Unit 2.

Conformal field theory: basics

conformal invariance in general d

Noether's theorem
Ward identities derived twice

complex coordinates

Ward identities in complex coordinates.

OPEs.

Conformal Invariance + Conformal Field Theory

Def. conformal invariance

invariance (of an action) under $g_{\mu\nu} \to \Lambda(x^\rho)\, g_{\mu\nu}$

We've already discussed the infinitesimal form,

$$g'^{\mu\nu} = g^{\mu\nu} + \partial^\mu \epsilon^\nu + \partial^\nu \epsilon^\mu = g^{\mu\nu} + \Lambda g^{\mu\nu}$$

and we showed that in 2d,

$$\partial^\alpha \epsilon^\beta + \partial^\beta \epsilon^\alpha = \Lambda \eta^{\alpha\beta}$$

is solved by $\epsilon = \epsilon^+(\xi^0 + \xi^1) + \epsilon^-(\xi^0 - \xi^1)$

(lightcone reparametrizations).

Let's redo this in more generality.

(i) $\quad \partial \partial^{\partial} \partial_\mu \epsilon_\rho = (2-D) \partial_\mu \partial_\rho \Lambda$

(ii) $\quad 2 \partial^{\partial} \partial_\rho \epsilon_\mu = (2-D) \partial_\rho \partial_\mu \Lambda$

$$\Rightarrow \quad 2 \partial^{2} (\partial_\mu \epsilon_\rho + \partial_\rho \epsilon_\mu) = 2(2-D) \partial_\mu \partial_\rho \Lambda$$

$$\Rightarrow \quad \partial^{2} (\partial_\mu \epsilon_\rho + \partial_\rho \epsilon_\mu) = \partial^{2} \Lambda \eta_{\mu\rho} \qquad \text{(def)}$$

but using (i), (ii), $\quad \partial^{2} \partial_\mu \epsilon_\rho = \partial^{2} \partial_\rho \epsilon_\mu$

$$\Rightarrow \quad 2 \partial^{2} \partial_\mu \epsilon_\rho = \partial^{2} \Lambda \eta_{\mu\rho}$$

$$\text{''} \quad (2-D) \partial_\rho \partial_\mu \Lambda$$

$$\Rightarrow \quad (2-D) \partial_\mu \partial_\nu \Lambda = \partial^\rho \partial_\rho \Lambda \, \eta_{\mu\nu}$$

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \Lambda(x)\, g_{\mu\nu}$$ ($= \Lambda(x)\eta_{\mu\nu}$ for simplicity)

$$\Rightarrow \quad 2\partial^\rho \epsilon_\rho = \Lambda \cdot d$$ eg $\partial_\alpha \partial_\beta = \partial_\beta \partial_\alpha$

$$\Lambda(x) = \frac{2}{d}\partial_\rho \epsilon^\rho$$

act on $\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \Lambda(x)\eta_{\mu\nu}$

with ∂_ρ:

(i) $\partial_\rho\partial_\mu \epsilon_\nu + \partial_\rho\partial_\nu \epsilon_\mu = \eta_{\mu\nu}\partial_\rho \Lambda(x)$

(ii) $\partial_\mu\partial_\nu \epsilon_\rho + \partial_\mu\partial_\rho \epsilon_\nu = \eta_{\nu\rho}\partial_\mu \Lambda(x)$

(iii) $\partial_\nu\partial_\mu \epsilon_\rho + \partial_\nu\partial_\rho \epsilon_\mu = \eta_{\mu\rho}\partial_\nu \Lambda(x)$

$\Rightarrow$ (ii) + (iii) − (i) $\Rightarrow$ (nb mixed partials are equal since $g=\eta$)

$$2\partial_\mu\partial_\nu \epsilon_\rho = \eta_{\mu\rho}\partial_\nu\Lambda + \eta_{\nu\rho}\partial_\mu\Lambda - \eta_{\mu\nu}\partial_\rho\Lambda$$

$$\Rightarrow \quad 2\partial^2 \epsilon_\rho = 2\partial_\rho\Lambda - d\,\partial_\rho\Lambda$$

$$\cancel{\Rightarrow \quad 2\partial^2\partial_\rho\epsilon^\rho = (2-d)\partial^2\Lambda}$$

$$\cancel{\text{but} \quad \partial^2\Lambda = \frac{2}{d}\partial^2\partial_\rho\epsilon^\rho}$$

$$\Rightarrow \quad 2\partial^2\partial_\mu\epsilon_\rho = (2-d)\partial_\mu\partial_\rho\Lambda = (2-d)\partial_\rho\partial_\mu\Lambda = 2\partial^2\partial_\rho\epsilon_\mu$$

but $$\underbrace{\partial^2(\partial_\mu\epsilon_\rho + \partial_\rho\epsilon_\mu)}_{2\partial^2\partial_\mu\epsilon_\rho} = \partial^2\Lambda\,\eta_{\mu\rho}$$

so $(2-d)\,\partial_\mu \partial_\nu \Lambda = \eta_{\mu\nu}\,\partial_\rho \partial^\rho \Lambda$

for $d=2$, $\nabla^2 \Lambda = 0 \Rightarrow \Lambda = \Lambda^+(\xi^+) + \Lambda^-(\xi^-)$
as before,
an infinite set of symmetries.

for $d>2$,

$$(2-d)\,\partial^2 \Lambda = d\,\partial^2 \Lambda$$
$$(2d-2)\,\partial^2 \Lambda = 0 \Rightarrow \partial^2 \Lambda = 0$$
$$\text{and} \Rightarrow (2-d)\,\partial_\mu \partial_\nu \Lambda = 0$$
$$\Rightarrow \Lambda = \Lambda_0 + \Lambda_\mu X^\mu$$

$\Rightarrow \quad \partial_\mu \partial_\nu \epsilon_\rho \sim \text{const}$ (using (iii)+(ii)−(i))

$$\Rightarrow \quad \epsilon_\mu = a_\mu + b_{\mu\nu} X^\nu + \underbrace{c_{\mu\nu\rho}}_{\text{sym}} X^\nu X^\rho$$

$\epsilon_\mu = a_\mu$ translation

$\epsilon_\mu = b_{\mu\nu} X^\nu$ dilation+rotation
because $b_{\mu\nu} + b_{\nu\mu} = \frac{2}{d}\,\underbrace{\partial^\rho (b_{\rho\sigma} X^\sigma)}_{b_\rho{}^\rho}\,\eta_{\mu\nu}$

$\Rightarrow b_{\mu\nu} = b_1 \eta_{\mu\nu} + b_2 m_{\mu\nu}$ ← antisym.

$G_\mu = C_{\mu\nu\rho} x^\nu x^\rho$ 'special conformal transformation'

One can show the SCT is the infinitesimal form of

$$x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2 x^2}$$

which in turn is

inversion $\circ$ translation by b^μ $\circ$ inversion

$$\frac{x^\mu - b^\mu x^2}{1 - 2b_\mu x^\mu + b^2 x^2} \longleftarrow \frac{x^\mu}{x^2} - b^\mu \longleftarrow \frac{x^\mu}{x^2} \leftarrow x^\mu$$

We will not have much need of these explicit forms, but just note that

translation	$-i\partial_\mu$
dilation	$-ix^\mu\partial_\mu$
rotation	$i(x_\mu\partial_\nu - x_\nu\partial_\mu)$
SCT	$-i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu)$

can be seen (after linear transformation) to satisfy the algebra of $SO(d+1,1)$.

conformal group in d Euclidean dimensions $= SO(d+1,1)$.

(3.1) (5

Last time, we solved

$$g'^{\mu\nu} = g^{\mu\nu} + \partial^{\mu}\epsilon^{\nu} + \partial^{\nu}\epsilon^{\mu} = g^{\mu\nu} + \Lambda g^{\mu\nu}$$

and found that for $D > 2$,

$$\epsilon_{\mu} = \underbrace{a_{\mu}}_{\text{translations}} + \underbrace{b_{\mu\nu} x^{\nu}}_{\text{rotations \& dilations}} + \underbrace{c_{\mu\nu\rho} x^{\nu} x^{\rho}}_{\text{SCT's}}$$

These transformations generate the conformal group in D Euclidean dimensions, $SO(D+1,1)$

and the associated infinitesimal generators obey the conformal algebra $so(D+1,1)$.

for $D = 2$,
$$\xi^{+} \to \xi^{+}(\xi^{+})$$
$$\xi^{-} \to \xi^{-}(\xi^{-})$$
is a conformal transformation, i.e. $g'^{\alpha\beta} = (1+\Lambda)\, g^{\alpha\beta}$.

