but $[dX\,dg]\,e^{-S[X,g]}$ is gauge-inv't

$$= [dX^{\zeta}\,dg^{\zeta}]\,e^{-S[X^{\zeta},g^{\zeta}]}$$

$$Z = \int \frac{[DX^{\zeta}\,Dg^{\zeta}\,D\zeta]}{V_{\text{diff}\times\text{Weyl}}}\,\Delta_{FP}(g^{\zeta})\,\delta(g^{\zeta}-\hat{g})\,e^{-S[X^{\zeta},g^{\zeta}]}$$

rename vbls.

$$= \int \frac{[DX\,Dg\,D\zeta]}{V_{\text{diff}\times\text{Weyl}}}\,\Delta_{FP}(g)\,\delta(g-\hat{g})\,e^{-S[X,g]}$$

$$= \int \frac{[DX\,D\zeta]}{V_{\text{diff}\times\text{Weyl}}}\,\Delta_{FP}(\hat{g})\,e^{-S[X,\hat{g}]}$$

but ζ parametrizes gauge group $\Rightarrow \int [D\zeta] = V_{\text{diff}\times\text{Weyl}}$

$$\Rightarrow Z[\hat{g}] = \int [DX]\,\Delta_{FP}(\hat{g})\,e^{-S[X,\hat{g}]}$$

So now we just compute Δ_{FP}.

For ζ near 1, $g^{\zeta} - g \equiv \delta g$ can be computed:

under diff, $\zeta'^{\gamma} = \zeta^{\gamma} + \epsilon^{\gamma}$ $\qquad \frac{\partial \zeta'^{\gamma}}{\partial \zeta^{\alpha}} = \delta^{\gamma}{}_{\alpha} + \partial_{\alpha} \epsilon^{\gamma}$

$$g_{\alpha\beta} = \frac{\partial \zeta'^{\gamma}}{\partial \zeta^{\alpha}} \frac{\partial \zeta'^{\delta}}{\partial \zeta^{\beta}} g'_{\gamma\delta}$$

NB

$$\Rightarrow -\delta g_{\alpha\beta} = g_{\gamma\beta} \partial_{\alpha} \epsilon^{\gamma} + g_{\gamma\alpha} \partial_{\beta} \epsilon^{\gamma}$$

$$= g_{\gamma\beta}\left(\nabla_{\alpha} \epsilon^{\gamma} - \Gamma^{\gamma}_{\alpha\beta'} \epsilon^{\beta'}\right) + g_{\gamma\alpha}\left(\nabla_{\beta} \epsilon^{\gamma} - \Gamma^{\gamma}_{\beta\beta'} \epsilon^{\beta'}\right)$$

for $\hat{g}_{\alpha\beta} = \delta_{\alpha\beta}$, Γ are $\mathcal{O}(\epsilon^2)$

$$\Rightarrow -\delta^{diff} g_{\alpha\beta} \approx \nabla_{\alpha} \epsilon_{\beta} + \nabla_{\beta} \epsilon_{\alpha}$$

$$\delta^{W} g_{\alpha\beta} = e^{2\delta\omega} g_{\alpha\beta} - g_{\alpha\beta} \approx 2\delta\omega\, g_{\alpha\beta}$$

$$\Rightarrow \delta^{diff \times Weyl} g_{\alpha\beta} = \left(2\delta\omega g_{\alpha\beta} - \nabla_{\alpha} \epsilon_{\beta} - \nabla_{\beta} \epsilon_{\alpha}\right)$$

In Joe's notation, $(\alpha\beta) \to (ab)$

and $\epsilon_\alpha \to \delta\sigma_a$

$$\delta g_{ab} = \left(2\,\delta\omega\, g_{ab} - \nabla_a \delta\sigma_b - \nabla_b \delta\sigma_a\right)$$

$$= \left(2\delta\omega - \nabla_c \delta\sigma^c\right) g_{ab} - 2\underbrace{\left[\frac{1}{2}\left(\nabla_a \delta\sigma_b + \nabla_b \delta\sigma_a - g_{ab}\nabla_c \delta\sigma^c\right)\right]}_{\equiv\, (P_1 \delta\sigma)_{ab}}$$

