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} C_2 \\ B_2 \end{pmatrix}$$

and similarly on F_3 and H_3, where $a, b, c, d \in \mathbb{Z}$, and $ad - bc = 1$. The other bosonic closed-string fields, namely $g_{\mu\nu}$ and C_4, are invariant.

On (p,q), it acts like:

$$\begin{pmatrix} p' \\ -q' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ -q \end{pmatrix}$$

Aside: clearly the two-forms transform as a doublet under SL(2, $\mathbb{Z}$). Correspondingly, the scalars transfrom as a *triplet*. To see this, define:

$$\varphi^{ij} \;\equiv\; \frac{1}{\tau_2} \begin{pmatrix} |\tau|^2 & \tau_1 \\ \tau_1 & 1 \end{pmatrix}$$

This is a *constrained* triplet, since $\frac{1}{2}\varphi^{ij}\varphi_{ij} = 1$, where indices are lowered using ε_{ij} with $\varepsilon_{12} = \varepsilon^{21} = +1$ (unconstrained triplet would not be an irrep). Then:

$$\varphi' \;=\; \Lambda \varphi \Lambda^T$$

The NS5 brane also has a tension which depends on C_0:

$$T_{\mathrm{NS5}} = \frac{\sqrt{e^{\Phi}\, C_0^2 + e^{-\Phi}}}{l_s\, (2\pi l_s)^5}$$

Here the story is the same: there are (p, q) 5-branes, with tension:

$$T_{(p,q)}^{(5)} = \frac{\sqrt{e^{\Phi}\, (p + q\, C_0)^2 + q^2\, e^{-\Phi}}}{l_s\, (2\pi l_s)^5}$$

where now the D5 brane is the $(1, 0)$ 5-brane, and the NS5 brane is the $(0, 1)$ 5-brane. Note that in general (p, q) strings may end on (p, q) 5-branes.

What about 7-branes? There are also $(p,\, q)$ 7-branes, but they are more interesting. The D7 brane couples magnetically to C_0. In fact, looping once around the seven-brane, we find that C_0 is multivalued, i.e. $C_0 \to C_0 + 1$. In terms of τ, $\tau \to \tau + 1$. But this is an $\mathrm{SL}(2, \mathbb{Z})$ transformation! Thus, the D7 brane gives rise to an $\mathrm{SL}(2, \mathbb{Z})$ monodromy.[2] $(p,\, q)$ seven branes will give more complicated monodromies, but not in fact the most general ones.

To be absolutely general, we should consider more complicated objects sometimes reffered to as "Q"-branes. Not much is known about the Q-branes, other than that BPS states do exist (check this...), and that they may be bound states of $(p,\, q)$ seven branes of different types. One understood example is that the $\mathrm{O7}^-$ plane is a bound state of $(1, 1)$ and $(1, -1)$ seven-branes, with monodromy:

$$\Lambda_{O7^-} = \begin{pmatrix} -1 & -4 \\ 0 & -1 \end{pmatrix}$$

Note that F-strings and D5-branes may end a D7 brane. Luckily, their tensions are independent of C_0, or else this would be ill defined!

$\mathbf{SL(2, \mathbb{Z})}$ on D3 branes

We have already seen that D3 branes are invariant under $\mathrm{SL}(2, \mathbb{Z})$. However, the world-volume gauge field which lives on the D3 is *not* invariant. Consider a stack of N D3 branes. The world-volume gauge theory is $U(N) = \mathrm{SU}(N) \times U(1)$, living in $3 + 1$ dimensions. Since the brane is half-BPS, we have $4d$ $\mathcal{N} = 4$ (rigid) SUSY. Ignoring the $U(1)$ factor, we thus have $\mathcal{N} = 4$ SYM theory on the D3 brane. This theory has a holomorphic gauge coupling given by $\tau = \frac{4\pi i}{g_{\mathrm{YM}}^2} + \frac{\theta}{2\pi}$. But this is actually the same τ we considered above. Thus, the IIB $\mathrm{SL}(2,\, \mathbb{Z})$ corresponds to/explains the $\mathrm{SL}(2,\, \mathbb{Z})$ "Montonen-Olive" duality of $\mathcal{N} = 4$ SYM. This setup (for large N), is the starting point for the AdS/CFT correspondence...

F theory

I will probably skip this topic...

More S-duals

Now we understand the strong coupling behavior of type IIB string theory. What about S-duals of type I and/or type IIA. Since both of these theories are related to type IIB, it seems reasonable to assume that they have relatively simple S-dual theories, but (considering the BPS states) we have no idea at present what these theories might be.