To understand those transformations, we'll change to complex coordinates.

$$z = \xi^0 + i\xi^1$$
$$\bar{z} = \xi^0 - i\xi^1$$

$$ds^2 = d\xi^{0\,2} + d\xi^{1\,2}$$

for any $V^a = \{V^0, V^1\}$

$$V^z \equiv V^0 + iV^1$$
$$V_z = \tfrac{1}{2}(V^0 - iV^1)$$
$$V^{\bar{z}} = V^0 - iV^1$$
$$V_{\bar{z}} = \tfrac{1}{2}(V^0 + iV^1)$$

$$g_{\cdot\cdot}: \quad \begin{array}{c|cc} & z & \bar{z} \\ \hline z & 0 & \tfrac{1}{2} \\ \bar{z} & \tfrac{1}{2} & 0 \end{array}$$

$$g^{\cdot\cdot}: \quad \begin{array}{c|cc} & z & \bar{z} \\ \hline z & 0 & 2 \\ \bar{z} & 2 & 0 \end{array}$$

i.e.
$$g_{zz} = g_{\bar{z}\bar{z}} = g^{zz} = g^{\bar{z}\bar{z}} = 0$$
$$g_{z\bar{z}} = g_{\bar{z}z} = \tfrac{1}{2}$$
$$g^{z\bar{z}} = g^{\bar{z}z} = 2$$

$$d^2z = 2d\xi^0 d\xi^1$$

$$\partial \equiv \partial_z \equiv \tfrac{1}{2}(\partial_0 - i\partial_1)$$
$$\bar{\partial} \equiv \partial_{\bar{z}} \equiv \tfrac{1}{2}(\partial_0 + i\partial_1)$$

so $\quad \partial z = 1 \qquad \partial \bar{z} = 0$

$\quad \bar{\partial} z = 0 \qquad \bar{\partial}\bar{z} = 1$

Our translations $x^\mu \to x^\mu + \epsilon^\mu(x)$

and conformal condition $\partial^\mu \epsilon^\nu + \partial^\nu \epsilon^\mu = \Lambda \delta^{\mu\nu}$

are now

$$z \to z + \epsilon^z(z,\bar z)$$
$$\bar z \to \bar z + \epsilon^{\bar z}(z,\bar z)$$

$$\partial^z \epsilon^z = \partial^{\bar z} \epsilon^{\bar z} = 0$$
$$\partial^{\bar z} \epsilon^z + \partial^z \epsilon^{\bar z} = \Lambda$$

now $\partial^z = g^{z\bar z} \partial_{\bar z} = 2\partial_{\bar z}$

$\partial^{\bar z} = \quad 2\partial_z$

$$\Rightarrow \begin{cases} \bar\partial \epsilon^z = 0 \\ \partial \epsilon^{\bar z} = 0 \end{cases} \quad \text{from } 1^{st} \text{ eq}$$

$\partial \epsilon^z + \bar\partial \epsilon^{\bar z} = \frac{1}{2}\Lambda$ (but, we saw Λ obeys

$\nabla^2 \Lambda = 0 \Rightarrow \partial\bar\partial \Lambda = 0$

$\Rightarrow \Lambda = \Lambda_{(z)}(z) + \Lambda_{(\bar z)}(\bar z)$)

$$\Rightarrow \quad \epsilon^z = \epsilon^z(z)$$
$$\epsilon^{\bar z} = \epsilon^{\bar z}(\bar z)$$

We conclude that in complex coords, infinitesimal conformal transformations

are

holomorphic reparametrizations

$$z \to z + \epsilon(z)$$

Let's derive this <u>crucial</u> result for finite transformations.

we want $g^{\alpha\beta} \to \Lambda g^{\alpha\beta}$

under $\xi^{\alpha} \to \xi'^{\alpha}$

$$g'^{\alpha\beta} = \frac{\partial \xi'^{\alpha}}{\partial \xi^{\gamma}} \frac{\partial \xi'^{\beta}}{\partial \xi^{\delta}} g^{\gamma\delta} = \Lambda g^{\alpha\beta}$$

$$\Rightarrow \left(\frac{\partial \xi'^{0}}{\partial \xi^{0}}\right)^2 + \left(\frac{\partial \xi'^{0}}{\partial \xi^{1}}\right)^2 = \left(\frac{\partial \xi'^{1}}{\partial \xi^{0}}\right)^2 + \left(\frac{\partial \xi'^{1}}{\partial \xi^{0}}\right)^2$$

and $$\frac{\partial \xi'^{0}}{\partial \xi^{0}} \frac{\partial \xi'^{1}}{\partial \xi^{0}} + \frac{\partial \xi'^{0}}{\partial \xi^{1}} \frac{\partial \xi'^{1}}{\partial \xi^{1}} = 0$$

Set $\xi'^{0} = u \quad \xi'^{1} = v$

$\xi^{0} = x \quad \xi^{1} = y$

Alternate derivation in complex coords:

$z, \bar{z} \to w, \bar{w}$

$g'^{w\bar{w}} = \frac{\partial w}{\partial z}\frac{\partial \bar{w}}{\partial \bar{z}} g^{z\bar{z}}$ other terms 0 because $g^{zz}, g^{\bar{z}\bar{z}} = 0$

$g'^{ww} = \frac{\partial w}{\partial z}\frac{\partial w}{\partial \bar{z}} g^{z\bar{z}}$

$g'^{\bar{w}\bar{w}} = \frac{\partial \bar{w}}{\partial z}\frac{\partial \bar{w}}{\partial \bar{z}} g^{z\bar{z}}$

$\exists$ require $\frac{\partial w}{\partial z}, \frac{\partial \bar{w}}{\partial \bar{z}} \neq 0$ (at least, not identically)

and $\frac{\partial w}{\partial \bar{z}}, \frac{\partial \bar{w}}{\partial z} = 0$

$\Rightarrow$ $w = w(z)$

$\bar{w} = \bar{w}(\bar{z})$

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2$$

(*)

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = 0$$

but recall the Cauchy–Riemann eqns

if $z = x+iy$
$\bar{z} = x - iy$
and $f(x,y) = u + iv$

then

$$\frac{\partial f}{\partial \bar{z}} = 0, \Leftrightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\Leftrightarrow$ (*)

So a finite transformation

$$z \to z'(z)$$
$$(\bar{z} \to \bar{z}'(z))$$

leads to $$g'^{z\bar{z}} = \left(\frac{\partial z'}{\partial z}\right)\left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right) g^{z\bar{z}}$$
$$= \left|\frac{\partial z'}{\partial z}\right|^2 g^{z\bar{z}}$$

So (even finite) holomorphic maps

$$z \to f(z)$$

are conformal transformations.

(We effectively have a conformal algebra acting on z and one acting on $\bar{z}$. So we've begun considering $\mathbb{C} \otimes \mathbb{C}$.)

The conformal algebra in complex coordinates:

$$z \to z' = z + \epsilon_n(z)$$

$$\bar{z} \to \bar{z}' = \bar{z} + \bar{\epsilon}_n(\bar{z})$$

take a <u>basis</u> $\epsilon_n = -z^{n+1}$

Then the corresponding <u>generators</u> are

$$l_n \equiv -z^{n+1}\partial_z$$

$$\bar{l}_n \equiv -\bar{z}^{n+1}\partial_{\bar{z}}$$

which obey

$$\begin{cases} [l_m, l_n] = (m-n) l_{m+n} & \text{"}A\text{"} \\ [\bar{l}_m, \bar{l}_n] = (m-n)\bar{l}_{m+n} & \text{"}\bar{A}\text{"} \\ [l_m, \bar{l}_n] = 0 & \end{cases}$$

(Witt algebra)

Note that $\mathcal{A}$ has a finite subalgebra

$\{l_0, l_{-1}, l_1\}$

$$l_0 = -z\partial_z \qquad \text{dilate/rotate} \quad \text{in } \mathbb{C}$$

$$l_1 = -z^2\partial_z \qquad \text{SCT} \quad \text{in } \mathbb{C}$$

$$l_{-1} = -\partial_z \qquad \text{translations} \quad \text{in } \mathbb{C}$$

similarly $\bar{\mathcal{A}}$ has $\{\bar{l}_{-1}, \bar{l}_0, \bar{l}_1\}$ acting on the other $\mathbb{C}$.

This is the algebra that makes CFT tractable.

Noether's Theorem + Ward Identities

cf §2.4 in DMS
§9.6 of Peskin

NT: to every continuous symmetry of the action we may associate a classically conserved current

WI: express effect of symmetry on correlation functions

We'll focus on symmetries under

$$x'^{\mu} = x^{\mu} + \omega_a \frac{\delta x^{\mu}}{\delta \omega_a}$$

ω_a: infinitesimal parameters

under which a scalar $\phi(x)$ transforms as

$$\phi'(x') = \phi(x)$$

$$\phi'(x) = \phi(x) - \omega_a \frac{\delta x^{\mu}}{\delta \omega_a} \partial_{\mu} \phi(x')$$

or, $\phi(x)$ in last term, to $O(\omega^2)$

we define the generator

$$i G_a \phi \equiv \left(\frac{\delta x^{\mu}}{\delta \omega_a} \partial_{\mu}\right) \phi$$

so $\phi'(x) = \phi(x) - i\omega_a G_a \phi$

Ex $\omega_a = \omega^\nu$

$$\frac{\delta x^\mu}{\delta \omega_a} = \delta^\mu_\nu$$

$$\phi'(x) = \phi(x) + \omega^\nu \partial_\nu \phi(x)$$

$$i G_\nu = \partial_\nu$$

$$G_\nu = -i \frac{\partial}{\partial x^\nu} \qquad (P_\nu)$$

Ex $x'^\mu = x^\mu + \underbrace{\omega^\mu{}_\nu x^\nu}_{\underbrace{\omega_{\rho\nu}\eta^{\rho\mu}}_{\text{with } \omega \text{ antisymm.}} x^\nu}$ (Lorentz)

$$\frac{\delta x^\mu}{\delta \omega_{\rho'\nu'}} = \frac{\delta}{\delta \omega_{\rho'\nu'}} \left(\omega_{\rho\nu} - \omega_{\nu\rho}\right) \frac{1}{2} \eta^{\rho\mu} x^\nu$$

$$= \frac{1}{2}\left(\eta^{\rho'\mu} x^{\nu'} - \eta^{\nu'\mu} x^{\rho'}\right)$$

$$\Rightarrow i G_{\rho\nu} = \frac{1}{2}\left(\eta^{\rho\mu} x^\nu \partial_\mu - \eta^{\nu\mu} x^\rho \partial_\mu\right)$$

$$= \frac{1}{2}\left(x^\nu \partial^\rho - x^\rho \partial^\nu\right)$$

need also additional term involving 'internal' transformation.