P_1 : vectors $\to$ ~~tensors~~ symmetric 2-tensors

$\delta\sigma_a$

So $\displaystyle\int [Dg]\, \delta(g^\zeta - \hat{g})$

$$\approx \int [D\delta\omega\, D\delta\sigma]\, \delta\left(-\left\{\left[2\delta\omega - \hat{\nabla}_c \delta\sigma^c\right]\hat{g} - 2\hat{P}_1 \delta\sigma\right\}_{ab}\right)$$

convenient sign; hats mean using $\hat{g} = \delta_{ab}$.

now we use the usual trick.

$$\delta(f_{ab}) = \int [D\beta^{ab}] \exp\left(2\pi i \int d^2\sigma \sqrt{\hat{g}}\, \beta^{ab} f_{ab}\right)$$

take β^{ab} symmetric if f_{ab} is.

This is a $\underline{\delta\text{-functional}}$ $\left(f_{ab}(\sigma_0, \sigma_1) = 0\right)$

$$\int [D\zeta]\, \delta(g^\zeta - \hat{g}) \equiv \Delta_{FP}^{-1}(\hat{g})$$

$$= \int [D\delta\omega\, D\delta\sigma][D\beta^{ab}] \exp\left(2\pi i \int d^2\sigma \sqrt{\hat{g}}\, \beta^{ab}\left[-(2\delta\omega - \hat{\nabla}_c\delta\sigma^c)\hat{g} + 2P_1\delta\sigma\right]_{ab}\right)$$

do $[D\delta\omega]$: requires $\beta^{ab}(\sigma)\hat{g}_{ab}(\sigma) = 0$

$\Rightarrow \beta^{ab}$ traceless

(kills $\hat{\nabla}_c \delta\sigma^c \hat{g}_{ab}\beta^{ab}$)

$$= \int [D\delta\sigma][D\beta'^{ab}_{\text{traceless}}] \exp\left(2\pi i \int d^2\sigma \sqrt{\hat{g}}\, \beta'^{ab}\left[2P_1\delta\sigma\right]_{ab}\right)$$

Another conventional trick:

for a bosonic path integral

$$\int [D\phi] \exp\Big(\int d^d x\ \phi \underbrace{\Delta}_{\text{some operator}} \phi\Big) = (\det\Delta)^{-1/2}$$

but

$$\int [D\psi] \exp\Big(\int d^d x\ \psi \Delta \psi\Big) = (\det\Delta)^{+1/2}$$

we can invert determinants by changing the statistics of our fields.

$\Rightarrow$ take $\delta\sigma_a \to c_a$ anticommuting ~~[illegible]~~ 'ghost'
$\beta'^{ab} \to b^{ab}$

$\Rightarrow$
$$\Delta_{FP}(\hat{g}) = \int [Db\, Dc] \exp\Big(2\pi i \int d^2\sigma \sqrt{\hat{g}}\ b_{ab} (\hat{P}_1 c)^{ab}\Big)$$

rescale fields for convenience, and use

$$b^{ab}\left(\nabla_a c_b + \nabla_b c_a - g_{ab} \hat{\nabla}_c c^c\right) = 2 b^{ab} \nabla_a c_b$$

and go to Euclidean signature.

$$\Rightarrow \quad \Delta_{FP}(\hat{g}) = \int [Db\, Dc] \exp(-S_g)$$

$$S_g = \frac{1}{2\pi}\int d^2\sigma \sqrt{\hat{g}}\; b_{ab} \hat{\nabla}^a c^b$$

Then

$$Z[\hat{g}] = \int [DX\, Db\, Dc] \exp(-S_p - \lambda\chi - S_g)$$

Last time:

$$Z = \int \frac{[Dx\, Dg]}{V_{\text{diff}\times\text{Weyl}}} e^{-S_p - \lambda\chi}$$

$$= \int [Dx\, Db\, Dc]\, e^{-S_p - \lambda\chi - S_g}$$

$$S_g = \frac{1}{2\pi} \int d^2\sigma \sqrt{\hat{g}}\, b_{ab} \hat{\nabla}^a c^b$$