What's the issue? There are two options: either their S-duals are not string theories, or we haven't found all possible superstring theories yet.

2. Thus, $\mathrm{SL}(2, \mathbb{Z})$ must be a discrete *gauge* symmetry.

Consider first the type I case. The fundamental string and D-string have the same tensions as in type IIB:

$$T_{F1} = \frac{g_s^{1/2}}{2\pi\alpha'}$$

$$T_{D1} = \frac{g_s^{-1/2}}{2\pi\alpha'}$$

However, the F-string is no longer BPS. It will disintegrate at strong coupling and disappear from the spectrum. The D-string, by constrast, is BPS and stable. There are no BPS states left which the D-string can end on (since the D3 brane, NS5 brane, and NS7 brane are no longer BPS). Thus, it appears that the S-dual of type I should be a *closed* string theory (without any analog of D-branes). The only other BPS state we have found is the D5 brane, which has tension:

$$T_{D5} = \frac{g_s^{1/2}}{(2\pi)^5\, l_s^6}$$

This becomes heavy at strong coupling, and will correspond to the NS5 brane of the closed string theory.

We will find this string theory (and its T-dual) in a minute. But what about type IIA? The BPS states are the D0, D2, D4, D6 and D8 branes, and the F-string and NS5 branes, with tensions:

$$T_{D0} = \frac{g_s^{-3/4}}{l_s}$$

$$T_{D2} = \frac{g_s^{-1/4}}{l_s\,(2\pi l_s)^2}$$

$$T_{D4} = \frac{g_s^{1/4}}{l_s\,(2\pi l_s)^4}$$

$$T_{D6} = \frac{g_s^{3/4}}{l_s\,(2\pi l_s)^6}$$

$$T_{D8} = \frac{g_s^{5/4}}{l_s\,(2\pi l_s)^8}$$

$$T_{F1} = \frac{g_s^{1/2}}{2\pi\alpha'}$$

$$T_{\mathrm{NS5}} = \frac{g_s^{-1/2}}{l_s\,(2\pi l_s)^5}$$

(The D0 "tension" is really what we would usually call a "mass"). At large coupling, $g_s \to \infty$, we see that the following objects become light (in Einstein frame): the D0, D2, and NS5 branes. All other BPS states become heavy, including the fundamental string. Thus it would appear that the resulting theory is *not* a string theory. Rather, it is a theory with light/massless particles, as well as light BPS two-branes and five-branes. What is this theory? Well, two-brane and five-brane BPS states are naturally explained by the electric and magnetic couplings of a 3-form in 11 dimensions, so this suggests that the strong coupling dual is 11-dimensional, and has 11D SUGRA as a low energy theory. This is "M-theory". Liam will talk more about this later...

Heterotic String Theories

What about the closed string theory which is supposedly dual to type I? Let's consider this more carefully. In particular, what is the spectrum of excitations of the D-string?

The D-string breaks SO(9, 1) → SO(1, 1) × SO(8). Thus, we classify the spectrum by left/right movers under SO(1, 1), as well as by SO(8) rep.

In the type IIB case, the bosonic excitations are the collective coordinates and the worldvolume gauge field; however the latter is nonpropagating in $1+1$D, so in fact we just have eight physical bosonic exicitations, which can be split in eight left-movers and eight right-movers. Not suprisingly, there are eight corresponding fermionic excitations for both the left-moving and right-moving cases. These furnish an **8** for left-movers, and an $\mathbf{8}'$ for the right-movers.

What happens when we orientifold? The world-volume gauge field is projected out, but the collective coordinates remain. The left-moving fermions are also projected out, but the right-movers remain. Moreover, in the type I theory there are D9 branes, so we will get additional excitations corresponding to D1-D9 F-strings. These carry a Chan-Paton factor at one end, and thus transform in the fundamental rep of SO(32). Turns out that the only massless excitations of this type are left-moving fermions which are SO(8) singlets.

So what are we left with? For the left-movers, there are more fermions than bosons, but for the right-movers there are equal numbers. This suggests that the worldsheet theory living on the D-string has $\mathcal{N} = (0, 1)$ SUSY. At strong coupling, the D-string should become the F-string of a different theory; in which case the F-string will have $\mathcal{N} = (0, 1)$ worldsheet SUSY. Such a theory is called a "heterotic" string theory, since the F-string is (in a sense) a fusion of the left-moving bosonic string and the right moving superstring.