given $S = \int d^d x \, \mathcal{L}(\phi(x), \partial_\mu \phi(x))$

and the symmetry $x'^\mu = x^\mu + \omega_a \frac{\delta x^\mu}{\delta \omega_a}$
$\phi'(x') = \phi(x)$

$$S = \int d^d x' \, \mathcal{L}(\underbrace{\phi'(x')}_{\phi(x)}, \partial'_\mu \underbrace{\phi'(x')}_{\phi(x)})$$ relabeling

$$\frac{\partial}{\partial x'^\mu} = \left(\frac{\partial x^\nu}{\partial x'^\mu}\right)\frac{\partial}{\partial x^\nu}$$

and there is a Jacobian $\left|\frac{\partial x'}{\partial x}\right|$ for the change of variables $x' \to x$

$$\Rightarrow S = \int d^d x \left|\frac{\partial x'}{\partial x}\right| \mathcal{L}\left(\phi(x), \frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu \phi(x)\right)$$

now since $x'^\mu = x^\mu + \omega_a \frac{\delta x^\mu}{\delta \omega^a}$

$$\frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu{}_\nu + \partial_\nu\left(\omega_a \frac{\delta x^\mu}{\delta \omega^a}\right)$$

$$\det\left(\frac{\partial x'}{\partial x}\right) \approx 1 + \mathrm{Tr}\left(\frac{\partial x'}{\partial x}\right) \approx 1 + \partial_\mu\left(\omega_a \frac{\delta x^\mu}{\delta \omega_a}\right)$$

finally $$\frac{\partial x^\nu}{\partial x'^\mu} = \delta^\nu{}_\mu - \partial_\mu\left(\omega_a \frac{\delta x^\nu}{\delta \omega_a}\right)$$

so that we have

$$S = \int d^d x \left[1 + \partial_\mu \left(\omega_a \frac{\delta X^\mu}{\delta \omega_a} \right) \right] \mathcal{L}\left(\phi(x), \partial_\mu \phi(x) - \partial_\nu \phi(x) \cdot \partial_\mu \left(\omega_a \frac{\delta}{\delta \omega} \right.\right.$$

$$\delta S \approx \int d^d x \left\{ \partial_\mu \left(\omega_a \frac{\delta X^\mu}{\delta \omega_a} \right) \mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (-1)\, \partial_\mu \left(\omega_a \frac{\delta X^\nu}{\delta \omega_a} \right) \partial_\nu \phi \right\}$$

now by hypothesis the action is invariant if $\omega_a = \text{const.}$

$$\delta S \approx \int d^d x \left\{ \mathcal{L}\, \delta^\mu{}_\nu - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi \right\} \frac{\delta X^\nu}{\delta \omega_a} \partial_\mu \omega_a$$

$\Rightarrow \quad \partial_\mu j^\mu_a = 0$ where

$$j^\mu_a = \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu{}_\nu \mathcal{L} \right\} \frac{\delta X^\nu}{\delta \omega_a}$$

That was the usual Noether story.
Now for the Ward identities, which are in some sense the QFT analogues of Noether's thm.

Suppose we have a global symmetry

$$\phi'(x) = \phi(x) + \omega^a \delta_a \phi(x)$$

where ω^a is constant.

$$\left(\text{eg, translation,} \quad \begin{matrix} \omega^a = \epsilon^\mu \\ \delta_a = \partial_\mu \end{matrix} \right)$$

then $\langle \mathcal{O} \rangle \equiv \int [D\phi]\, \mathcal{O} \exp(-S[\phi])$

is invariant.

but under

$$\phi'(x) = \phi(x) + \rho(x)\, \omega^a \delta_a \phi(x)$$

we must have

$$\cancel{\delta\left(\int [D\phi] \exp(-S[\phi])\right) = \int \delta[D\phi] \exp(-S[\phi]) + \int [D\phi] \exp(-\delta S[\phi])}$$

$$\cancel{\text{with} \quad \delta S[\phi] = \int d^d x\, j^\mu_a\, \partial_\mu \rho(x)\, \omega^a}$$

$$\delta\left(\int[D\phi]\exp(-S[\phi])\right) = \int[D\phi]\exp(-S[\phi])\int d^dx\; j^\mu_a\,\partial_\mu[\rho\omega^a$$

Now in reality this variation comes from

δS and $\delta[D\phi]$.

the variation of the measure causes substantial complication (really just subtlety..), and we'll gloss over it.

Paper treatment in DMS. chapter 2.

Now, note that $\phi(x) \to \phi(x) + \omega^a\rho(x)\delta_a\phi(x)$ is a change of variables in the functional integral

$$\Rightarrow \quad \delta\left(\int[D\phi]\exp(-S)\right) = 0 \quad \Rightarrow \partial_\mu j^\mu_a = 0 \quad \text{Noether}$$

$$\delta\left(\int[D\phi]\,\mathcal{O}\exp(-S)\right) = 0 \qquad \mathcal{O} \text{ any local operator}$$

$$\Rightarrow \delta\langle\mathcal{O}\rangle = 0$$

$\Rightarrow \langle\delta\mathcal{O}\rangle$ constrained as follows.

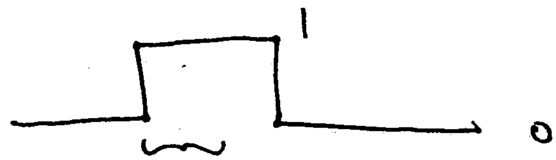
$$0 = \delta\left(\int [D\phi]\, \mathcal{O} \exp(-S)\right)$$

$$= \underbrace{\int \delta[D\phi]\, \mathcal{O} e^{-S}}_{\to 0 \text{ by assumption}} + \underbrace{\int [D\phi]\, \delta\mathcal{O}\, e^{-S}}_{} + \int [D\phi]\, \mathcal{O} e^{-S} (-\delta S)$$

$$0 = 0 + \langle \delta \mathcal{O} \rangle + \int [D\phi]\, e^{-S} \int d^d x\, \mathcal{O}\, j^\mu_a\, \omega^a \partial_\mu \rho(x)$$

$$\Rightarrow \int [D\phi]\, e^{-S} \left\{ \delta\mathcal{O} + \int d^d x\, \mathcal{O}\, \omega^a [\partial_\mu \rho(x)]\, j^\mu_a \right\} = 0$$

set $\rho(x) =$



$$\Rightarrow \boxed{\langle \delta \mathcal{O}(x_0) \rangle = \left\langle \int d^d x\, \omega^a\, \partial_\mu \left[j^\mu_a\, \mathcal{O}(x_0) \right] \right\rangle}$$

this is the Ward identity.

$$\left(\Rightarrow \omega^a \partial_\mu j^\mu_a(x) \cdot \mathcal{O}(x_0) = \delta^d(x - x_0)\, \delta\mathcal{O}(x_0) \right)$$

as operator eqn, ie. in $\langle\ \rangle$

To learn what this is telling us, we will reformulate the result for the W.I. in two dimensions.

(21)

[NB (20) likely not helpful]

divergence theorem:

$$\int_R d^2z \left(\partial_z V^z + \partial_{\bar z} V^{\bar z}\right) = i \oint_{\partial R} V^z d\bar z - V^{\bar z} dz$$

Word identity: (operator eqn, i.e. inside $\langle\rangle$)

$$\delta\mathcal{O} = \int d^2z \underbrace{\partial_\mu j^\mu(z,\bar z)}_{\left(\partial_z j^z + \partial_{\bar z} j^{\bar z}\right)\cdot \frac{i\epsilon}{\cancel{4\pi}}} \mathcal{O}(z_0,\bar z_0)$$

to conform to bk.
(absorb in j.)

$$\Rightarrow -\frac{2\pi i}{\epsilon}\delta\mathcal{O} = \frac{1}{2} i \oint_{\partial R} \left(j^z d\bar z - j^{\bar z} dz\right)\mathcal{O}$$

$g^{z\bar z} j_{\bar z}$ $\quad$ $g^{z\bar z} j_z$

$$\frac{2\pi}{\epsilon}\delta\mathcal{O} = \oint \left(j\, dz - \tilde j\, d\bar z\right)\mathcal{O}(z_0,\bar z_0)$$

now if $j = j(z)$

$\tilde j = \tilde j(\bar z)$

→ check ✓

$$\frac{2\pi}{\epsilon}\delta\mathcal{O} = 2\pi i \operatorname*{Res}_{z\to z_0} j(z)\mathcal{O}(z_0,\bar z_0) + \overline{2\pi i}\,(-1)\operatorname*{Res}_{\bar z\to \bar z_0} \tilde j(\bar z)\mathcal{O}(z_0,\bar z_0)$$

$$\Rightarrow \frac{\delta\mathcal{O}}{i\epsilon} = \operatorname{Res} j\mathcal{O} + \operatorname{Res}\tilde j\mathcal{O}$$

This will serve us well once we know

(1) which j to use, ie. what conserved current?
(equiv: what transformation are we interested in?)

(2) how to evaluate $\operatorname{Res} j\mathcal{O}$.

Concerning (1), our main interest is in local conformal transformations (holomorphic reparametrizations)

$$z \to z + \epsilon(z).$$

So let's find the associated conserved current.