Ghost action in conformal gauge $\hat{g}_{ab} = e^{2\omega}\delta_{ab}$

in complex coords:

$$S_g = \frac{1}{2\pi}\int \frac{d^2z}{2}\, \underbrace{e^{2\omega}}_{\text{from } \sqrt{\hat{g}}} \left(b_{zz}\hat{\nabla}^z c^z + b_{\bar z\bar z}\hat{\nabla}^{\bar z} c^{\bar z} \right)$$

but $\hat{\nabla}^z = \hat{g}^{z\bar z}\nabla_{\bar z} = e^{-2\omega}\cdot 2\nabla_{\bar z}$

($\underline{\text{and}}$, $\nabla_{\bar z} c^z = \partial_{\bar z} c^z$)

[$\underline{\text{used}}$ $b_{z\bar z} = 0$ which follows from $b_{ab}g^{ab} = 0$.]

$$= \frac{1}{2\pi}\int d^2z \left(b_{zz}\nabla_{\bar z} c^z + b_{\bar z\bar z}\nabla_z c^{\bar z} \right)$$

= (exercise, check $\Gamma^z_{\bar z z}$ etc)

$$\frac{1}{2\pi}\int d^2z \left(b_{zz}\partial_{\bar z} c^z + b_{\bar z\bar z}\partial_z c^{\bar z} \right)$$

no ω!

$\Rightarrow$ $\underline{\text{Weyl inv't}}$

~~So $b_{zz}\partial_{\bar z}c^z$ must be $(h,\tilde h) = (1,1)$~~

~~$\Rightarrow (h_b, h_c) = (\lambda, 1-\lambda)$ (+ ~ part)~~

But now recall that <u>conformal</u> transforms (<u>diff + Weyl</u>)

$$ds^2 = dz\,d\bar{z}$$

diff $\quad z' = \sigma'^1 + i\sigma'^2 = f(\sigma^1 + i\sigma^2) = f(z)$

$\partial_z f = \dfrac{\partial z'}{\partial z}$

Weyl $\quad \omega = \ln|\partial_z f|$

$\Rightarrow dz' = \partial_z f\, dz$

$$ds'^2 = e^{2\omega}|\partial_z f|^{-2} dz'\,d\bar{z}' = dz'\,d\bar{z}'$$

$\Rightarrow$ metric invariant. see facing page for details

but b_{zz}, c^z are Weyl-invariant

$\Rightarrow$ transform under conformal map via diff, $z' = f(z)$

$$b_{zz} \to \left(\frac{\partial z}{\partial z'}\right)^2 b_{zz} = b_{z'z'} \qquad b' = \left(\frac{\partial z'}{\partial z}\right)^{-2} b \qquad h = 2$$

$$c^z \to \frac{\partial z'}{\partial z} c^z = c^{z'} \qquad c' = \left(\frac{\partial z'}{\partial z}\right)^{1} c \qquad h = -1$$

recall $\mathcal{O}'(z', \bar{z}') = \left(\dfrac{\partial z'}{\partial z}\right)^{-h} \mathcal{O}(z, \bar{z})$

		h	$\tilde{h}$
$\Rightarrow$	b_{zz}	2	0
	c^z	-1	0
	$b_{\bar{z}\bar{z}}$	0	2
	$c^{\bar{z}}$	-1	0

$$S_g = \frac{1}{2\pi}\int d^2z \; b\bar{\partial}c \; + \; \frac{1}{2\pi}\int d^2z \; \tilde{b}\partial\tilde{c}$$

bc CFT $\quad (h_b, h_c) = (2,-1)$

$\tilde{b}\tilde{c}$ CFT $\quad (\tilde{h}_b, \tilde{h}_c) = (2,-1)$

But we computed $c_{bc} = -3(2\lambda-1)^2+1$

and this was for $(h_b, h_c) = (\lambda, 1-\lambda)$

$\Rightarrow \lambda = 2$

$\Rightarrow c_{bc} = -3(3)^2+1 = -26.$

also $a^g = \frac{1}{2}\lambda(1-\lambda) = -1.$

Assemble all the pieces:

we have the CFT

$$S = \frac{1}{2\pi}\int d^2z \left[\partial X^\mu \bar{\partial} X_\mu + b \bar{\partial} c \right] \quad + \sim$$

with central charge

$$c = c^X + c_{bg} = D + (-26)$$

and vacuum energy

$$A = a^X + a^g = 0 + (-1)$$

$$\Rightarrow \boxed{\begin{array}{ll} A = -1 & \tilde{A} = -1. \\ c = D - 26 & \tilde{c} = D - 26 \end{array}}$$

①

Our classical theory is diff, Weyl, and Poincaré invariant.