This motivates us to pursue a general classification of heterotic string theories, in the hope of finding the S-dual we are after.

$\mathcal{N} = (0, 1)$ worldsheet theories

Up till now, we have worked with $\mathcal{N} = (1, 1)$ worldsheet SUSY, with action:

$$S = \frac{1}{4\pi}\int \mathrm{d}^2 z \left(\frac{2}{\alpha'}\,\partial X^\mu\,\bar\partial X^\nu + \psi^\mu\,\bar\partial\psi^\nu + \tilde\psi^\mu\,\partial\tilde\psi^\nu\right)\eta_{\mu\nu}$$

(Except in the case of the type I string, where the worldsheet was unoriented, and had $\mathcal{N} = 1$ SUSY).

But what if we drop the left-moving fermions (and their corresponding ghosts) from the action? This is the right direction to move in to obtain a heterotic theory, but we have a problem: the central charge of the left-moving sector is no longer zero... will get a Weyl anomaly. In the bosonic string, this was fixed by adding 16 extra spacetime dimensions, i.e. 16 extra worldsheet scalars. We could do this, i.e. add *left*-moving scalars to the action, cancelling the Weyl anomaly. (These scalars can no longer be interpreted as physical dimensions, since they are left-moving only.)

Alternately, and (as it turns out) equivalently, we can add 32 extra worldsheet fermions to the left-moving sector. Thus, the worldsheet action becomes:

$$S = \frac{1}{4\pi}\int \mathrm{d}^2 z \left(\frac{2}{\alpha'}\,\partial X^\mu\,\bar\partial X^\nu + \delta_{AB}\,\lambda^A\,\bar\partial\lambda^B + \tilde\psi^\mu\,\partial\tilde\psi^\nu\right)\eta_{\mu\nu}$$

where $A = 1...32$ parameterizes the internal fermionic directions. Since δ_{AB} must signature $(+++....)$, there is an SO(32) symmetry acting on the λ^A's. This is already looking hopeful...

Periodicity condition on the λ^A's can be quite complicated:

$$\lambda^A(w+2\pi) = O^A{}_B\,\lambda^C(w)$$

where $O^A{}_B \in O(32)$. Acc. to Polchinksi, not currently feasible to classify *all* possible consistent theories of this type. Rather, can describe the known examples.

Actually, it turns out that the known interesting (consistent and tachyon-free) examples are all quite simple. We split the λ^A into two groups for $A=1,\dots 16$ and $A'=17,\dots 32$ and define the periodicities:

$$\begin{aligned}\lambda^A(w+2\pi) &= -\exp(\pi i\alpha_1)\,\lambda^A(w)\\ \lambda^{A'}(w+2\pi) &= -\exp(\pi i\alpha_2)\lambda^{A'}(w)\\ \tilde{\psi}^\mu(w+2\pi) &= -\exp(\pi i\,\tilde{\alpha})\,\tilde{\psi}^\mu(w)\end{aligned}$$

where the α's are zero for NS boundary conditions, and 1 for R boundary conditions. In addition, define separate world-sheet fermion numbers $\exp(\pi i\,F_1)$ and $\exp(\pi i\,F_2)$ for the two blocks of λ's, as well as $\exp(\pi i\,\tilde{F})$ for the right-movers.

The zero-point energy for the left-movers is:

$$A = -1+\alpha_1+\alpha_2$$

and for the right-movers

$$A = -1+\tilde{\alpha}$$

(As with the type I/II superstring).

We now consider three different GSO projections.

SO(32)

The first projection we consider is simply:

$$\begin{aligned}\exp(\pi i\,(F_1+F_2)) &= 1\\ \exp(\pi i(\alpha_1+\alpha_2)) &= 0\\ \exp(\pi i\,\tilde{F}) &= 1\end{aligned}$$

Consider the left-movers first. In the NS,NS sector, we find a tachyon, a massless vector, and a massless adjoint scalar:

$$|0\rangle_{\mathrm{NS,NS}}\quad \alpha^i_{-1}\,|0\rangle_{\mathrm{NS,NS}}\quad \lambda^A_{-1/2}\,\lambda^B_{-1/2}\,|0\rangle_{\mathrm{NS,NS}}$$

Or, in terms of SO(8) $\times$ SO(32) quantum numbers, $(\mathbf{1},\mathbf{1})\oplus(\mathbf{8}_v,\mathbf{1})\oplus(\mathbf{1},\mathbf{496})$ where $\mathbf{496}$ is the antisymmetric tensor rep. SO(32), i.e. the adjoint rep.