We have seen that under

$$x'^{\mu} = x^{\mu} + \omega_a \frac{\delta x^{\mu}}{\delta \omega_a}$$

$$\delta S = \int d^d x \left\{ \mathcal{L}\, \delta^{\mu}_{\ \nu} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi \right\} \frac{\delta x^{\nu}}{\delta \omega_a} \partial_\mu \omega_a$$

take translations $\omega_a = \epsilon^{\nu}$

$$\frac{\delta x^{\mu}}{\delta \epsilon^{\nu}} = \delta^{\mu}_{\ \nu}$$

$$x'^{\mu} = x^{\mu} + \epsilon^{\mu}$$

Then $\delta S = \int d^d x \underbrace{\left\{ \mathcal{L}\, \delta^{\mu}_{\ \nu} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi \right\}}_{T^{\mu}_{\ \nu}} \partial_\mu \epsilon^{\nu}$

so $T^{\mu}_{\ \nu}$ (or $\epsilon^{\nu} T^{\mu}_{\ \nu}$) is the conserved current.

$T^{\mu\nu}$ generates translations

loosely "$T^{\alpha\beta}$ " worldsheet translations."

really ~~with~~ $j^{\alpha} = \epsilon^{\beta} T^{\alpha}_{\ \beta}$

~~this~~ generates $\delta \xi^{\alpha} = \epsilon^{\alpha}$

$T^{\alpha\beta}$ in complex coords

$T^\alpha{}_\alpha = 0.$

$$T^z{}_{\bar z} = g^{z\bar z} T_{\bar z\bar z} \doteq 2T_{\bar z\bar z}$$
$$T^{\bar z}{}_z = \qquad 2T_{zz}$$
$$T^z{}_z = 2T_{\bar z z}$$
$$T^{\bar z}{}_{\bar z} = 2T_{z\bar z}$$

$\Rightarrow \; T_{z\bar z} = T_{\bar z z} = 0.$

Conservation of T: $\partial^\alpha T_{\alpha\beta} = 0$

$$\partial^{\bar z} T_{zz} = 0 \quad \Rightarrow \quad T_{zz} = T_{zz}(z). \equiv T(z)$$
$$\partial^z T_{\bar z\bar z} = 0 \qquad T_{\bar z\bar z} = T_{\bar z\bar z}(\bar z). \equiv \tilde T(\bar z).$$

So $T, \tilde T$ must generate (local) conformal transformations,

i.e. just as $j^\alpha = \epsilon^\beta T^\alpha{}_\beta$ generates $\sigma^\alpha \to \sigma^\alpha + \epsilon^\alpha(\sigma)$

$j(z) \equiv \epsilon(z) T(z)$ generates $z \to z + \epsilon(z)$

$\tilde j(\bar z) \equiv \epsilon^*(\bar z) \tilde T(\bar z)$ $\qquad \bar z \to \bar z + \epsilon^*(\bar z).$

[$\epsilon(z)$ is what Joe calls $\epsilon v(z)$]

Now we'll use the Ward identity in complex coords to rewrite this in a very efficient form.

Important special case:

$$j_\omega(z) = i v(z)\, T_{zz}(z)$$
$$\tilde{j}_\omega(\bar{z}) = i v(z)^* \, T_{\bar{z}\bar{z}}(\bar{z})$$

corresponding to the symmetry $z' = z + \epsilon v(z)$
$\bar{z}' = z + \epsilon v^*(z)$

(conformal holomorphic reparametrization ω)

$$\Rightarrow \quad \frac{\delta_\omega \mathcal{O}}{i\epsilon} = \operatorname{Res} j_\omega \mathcal{O} + \operatorname{Res} \tilde{j}_\omega \mathcal{O}$$

so if we knew the $\frac{1}{z-z_0}$ term in $T_{zz}(z)\,\mathcal{O}(z_0,\bar{z}_0)$

$\frac{1}{\bar{z}-\bar{z}_0}$ term in $T_{\bar{z}\bar{z}}(\bar{z})\,\mathcal{O}(z_0,\bar{z}_0)$

we would know how $\mathcal{O}$ transformed.

$\Rightarrow$ residues of coincidence of $\mathcal{O}$ with T give, via Ward identity, the transformation of $\mathcal{O}$ under a conformal transformation.

$\Rightarrow$ T_{zz} generates holomorphic reparametrizations (conformal rescalings)

What does "residues of coincidence of Θ' with T" mean?

In QFT one typically wants to understand

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle$$

correlation fn of n local operators.

Such correlators are usually singular when $x_i \to x_j$,

eg $\langle \varphi(x)\varphi(y) \rangle$ is divergent as $x \to y$
(cf HW).

An efficient way to track such behavior is to reexpand in a basis of local operators near the coincident point:

$$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) \to \sum_i c_i(x_1-x_2)\,\mathcal{O}_i(x_2)$$

where $c_i(x_1-x_2)$ are some coeff. fns.

This is the OPE.

In (free) 2d CFT, such expansions actually converge at least as far as the distance to the next problematic operator.

K. Wilson's comment on OPE.

static charge distribution

$$V(\vec{r}) = \int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3 r'$$

for $r \gg$ size of charge distr., we may write

$$V(\vec{r}) = \frac{Q}{r} + \frac{\vec{r}\cdot\vec{d}}{r^3} + \dots$$

Points to notice) (i) always get $\frac{1}{r}, \frac{\vec{r}}{r^3}, \dots$

no matter what ρ is

(ii) definite order of importance

(iii) dimensional analysis $\Rightarrow$ dependence on $\vec{x}-\vec{x}\,'$ for point charges at $x, \vec{x}$.

(eg $\vec{d} \propto (\vec{x}-\vec{x}\,')^1$

$Q \propto (\vec{x}-\vec{x}\,')^0$.

most important: lowest power (0) of $(\vec{x}-\vec{x}\,')$, namely $Q \propto (\vec{x}-\vec{x}\,')^0$.

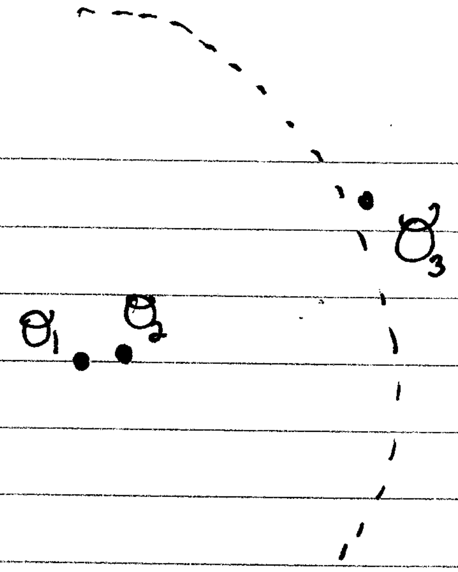
(i) $\Leftrightarrow$ fixed basis of local operators

(ii) $\Leftrightarrow$ most singular (as $\overset{(\vec{x}-\vec{x}\,'\to 0)}{z \to 0}$) terms dominate

(iii) $\Leftrightarrow$ can read off $\vec{x}-\vec{x}\,'$ $(z-z')$ dependence of coeffs C^i

in 2D CFT, (iii) works better than in general OPE

(ii): have $\frac{1}{z-z'}, \frac{1}{(z-z')^2}$ etc terms in Laurent expansion.



This also gives a useful way of defining products of fields at a point (w/o more honest renormalization).

Key: OPE gives leading short-distance behavior.

But, recall that normal ordering (putting all annihilation ops to the right), which by construction ensures a vanishing vacuum expectation value), also effectively subtracts the leading divergence.

$$\mathcal{O}_1(x)\,\mathcal{O}_2(y) = C_i(x-y)\,\mathcal{O}_i(y)$$

$$\lim_{x\to y} \mathcal{O}_1(x)\,\mathcal{O}_2(y) = \frac{C_k}{(x-y)^p}\,\mathcal{O}_k(y) \quad \text{for some } C_k,\ p.$$

$$\langle 0|\,{:}\mathcal{O}_1(x)\,\mathcal{O}_2(y){:}\,|0\rangle = 0$$

$$\Rightarrow\ {:}\mathcal{O}_1(x)\,\mathcal{O}_2(y){:} = \mathcal{O}_1(x)\,\mathcal{O}_2(y) - \langle \mathcal{O}_1(x)\,\mathcal{O}_2(y)\rangle$$

$$\underset{\text{in } \lim x\to y}{=} \mathcal{O}_1(x)\,\mathcal{O}_2(y) - \frac{C_k}{(x-y)^p}\,\mathcal{O}_k(y)$$

So we'll study the leading behavior in $\frac{1}{z}$, and can relate to :: eventually.