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{\gamma}\left[\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - \alpha' R \lambda\right]$$

diff + Poincaré are easily preserved in the quantum theory.
Weyl is not obviously preserved.

eg suppose we regulate a UV divergence with a Pauli-Villars field Y^μ,

$$\Delta S = \mu^2 \int d^2\sigma \sqrt{\gamma}\, Y^\mu Y_\mu$$

diff x Poincaré obvious.
Weyl violated.

So does there exist a Weyl-invariant regulator?
Can we impose some conditions on our theory s.t. Weyl symmetry is non-anomalous?

We'd better: a nonvanishing Weyl anomaly means that $Z[\hat{g}]$ depends on the choice of $\hat{g}$ and that unitarity or covariance may fail to persist.

$$\langle \cdots \rangle_g = \int [Dx\, Db\, Dc]\, e^{-S[x,b,c,g]}$$

we want $\langle \dots \rangle_g = \langle \dots \rangle_{g^\gamma}$ for any $\gamma \in$ Weyl group

$$\delta \langle \cdots \rangle_g = \int [Dx\, Db\, Dc] e^{-S} \left(- \frac{\delta S}{\delta g_{ab}} \delta g_{ab} \right) (\cdots)$$

but $$\frac{\delta S}{\delta g_{ab}(\sigma)} = \frac{\sqrt{g(\sigma)}}{4\pi} T^{ab}(\sigma)$$

$$\delta \langle \cdots \rangle_g = - \int [Dx\, Db\, Dc] e^{-S} \int d^2\sigma \frac{\delta S}{\delta g_{ab}(\sigma)} \delta g_{ab}(\sigma) (\cdots)$$

$$= - \frac{1}{4\pi} \int [Dx\, Db\, Dc]\, e^{-S} \int d^2\sigma \sqrt{g(\sigma)} \left(T^{ab}(\sigma) \cdots \right) \delta g_{ab}(\sigma)$$

$$= - \frac{1}{4\pi} \int d^2\sigma \sqrt{g(\sigma)}\, \delta g_{ab}(\sigma) \langle T^{ab}(\sigma) \cdots \rangle$$

for now, we've imagined that $\delta g_{ab}(\sigma)$ vanishes near the insertions "$\dots$"

Later we'll understand that the insertions must also be Weyl inv't.

for a Weyl transform, $\delta g_{ab}(\sigma) = 2\delta\omega\, g_{ab}(\sigma)$

$$\Rightarrow \delta_W \langle \cdots \rangle_g = -\frac{1}{2\pi}\int d^2\sigma \sqrt{g(\sigma)}\, \langle T^a{}_a(\sigma) \cdots \rangle$$

$\Rightarrow$ Weyl inv't if $T^a{}_a = 0$.

Now we know $T^a{}_a = 0$ for flat worldsheet (unit gauge $g_{ab} = \delta_{ab}$).

This is how we got $T^{ab} \Leftrightarrow \{T(z), \tilde{T}(\bar{z})\}$

(CFT is diff + conformally inv't)

So any anomaly must be measured by the curvature of the worldsheet.

$[T] \sim M^2$ so $T^a{}_a = a_1 R + \cdots$

$[R] \sim M^2$

higher terms like $a_2 \nabla^2 R$

would have, on dimensional grounds, $a^2 \sim L^2$

$\Rightarrow$ in UV limit $L \to 0$, no contribution.

In HW you have shown that in fact

$$a_1 = -\frac{c}{12} \qquad c = c_{tot} = c^X + c^{gh}$$
$$= D - 26$$

$\Rightarrow$ Weyl invariance (vanishing Weyl anomaly) requires $D = 26$

$(c_{tot} = 0)$.

Now let's consider more general backgrounds for string propagation.