The R,NS and NS,R sectors are projected out, and the R,R sector has a positive normal ordering constant, and therefore no massless states.

Now consider the right movers. As before, we obtain a massless $\mathbf{8}_v+\mathbf{8}$ and no tachyon. Now consider the level-matching condition: even though there is a left-moving tachyon, there is no right-moving tachyon to match it with, so there are no tachyons in the spectrum. Tensoring together the SO(32) singlets, we find the $\mathcal{N}=1$ SUGRA multiplet:

$$(\mathbf{8}_v,\mathbf{1})\otimes(\mathbf{8}_v\oplus\mathbf{8}) = (\mathbf{1},\mathbf{1})\oplus(\mathbf{28},\mathbf{1})\oplus(\mathbf{35},\mathbf{1})\oplus(\mathbf{56},\mathbf{1})\oplus(\mathbf{8}',\mathbf{1})$$

The SO(32) adjoints give an $\mathcal{N}=1$ gauge multiplet:

$$(\mathbf{1},\mathbf{496})\otimes(\mathbf{8}_v\oplus\mathbf{8}) = (\mathbf{8}_v,\mathbf{496})\oplus(\mathbf{8},\mathbf{496})$$

We recognize the spectrum of $\mathcal{N}=1$, Spin(32)/$\mathbb{Z}_2$ SUGRA.

$\boldsymbol{E_8 \times E_8}$

Now consider the projection:

$$\begin{aligned} \exp(\pi i F_1) &= 1 \\ \exp(\pi i F_2) &= 1 \\ \exp(\pi i \tilde{F}) &= 1 \end{aligned}$$

The right moving sector again gives the massless spectrum $\mathbf{8}_v \oplus \mathbf{8}$, with the tachyon removed. Now consider the left-moving NS,NS sector. As before, we find the states

$$|0\rangle_{\mathrm{NS,NS}} \quad \alpha^i_{-1}|0\rangle_{\mathrm{NS,NS}} \quad \lambda^A_{-1/2}\lambda^B_{-1/2}|0\rangle_{\mathrm{NS,NS}}$$

However, this time the projection forces us to take A and B from the same subgroup, breaking SO(32) to SO(16) $\times$ SO(16) (i.e. the (NS $-$, NS $-$) sector is absent). Thus, in terms of SO(16) $\times$ SO(16) quantum numbers, we find:

$$(\mathbf{1},\mathbf{1},\mathbf{1}) \oplus (\mathbf{8}_v,\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{120},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{120})$$

In addition, the NS-R and R-NS ground states are massless and allowed by the projection. They transform under the 16 λ^A zero modes, which can be rewritten as 8 raising and 8 lowering operators. Thus, we find a **256** spinor rep of SO(16), which splits into irreps **128** and **128′**, the later of which is removed by the projection. Thus, we find the additional massless states:

$$(\mathbf{1},\mathbf{128},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{128})$$

We expect to find only adjoint reps in a 10D $\mathcal{N}=1$ SUGRA theory. Thus, for consistency $\mathbf{120} \oplus \mathbf{128}$ must be the adjoint rep of some gauge group which contains SO(16). In fact, this group is E_8, with adjoint rep **248**. Thus, in terms of SO(8) $\times E_8 \times E_8$ quantum numbers we have found the left movers:

$$(\mathbf{1},\mathbf{1},\mathbf{1}) \oplus (\mathbf{8}_v,\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{248},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{248})$$

Tensoring the singlets together with the right-movers, we get the $\mathcal{N}=1$ SUGRA multiplet as before:

$$(\mathbf{8}_v,\mathbf{1},\mathbf{1}) \otimes (\mathbf{8}_v \oplus \mathbf{8}) = (\mathbf{1},\mathbf{1},\mathbf{1}) \oplus (\mathbf{28},\mathbf{1},\mathbf{1}) \oplus (\mathbf{35},\mathbf{1},\mathbf{1}) \oplus (\mathbf{56},\mathbf{1},\mathbf{1}) \oplus (\mathbf{8}',\mathbf{1},\mathbf{1})$$

Tensoring the others, we get:

$$((\mathbf{1},\mathbf{248},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{248})) \otimes (\mathbf{8}_v \oplus \mathbf{8}) = (\mathbf{8}_v,\mathbf{248},\mathbf{1}) \oplus (\mathbf{8}_v,\mathbf{1},\mathbf{248}) \oplus (\mathbf{8},\mathbf{248},\mathbf{1}) \oplus (\mathbf{8},\mathbf{1},\mathbf{248})$$

Thus, we find the same spectrum as $\mathcal{N}=1$, $E_8 \times E_8$ SUGRA.