As shown in Polchinski §2.1 and indirectly verified in your HW,

$$:X^\mu(z_1,\bar{z}_1)\, X^\nu(z_2,\bar{z}_2): \; = X^\mu(z_1)\, X^\nu(z_2) + \frac{\alpha'}{2}\eta^{\mu\nu} \ln |z_1 - z_2|^2$$

i.e. $X^i(z_1, \bar{z}_1)$ obeys

$$\langle X^i(z_1,\bar{z}_1)\, X^i(z_2,\bar{z}_2)\rangle \overset{(\text{no sum on } i)}{=} -\frac{\alpha'}{2}\left\{ \ln(z_1 - z_2) + \ln(\bar{z}_1 - \bar{z}_2)\right\}$$

hit with $\partial_{z_1}\partial_{z_2}$:

$$\langle \partial X^i(z_1,\bar{z}_1)\, \partial X^i(z_2,\bar{z}_2)\rangle = -\frac{\alpha'}{2}\frac{1}{(z_1 - z_2)^2}$$

$$\langle \bar{\partial} X^i(z_1,\bar{z}_1)\, \bar{\partial} X^i(z_2,\bar{z}_2)\rangle = -\frac{\alpha'}{2}\frac{1}{(\bar{z}_1 - \bar{z}_2)^2}$$

$$\langle \partial X^\mu(z)\, \partial X^\nu(w)\rangle = -\frac{\alpha'}{2}\frac{1}{(z-w)^2}\eta^{\mu\nu}$$

Remark/clarification:

Conserved currents, $T, \tilde{T}, v, v^*$:

current $j_b = V^a T_{ab}$

$$j_z = V^z T_{zz}$$
$$j_{\bar{z}} = V^{\bar{z}} T_{\bar{z}\bar{z}}$$

conservation of T: $\partial_z T_{\bar{z}\bar{z}} = \partial_{\bar{z}} T_{zz} = 0$

conservation of j:

$$\partial^z j_z + \partial^{\bar{z}} j_{\bar{z}} = 0$$

$$\bar{\partial}(V^z T_{zz}) + \partial(V^{\bar{z}} T_{\bar{z}\bar{z}}) = 0$$

if we take $V^z = V^z(z)$, $V^{\bar{z}} = V^{\bar{z}}(\bar{z})$, this is conserved.

This current, $\left(V^z T(z), V^{\bar{z}} \tilde{T}(\bar{z})\right)$,

generates

$$z' = z + \epsilon V^z(z)$$
$$\bar{z}' = \bar{z} + \epsilon V^{\bar{z}}(\bar{z})$$

so we remain on the real slice iff $V^{\bar{z}}(\bar{z}) = V^z(z)^*$

$\Rightarrow$ current should be $\left(v(z) T(z), v^*(z) \tilde{T}(\bar{z})\right)$. ☒

Unit 3.

Conformal field theory: developments.

Wick's theorem
TT OPE + central charge
Contours + charges
State-operator correspondence
Vertex operators
Virasoro algebra
Ghost CFTs

Last time we learned:

$T_{\alpha\beta}$ describable as $T(z), \tilde{T}(\bar{z})$.

The currents
$$j(z) = i v(z) T(z)$$
$$\tilde{j}(\bar{z}) = i v^*(\bar{z}) \tilde{T}(\bar{z})$$
generate conformal transformations (holomorphic reparametrizations)
$$z' = z + \epsilon v(z) \qquad \text{with finite form } z' = z'(z)$$

The Ward identity
$$\frac{1}{i\epsilon}\delta\mathcal{O} = \operatorname*{Res}_{z\to z_0} j(z)\mathcal{O}(z_0,\bar{z}_0) + \operatorname*{Res}_{\bar{z}\to\bar{z}_0} \tilde{j}(\bar{z})\,\mathcal{O}(z_0,\bar{z}_0)$$
then determines conformal transformations of $\mathcal{O}$ in terms of

$$\lim_{z\to w} T(z)\mathcal{O}(w,\bar{w}) \qquad T\mathcal{O} \text{ OPE}$$
$$\lim_{\bar{z}\to\bar{w}} \tilde{T}(\bar{z})\mathcal{O}(w,\bar{w}) \qquad \tilde{T}\mathcal{O} \text{ OPE}$$

We'll now see that for the cases of interest to us, we can compute the OPE by considering normal ordering.

N.B. package everything about a CFT in the OPEs of its basic fields.

We argued that

$:\mathcal{O}(z)\,\mathcal{O}(w):$ is nonsingular (and has zero vev).

if $\quad \mathcal{O}(z)\,\mathcal{O}(w) \sim \displaystyle\sum_{n=-1}^{\infty} C_n (z-w)^n \mathcal{O}_n(w)$

(i.e. only <u>one</u> singular term)

then $\quad \mathcal{O}(z)\mathcal{O}(w) - C_{-1}(z-w)^{-1}\mathcal{O}_{-1}(w) \quad$ is nonsingular.

$$\Rightarrow \quad C_{-1}(z-w)^{-1}\mathcal{O}_{-1}(w) = \mathcal{O}(z)\,\mathcal{O}(w) - :\mathcal{O}(z)\,\mathcal{O}(w):$$

Free fields do have only <u>one</u> singular term $\Rightarrow$ free field OPEs can be computed using normal ordering.

This will suffice for our purposes.

For more general results look at Polchinski (2.7.12) and at §6.5 in DMS.

Now $T_{\alpha\beta} = -\frac{1}{\alpha'} :\left(\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2}\cancel{\delta_{\alpha\beta}} \partial_\gamma X^\mu \partial^\gamma X_\mu\right):$

$\delta_{zz} = 0$

$$\Rightarrow T(z) = -\frac{1}{\alpha'} :\partial X \partial X:$$

$$\tilde{T}(\bar{z}) = -\frac{1}{\alpha'} :\bar{\partial} X \bar{\partial} X:$$

<u>Wick's Theorem</u>

$$:\phi_1(z_1) \cdots \phi_N(z_N): = \phi_1(z_1) \cdots \phi_N(z_N) + \sum \text{subtractions}$$

subtraction / contraction

subtraction takes one or more <u>pairs</u> of fields + replaces them with (minus the) propagator $+\frac{\alpha'}{2}\eta^{\mu\nu} \ln|z_i - z_j|^2$.

cf Polchinski eq 2.2.5

eg

$$:X^\mu(z_1) X^\nu(z_2) X^\rho(z_3):$$

check sign? ok

$$= X^\mu(z_1) X^\nu(z_2) X^\rho(z_3) + \frac{\alpha'}{2}\eta^{\mu\nu} \ln|z_{12}|^2 X^\rho(z_3)$$

$$+ \frac{\alpha'}{2}\eta^{\mu\rho} \ln|z_{13}|^2 X^\nu(z_2)$$

$$+ \frac{\alpha'}{2}\eta^{\nu\rho} \ln|z_{23}|^2 X^\mu(z_1)$$

$$z_{ij} = z_i - z_j$$

So let's compute

$$:T_{zz}(z)\!:\,:\!\partial_z X^\nu(w,\bar{w}) : \qquad \text{"}:T::\partial X:\text{"}$$

$$T(z) = -\frac{1}{\alpha'} : \partial X^\mu(z)\, \partial X_\mu(z) :$$

$$= -\frac{1}{\alpha'}\eta_{\mu\rho} : \partial X^\mu(z)\, \partial X^\rho(z) :$$

$$:T(z)::\partial_z X^\nu(w):$$

$$= :T(z)\,\partial_z X^\nu(w):$$

$$+ \left(-\frac{1}{\alpha'}\eta_{\mu\rho}\right)\left\{ :\partial X^\mu(z): \underbrace{\langle \partial X^\rho(z)\, \partial X^\nu(w)\rangle}_{-\frac{\alpha'}{2}\frac{1}{(z-w)^2}\eta^{\rho\nu}} + :\partial X^\rho(z): \underbrace{\langle \partial X^\mu(z)\, \partial X^\nu(w)\rangle}_{-\frac{\alpha'}{2}\frac{1}{(z-w)^2}\eta^{\mu\nu}} \right.$$

note

$$= \text{regular} - \frac{1}{\alpha'}\left\{ :\partial X^\nu(z): \left(-\frac{\alpha'}{2}\right)\frac{1}{(z-w)^2} + :\partial X^\nu(z): \left(-\frac{\alpha'}{2}\right)\frac{1}{(z-w)^2}\right\}$$

$$= \text{reg} + :\partial X^\nu(z): \frac{1}{(z-w)^2}$$

but $\partial X^\nu(z) = \partial X^\nu(w) + \partial^2 X^\nu(w)\,(z-w) + \cdots$

$$\text{ans} = \text{reg} + \frac{\partial X^\nu(w)}{(z-w)^2} + \frac{\partial^2 X^\nu(w)}{(z-w)} + \cdots$$

$\Rightarrow$ OPE of T with ∂X is

$$T(z)\partial X(w) \sim \frac{\partial X(w)}{(z-w)^2} + \frac{\partial_w \partial X(w)}{(z-w)} + \cdots$$

Back to the Ward identity

$$\frac{1}{i\epsilon}\delta \mathcal{O} = \operatorname*{Res}_{z\to w} j(z)\mathcal{O}(w) + \operatorname*{Res}_{\bar z\to \bar w} \tilde{j}(\bar z)\mathcal{O}(\bar w)$$

where $j(z)$ is the current that generates the symmetry transform δ.

Take
$$j(z) = i v(z) T(z)$$
$$\tilde{j}(\bar z) = i v^*(\bar z)\tilde{T}(\bar z)$$
$$\mathcal{O}(w) = \partial X^\mu(w)$$

$$v(z) = v_0 + v_1 z + \cdots$$

Use $T\partial X \sim \dfrac{\partial X}{(z-w)^2} + \dfrac{\partial^2 X}{z-w}$

and choose $w=0$ for convenience.

$$j(z)\,\mathcal{O}(w) \sim i\left(v_0 + v_1 z + v_2 z^2 + \dots\right)\left(\frac{\partial X}{z^2} + \frac{\partial^2 X}{z} + \cdots\right)$$

$$\sim i\left[v_1 z \frac{\partial X}{z^2} + v_0 \frac{\partial^2 X}{z} + \text{non-residue}\right]$$

and $v_1 = \frac{\partial v}{\partial z}$

$$\text{Res } j(z)\,\mathcal{O}(w) = i\left(\partial v\, \partial X + v\, \partial^2 X\right) = \frac{1}{i\epsilon}\,\delta\, \partial X$$

$$\delta(\partial X) = \left(-i\epsilon\, \partial_w v\right)\partial X - i\epsilon\, v\, \partial_w\, \partial X$$

The transformation in question is $z \to z + \epsilon v(z)$
$\bar{z} \to \bar{z} + \epsilon v^*(\bar{z})$

check i
ok

and we have found

$$\delta\left[\partial X\right] = -i\epsilon\left[\frac{\partial v}{\partial w}\,\partial X + v\frac{\partial}{\partial w}\,\partial X\right]$$

~~This is the infinitesimal form of~~

$$\cancel{\partial X \to \left(\frac{\partial z}{\partial z'}\right)\partial X}$$

Now if we suppose a field ϕ transforms under conformal transformations

$$z \to z + \epsilon v(z) \equiv w$$
$$\bar{z} \to \bar{z} + \epsilon v^*(\bar{z}) \equiv \bar{w}$$

as
$$\phi(w,\bar{w}) = \left(\frac{dw}{dz}\right)^{-h}\left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\tilde{h}} \phi(z,\bar{z})$$

$h, \tilde{h}$ constants "weights"

then for small ϵ

$$\phi(w,\bar{w}) = \left(1+\epsilon\frac{\partial v}{\partial z}\right)^{-h}\left(1+\epsilon\frac{\partial v^*}{\partial \bar{z}}\right)^{-\tilde{h}} \underbrace{\phi\left(w-\epsilon v(z), \bar{w}-\epsilon v^*(\bar{z})\right)}_{\approx\, \phi(w,\bar{w}) - \epsilon v \partial_z \phi - \epsilon v^* \partial_{\bar{z}} \phi}$$

$$\approx \phi(w,\bar{w}) - h\epsilon\frac{\partial v}{\partial z}\phi - \epsilon v \partial_z \phi - \tilde{h}\epsilon\frac{\partial v^*}{\partial \bar{z}}\phi - \epsilon v^* \partial_{\bar{z}}\phi$$

$$\Rightarrow \delta_\epsilon \phi \approx \epsilon\left[-h\phi\frac{\partial v}{\partial z} - v\frac{\partial \phi}{\partial z}\right] + \text{antihol.}$$

matching what we found for ∂X, if $h=1$, $\tilde{h}=0$

(sign issue, w/ ~~never mind~~ fixed)

An important definition is that of a primary field:

$\mathcal{O}$ is primary iff

$$\mathcal{O}'(z',\bar{z}') = (\partial_z z')^{-h} (\partial_{\bar{z}} \bar{z}')^{-\tilde{h}} \mathcal{O}(z,\bar{z})$$

You will show in homework that

$\mathcal{O}$ is primary $\Leftrightarrow$

$$T(z)\,\mathcal{O}(0,0) = \frac{h}{z^2}\mathcal{O}(0,0) + \frac{1}{z}\partial\mathcal{O}(0,0) + \cdots$$

we effectively just did this for ∂X.

Primary fields have very nice conformal transform properties
$\Leftrightarrow$ very nice OPE with T.

Serve as a great basis of operators.

Recap

Word identity relates

$$\delta \mathcal{O} \Leftrightarrow \text{Res } v(z) T(z) \mathcal{O}$$

$$\Leftrightarrow T\mathcal{O} \text{ OPE.}$$

So the $T\mathcal{O}$ OPE tells us about conformal transformat of $\mathcal{O}$.

We can compute the OPE using Wick's theorem.

We've defined primary fields $\mathcal{O}$ to have nice transformation properties $\Leftrightarrow$ nice $T\mathcal{O}$ OPE.

$$T(z) = -\frac{1}{\alpha'} :\partial X^\mu(z)\,\partial X_\mu(z):$$

So let's see if $T(z)$ is a primary field.

$$T(z)T(w) = \frac{1}{\alpha'^2} :\partial X_\mu(z)\partial X_\nu(z): \eta^{\mu\nu} \times$$

$$\times :\partial X_\rho(w)\,\partial X_\sigma(w): \eta^{\rho\sigma}$$

for each contraction, insert $-\frac{\alpha'}{2}\eta_{\cdot\cdot}\frac{1}{(z-w)^2}$

$$= \frac{1}{\alpha'^2}\eta^{\mu\nu}\eta^{\rho\sigma}\Big\{\Big[-\frac{\alpha'}{2}\frac{1}{(z-w)^2}\Big]^2\Big[\eta_{\nu\rho}\eta_{\mu\sigma} + \eta_{\nu\sigma}\eta_{\mu\rho}\Big]$$

$$- \frac{\alpha'}{2}\frac{1}{(z-w)^2}:\Big[\partial X_\mu \partial X_\rho \eta_{\nu\sigma} + \partial X_\nu \partial X_\rho \eta_{\mu\sigma} + \partial X_\mu \partial X_\sigma \eta_{\nu\rho} + \partial X_\nu \partial X_\sigma \eta_{\mu\rho}\Big]:\Big\}$$

trg

$$= \frac{1}{4}\frac{1}{(z-w)^4}\cdot 2\eta^{\rho\sigma}\eta_{\rho\sigma} - \frac{1}{2\alpha'}\frac{4}{(z-w)^2}:\partial X_\mu(z)\partial X^\mu(w):$$

$$\partial X_\mu(z) = \partial X_\mu(w) + (z-w)\partial^2 X_\mu(w)$$

$$= \frac{1}{2}D\frac{1}{(z-w)^4} \underbrace{- \frac{2}{\alpha'}\frac{:\partial X(w)\,\partial X(w):}{(z-w)^2}}_{\frac{2}{(z-w)^2}T(w)} \underbrace{- \frac{2}{\alpha'}\frac{1}{(z-w)} :\partial^2 X(w)\,\partial X(w):}_{\frac{1}{z-w}\partial T(w)}$$

So $T(z)\,T(0) \sim \frac{D}{2z^4} + \frac{2}{z^2}T(0) + \frac{1}{z}\partial T(0)$

like a primary field with $h=2$, except for leading term $\sim \frac{1}{z^4}$.

D is called the central charge.

general CFT

$$T(z)\,T(0) \sim \frac{c}{2z^4} + \frac{2}{z^2}T(0) + \frac{1}{z}\partial T(0)$$

additive: D free scalars $X^\mu \Rightarrow c = D$.

One can easily show

$$\tilde{T}(\bar{z})\,\tilde{T}(0) \sim \frac{D}{2\bar{z}^4} + \frac{2}{\bar{z}^2}\tilde{T}(0) + \frac{1}{\bar{z}}\bar{\partial}\tilde{T}(0)$$

generally, $\tilde{c}$.

So what?

We will soon see that $c \neq 0$ is an obstruction to a consistent quantum theory of strings:

c is proportional to an anomaly in Weyl symmetry.

Worth repeating:

we will soon see, in the path-integral approach, that

the quantum theory of strings possesses the Weyl invariance enjoyed by the classical theory, only if the total central charge $c_{tot} = 0$.

Breakdown of Weyl invariance is a true inconsistency (leads to negative probabilities / Lorentz anomaly / results depend on gauge choice.)

NB we argued that quantum theory looks ok from LCG, iff $D = 26 \Rightarrow c_X = 26$.

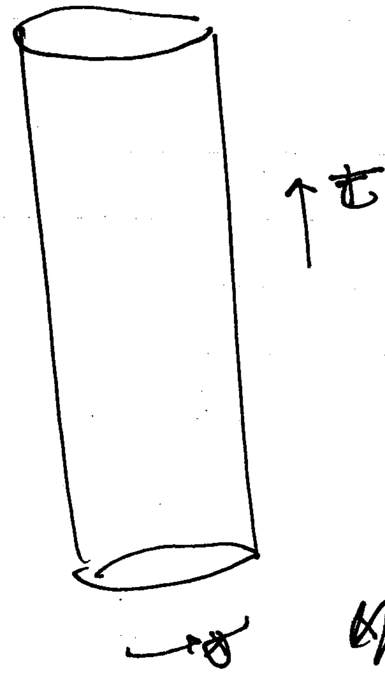
The path-integral approach will agree: $\exists$ another "ghost" sector with $c_{ghost} = -26$, so $c_{tot} = D - 26$.

So far we have studied conformal transformations, generated by the all-important $T(z)$, from the viewpoint of the OPE + Ward identities. (current)

We will also need an explicit realization of the associated charges.

To do this, we take $\int j^0 \, d^{d-1}x$ in general.

What is the appropriate spatial integral in 2 Euclidean dimensions?



$$-\infty < \tau < \infty$$
$$\sigma \sim \sigma + 2\pi.$$

To complexify this, take

$$w = \sigma + i\tau$$
$$(w \sim w + 2\pi).$$

Or, we may use $z \equiv \exp(-iw)$
$= \exp(\tau - i\sigma)$

$z = 0$ infinite past
$|z| \to \infty$ future
$|z| = \text{const}$ region of constant τ (spatial slice)

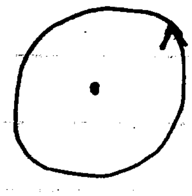
So constant-time slices are circles in the z plane.

Using such slices to define time-ordering is called radial quantization.

time ordering $\to$ radial ordering

Note: suppose we have a hol. conserved current $j(z)$

fixed-time, all-space integral = circular contour $\int$.