SO(16) $\times$ SO(16)

We consider the projection:

$$\begin{aligned} \exp(\pi i (F_1 + \alpha_2 + \tilde{\alpha})) &= 1 \\ \exp(\pi i (F_2 + \alpha_1 + \tilde{\alpha})) &= 1 \\ \exp(\pi i (\tilde{F} + \alpha_1 + \alpha_2)) &= 1 \end{aligned}$$

Note that these equations imply:

$$\exp(\pi i (F_1 + F_2 + \tilde{F})) = 1$$

As well as:

$$\begin{aligned} \exp(\pi i (F_1 + \alpha_1)) &= \exp(\pi i (F_2 + \alpha_2)) \\ &= \exp(\pi i (\tilde{F} + \tilde{\alpha})) \end{aligned}$$

Thus, the allowed sectors are:

$$\begin{array}{rl} (\mathrm{NS}+, \mathrm{NS}+, \mathrm{NS}+) & (R+, R+, R+) \\ (\mathrm{NS}+, R-, R-) & (R+, \mathrm{NS}-, \mathrm{NS}-) \\ (R-, \mathrm{NS}+, R-) & (\mathrm{NS}-, R+, \mathrm{NS}-) \\ (R-, R-, \mathrm{NS}+) & (\mathrm{NS}-, \mathrm{NS}-, R+) \end{array}$$

where the three entries are for $\tilde{\psi}^\mu$, λ^A, $\lambda^{A'}$ respectively. Note that the sectors $(R+, R+, R+)$, $(\mathrm{NS}+, R-, R-)$, $(\mathrm{NS}-, R+, \mathrm{NS}-)$, $(\mathrm{NS}-, \mathrm{NS}-, R+)$ contain no massless states. The remaining sectors break down as follows: in $(\mathrm{NS}+, \mathrm{NS}+, \mathrm{NS}+)$, we find the "universal" $\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}$ bosonic sector from:

$$\alpha^i_{-1} |0\rangle_{\mathrm{NS,NS}} \otimes \tilde{\psi}^j_{-1/2} |0\rangle_{\mathrm{NS}}$$

We also find the SO(16) gauge bosons $(\mathbf{8}_v, \mathbf{120}, \mathbf{1}) \oplus (\mathbf{8}_v, \mathbf{1}, \mathbf{120})$ from:

$$\lambda^A_{-1/2} \lambda^B_{-1/2} |0\rangle_{\mathrm{NS,NS}} \otimes \tilde{\psi}^i_{-1/2} |0\rangle_{\mathrm{NS}}$$

where A and B must reside in the same subblock.

In the $(R+, \mathrm{NS}-, \mathrm{NS}-)$ sector, we obtain a bifundamental fermion $(\mathbf{8}, \mathbf{16}, \mathbf{16})$ via:

$$\lambda^A_{-1/2} \lambda^{B'}_{-1/2} |0\rangle_{\mathrm{NS,NS}} \otimes |0\rangle_R$$

Finally, in the $(R-, R-, \mathrm{NS}+)$ and $(R-, \mathrm{NS}+, R-)$ sectors we obtain $(\mathbf{8}', \mathbf{128}', \mathbf{1})$ and $(\mathbf{8}', \mathbf{1}, \mathbf{128}')$ fermions respectively.

This theory is nonsupersymmetric, consistent, and tachyon-free. Perhaps the only known example...? It is perturbatively stable (because of the absence of a tachyon), but will acquire a cosmological constant at 1-loop order.

It is possible that this theory is related to the supersymmetric heterotic string theories, but difficult to prove anything or even demonstrate convincingly in the absence of SUSY.

Heterotic T-duality

Leaving this issue aside, let's return considering the two $\mathcal{N} = 1$ heterotic superstring theories we have found. We have already seen that the SO(32) theory is dual to type I, via $g_I = 1/g_h$, $C_2 \leftrightarrow B_2$, and the Einstein-frame metric and gauge field are invariant.

However, now we want to address: are the two heterotic theories related to each other? It's plausible, since they have the same number of states, and gauge groups of the same rank both containing SO(16) $\times$ SO(16).

... anyway, they are in fact T-dual, but will have to turn on Wilson lines to see this; see Polchinksi...