C

$$\oint_C j \frac{dz}{2\pi i} \equiv \varphi$$

independence of radius of $C \Leftrightarrow$ time-independence

$\Rightarrow \varphi$ is a conserved charge,

contour integrals of currents give conserved charges

$$Q_i\{C\} \equiv \oint_C \frac{dz}{2\pi i}\, j_i$$

An operator commutator

$$[A, B]$$

where $A = \oint a(z)\,dz$

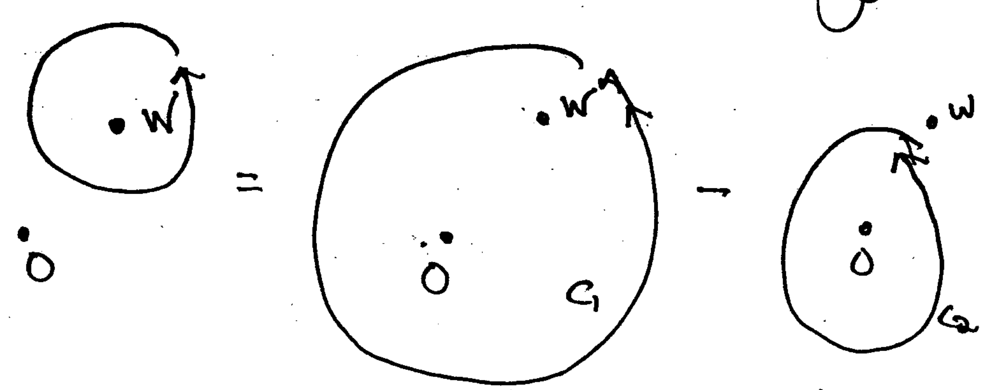
$B = \oint b(z)\,dz$

may now be defined.

$$\oint_w dz\ a(z)\, b(w)$$

$\uparrow$ ie around w

has operator meaning iff a, b are radially ordered (just as one considers time-ordered operators usually)



both fixed-time circles.

$$\Rightarrow \oint_w dz\, a(z) b(w) = \oint_{C_1} dz\, a(z) b(w) - \oint_{C_2} dz\, b(w)\, a(z)$$

since $|z| < |w|$ on C_2.

take limit $C_1 \to C_2$, so shrinking $\oint_w$ to be <u>small</u>.

Now on the one hand,

$$\oint_w dz\, a(z)\, b(w) = \left(\oint_{C_1} dz\, a(z)\right) b(w) - b(w) \oint_{C_2} dz\, a(z)$$

$$= [A, b(w)]$$

but also for $C_1 \to C_2$, we pick up the leading terms as $\lim_{z \to w} a(z) b(w)$

which is to say, the leading ab OPE.

Also, $\oint_0 dw \oint_w dz\, a(z)\, b(w) = \oint dw\, [A, b(w)] = [A, B]$

$$[A, B] = \oint_0 dw \oint_w dz\, a(z)\, b(w)$$

$$= \oint_0 dw\, \operatorname*{Res}_{z \to w} a(z) b(w) \cdot 2\pi i$$

Polchinski puts $\frac{1}{2\pi i}$ in his defs of $Q_1, Q_2, [Q_1, Q_2]$

$$\begin{cases} [Q_1, Q_2] = \oint \frac{dz}{2\pi i} \operatorname{Res} j_1 \cdot j_2 \\ Q = \oint \frac{dz}{2\pi i}\, j(z) \end{cases}$$

This means that:

the OPE of the conserved currents determines the (operator) algebra of the conserved charges.

Pretty amazing.

All-important current: $j = v(z)\,T(z)$.

take $v_m = z^{m+1}$ so $j_m = z^{m+1}\,T(z)$

charge: $$Q_m = \oint \frac{dz}{2\pi i}\, j_m = \oint \frac{dz}{2\pi i}\, z^{m+1}\, T(z) \equiv L_m$$

"Virasoro generator" = Laurent coeff of T.

$$[L_m, L_n] = \int \frac{dw}{2\pi i} \operatorname*{Res}_{z\to w} z^{m+1}\, T(z)\, w^{n+1}\, T(w)$$

$$= \int \frac{dw}{2\pi i} \operatorname*{Res}_{z\to w} z^{m+1}\, w^{n+1} \left\{ \frac{c}{2(z-w)^4} + \frac{2}{(z-w)^2}\, T(w) + \frac{1}{z-w}\, \partial T(w) \right\}$$

Use $z = w + (z-w)$

$$z^{m+1} = \left[w+(z-w)\right]^{m+1} = w^{m+1}\left[1+\frac{z-w}{w}\right]^{m+1}$$

$$\approx w^{m+1}\left[1+(m+1)\frac{z-w}{w} + \frac{1}{2!}\left(\frac{z-w}{w}\right)^2 m(m+1) + \frac{1}{3!}\left(\frac{z-w}{w}\right)^3 (m-1)m(m+1) + \cdots\right]$$

$$\Rightarrow \operatorname*{Res}_{z\to w}\left(w^{m+1+n+1}\frac{1}{z-w}\partial T(w) + w^{m+n+1}(m+1)\,2\frac{1}{z-w}T(w) + \frac{1}{3!}\cdot\frac{c}{2}(m^2-1)m\frac{1}{z-w}w^{m+n-1}\right) + \text{things w/o res.}$$

$$[L_m, L_n] = \int\frac{dw}{2\pi i}\left\{ w^{m+n+2}\,\partial T(w) + w^{m+n+1}\cdot 2(m+1)\,T(w) + w^{m+n-1}\frac{c}{12}(m^3-m)\right\}$$

$\int$ by parts, 1st line becomes

$$w^{m+n+1}T(w)\Big[\underbrace{2m+2-(m+n+2)}_{(m-n)}\Big]$$

$$[L_m, L_n] = \underbrace{\frac{c}{12}(m^3-m)\,\delta_{m,-n}}_{\text{anomalous part}} + \underbrace{(m-n)L_{m+n}}_{\text{classical Virasoro algebra}}$$

7.1

$[\alpha^{\mu}_{m}, \alpha^{\nu}_{n}]$

State-operator map

BG CFT

So we have an ∞ algebra of conserved charges (Virasoro generators = Laurent coeffs of T)

Let's use it, first to determine the normal ordering constant that we fudged in LCG.

In unit gauge + z coords, Polyakov action is

$$S = \frac{1}{2\pi\alpha'} \int d^2z \, \partial X_\mu \bar{\partial} X^\mu$$

with eom $\partial\bar{\partial} X^\mu = 0$

$$\Rightarrow \partial X^\mu = \partial X^\mu(z)$$
$$\bar{\partial} X^\mu = \bar{\partial} X^\mu(\bar{z})$$

define

$$\alpha_m^\mu = \sqrt{\frac{2}{\alpha'}} \oint \frac{dz}{2\pi} z^m \partial X^\mu(z) \Leftrightarrow \partial X^\mu(z) = -i\sqrt{\frac{\alpha'}{2}} \sum_{m=-\infty}^{\infty} \frac{\alpha_m^\mu}{z^{m+1}}$$

$$\tilde{\alpha}_m^\mu = -\sqrt{\frac{2}{\alpha'}} \oint \frac{d\bar{z}}{2\pi} \bar{z}^m \bar{\partial} X^\mu(\bar{z}) \Leftrightarrow \bar{\partial} X^\mu(\bar{z}) = -i\sqrt{\frac{\alpha'}{2}} \sum_{m=-\infty}^{\infty} \frac{\tilde{\alpha}_m^\mu}{\bar{z}^{m+1}}$$

One can compute the spacetime-translation current to justify

$$\alpha_0^\mu = \tilde{\alpha}_0^\mu \equiv \sqrt{\frac{\alpha'}{2}}\, p^\mu$$

consistent with our earlier LCG mode analysis.

Then, integrating,

$$X^\mu(z,\bar{z}) = x^\mu - i\frac{\alpha'}{2} p^\mu \ln|z|^2 + i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{1}{m}\left(\frac{\alpha_m^\mu}{z^m} + \frac{\tilde{\alpha}_m^\mu}{\bar{z}^m}\right)$$

To find $[\alpha,\alpha]$ we can use the XX OPE.
~~Left as an exercise~~

$$\partial X^\mu(z)\,\partial X_\mu(w) \sim -\frac{\alpha'}{2}\frac{1}{(z-w)^2}\eta^\mu{}_\mu \sim -\frac{D\alpha'}{2}\frac{1}{(z-w)^2}$$

$$\partial X^\mu(z)\,\partial X^\nu(w) \sim -\frac{\alpha'}{2}\frac{1}{(z-w)^2}\eta^{\mu\nu}$$

$$[\alpha^\mu_m, \alpha^\nu_n] = \oint \frac{dw}{2\pi i} \operatorname*{Res}_{z\to w} \underbrace{z^m \partial X^\mu(z)\, w^n \partial X^\nu(w) \left(\frac{2}{\alpha'}\right)\left(\frac{i^2}{1}\right)}_{\left(-\frac{\cancel{2}}{\cancel{\alpha'}}\right) z^m w^n \left(-\frac{\cancel{\alpha'}}{\cancel{2}}\right)\frac{1}{(z-w)^2}\eta^{\mu\nu}}$$

from def of Φ.

cf (2.6.11), (2.6.14)

$(w+z-w)^m\, w^n$

$\sim m\left(\frac{z-w}{w}\right) w^m\, w^n$

$$= \oint \frac{dw}{2\pi i}\, m\, w^{m+n-1}\eta^{\mu\nu} = m\,\delta_{m+n}\,\eta^{\mu\nu}$$

$$[\alpha^\mu_m, \alpha^\nu_n] = m\,\delta_{m+n}\,\eta^{\mu\nu}$$

Now let's write L_m in terms of α_n.

$$T = -\frac{1}{\alpha'} :\partial X_\mu \partial X^\mu:$$

$$L_m = \oint \frac{dz}{2\pi i} z^{m+1} \underbrace{T(z)}_{\left[-\frac{1}{\alpha'}(-1)\left(\frac{\alpha'}{2}\right)\right] \sum_{n=-\infty}^{\infty} \frac{\alpha_n^\mu}{z^{n+1}} \sum_{p=-\infty}^{\infty} \frac{\alpha_{p\mu}}{z^{p+1}}}$$

$$= \sum_{p,n} \frac{1}{2} \oint \frac{dz}{2\pi i} z^{m+1-n-1-p-1} \alpha_n^\mu \alpha_{p\mu}$$

$$\text{so } L_m = \frac{1}{2} \sum_{\substack{p,n \\ p+n=m}} \alpha_n^\mu \alpha_{p\mu} = \sum_{n=-\infty}^{\infty} \frac{1}{2} \alpha_n^\mu \alpha_{m-n\,\mu}$$

now for $m \neq 0$, ordering is ~~clear~~ irrelevant: they commute.

$$\cancel{L_m = \frac{1}{2} \sum_{n=-\infty}^{\infty}}$$

How about L_0? after all $\alpha_n^\mu \alpha_{-n\,\mu} = \alpha_{-n}^\mu \alpha_{n\,\mu} + \underbrace{n\,\eta^\mu_\mu}_{\propto D n.}$

The Virasoro algebra gives

$$L_1 L_{-1} - L_{-1} L_1 = \frac{c}{12}(1^3 - 1)\,\delta_{1,1} + (1-(-1))\,L_0$$

$$= 2L_0$$

but $\alpha_n^\mu |0;k\rangle = 0 \quad \forall n>0$

$\Rightarrow \quad L_1 |0;0\rangle = 0$ b/c each term has a lowering op (or $\alpha_0^\mu = p^\mu$)

$L_{-1} |0;0\rangle = 0$ similarly

$$L_0 = \underbrace{\frac{1}{2}\alpha_0^\mu \alpha_{0\mu}}_{\frac{1}{2}\left(\sqrt{\frac{\alpha'}{2}}\,p^\mu\right)^2 = \frac{\alpha' p^2}{4}} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \alpha_{-n}^\mu \alpha_{n\mu} + a_0^X$$

a_0^X ↳ n.o. constant TBD

$$\frac{1}{2}\left(\sqrt{\frac{\alpha'}{2}}\,p^\mu\right)^2 + \sum_{n=1}^{\infty} \alpha_{-n}^\mu \alpha_{n\mu} + a^X \quad \text{new n.o. const}$$

but $\underbrace{(L_1 L_{-1} - L_{-1} L_1)|0;0\rangle}_{0-0} = 2L_0 |0;0\rangle$

$= a^X |0;0\rangle$

$\Rightarrow a^X = 0.$

not yet what we called A in LCG.
[we still have to include the ghosts]

State-operator map

We've identified the radius $r = |z|$ with "time".
What generates time translations?

$$z \to \lambda z$$

is of our familiar conformal form

$$z \to z + \epsilon v(z)$$

with $\epsilon v = \lambda z$

(and, $\bar{z} \to \bar{z} + \epsilon v^*(\bar{z})$ $\qquad \lambda \in \mathbb{R}$
$\epsilon v^* = \lambda \bar{z}$

The associated conserved current is

$$\epsilon v(z) T(z) = \lambda z T(z) \qquad \text{hol.}$$
$$\lambda \bar{z} \tilde{T}(\bar{z}) \qquad \text{antihol.}$$

so the charge is

$$Q^{\text{dilation}}_{\lambda} = \oint \frac{dz}{2\pi i} \lambda z T(z) \equiv \lambda L_0$$

$$\tilde{Q}^{\text{dilation}}_{\lambda} = \oint \frac{d\bar{z}}{2\pi i} \lambda \bar{z} \tilde{T}(\bar{z}) \equiv \lambda \tilde{L}_0$$

$\Rightarrow$ dilation by λ is generated by the charge
$\lambda(L_0 + \tilde{L}_0)$.

$$\Rightarrow \quad H = a(L_0 + \tilde{L}_0)$$ for some constant a.

$$\boxed{H = L_0 + \tilde{L}_0}$$ take $a = 1$ following Joe.

we've just seen that

$$L_0 |0,0\rangle = 0$$ (nb $|0,0\rangle \neq |0,0;k\rangle$)

$$\Rightarrow H|0,0\rangle = 0$$ no zero-point energy in radial quantization.

We have seen that

$$\partial X = \sum \frac{\alpha_n}{z^{n+1}}\left(-i\sqrt{\frac{\alpha'}{2}}\right)$$

$$\bar\partial X = \sum \frac{\tilde\alpha_n}{z^{n+1}}\left(-i\sqrt{\frac{\alpha'}{2}}\right)$$

with $[\alpha_m, \alpha_n] = m\,\delta_{m+n}$

$[\tilde\alpha_m, \tilde\alpha_n] = m\,\delta_{m+n}$

and we have the Fock space spanned by

$$\left\{ \prod_{n=1}^{\infty} (\alpha_{-n})^{N_n} \otimes \prod_{m=1}^{\infty} (\tilde\alpha_{-m})^{N_m} \right\} |0,0;k\rangle$$

This is our Hilbert space.

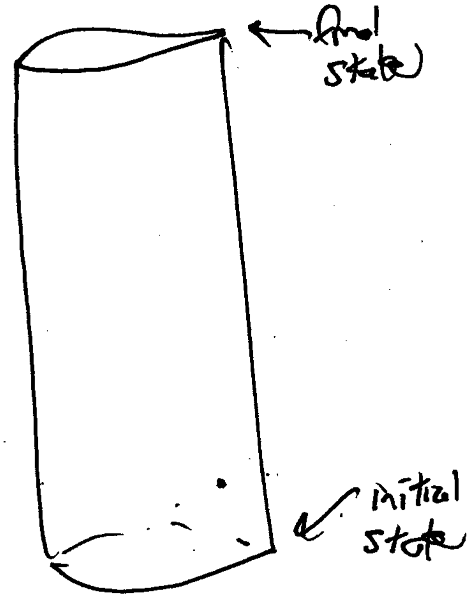
What about the space of local operators $\mathcal{O}_i(z,\bar z)$?

This, it turns out, is spanned by polynomials in

$$\{\partial X, \bar\partial X; \partial^2 X, \bar\partial^2 X; \cdots\} \times \underbrace{e^{ikX(z,\bar z)}}_{\text{not in polynomial}}$$

This makes sense: after all X is the only field in the game.

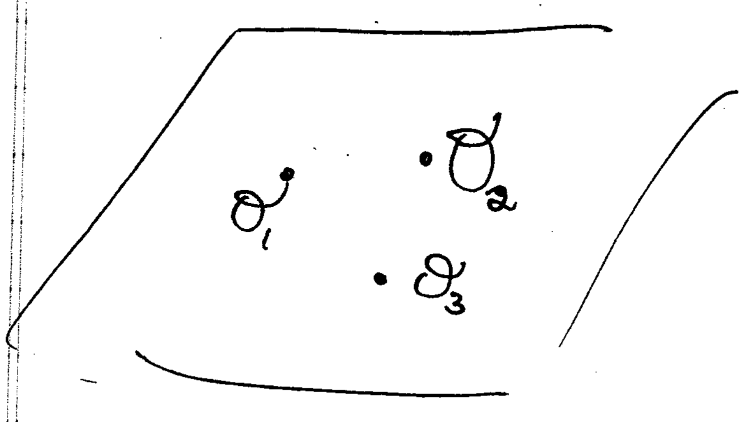
Hilbert space useful for



cylinder

$L_0 + \tilde{L}_0 \uparrow$

operators useful for



plane

$L_0 + \tilde{L}_0$ gives radial translation.

Let's think some more about the correspondence between states + operators.

Hilbert space vs space of local operators

$$S = \int_\Sigma \partial X \bar{\partial} X$$ focusing on one of the X^μ.

path integral:

$$\int \underbrace{\mathcal{D}X(z,\bar{z})}_{\text{maps } \Sigma \to \mathbb{R}} e^{-S[X(z,\bar{z})]}$$

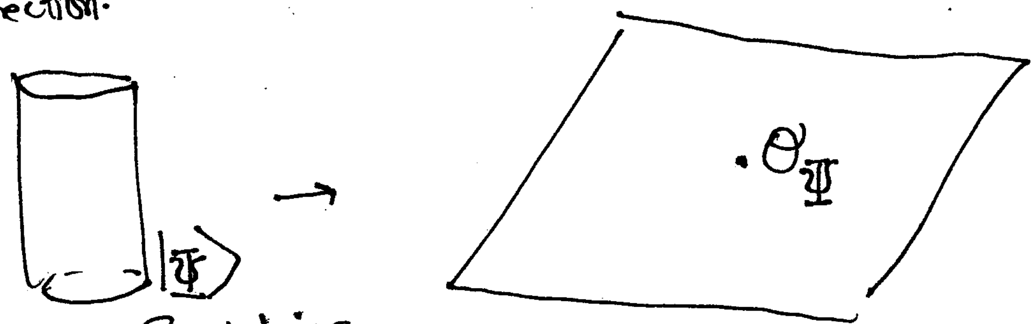
(generally, $\Sigma \to M$ ← spacetime)

objects of interest

$$\langle \mathcal{O}_1(z_1,\bar{z}_1) \cdots \mathcal{O}_n(z_N,\bar{z}_N) \rangle$$

$$\equiv \int \mathcal{D}X(z,\bar{z})\, e^{-S[X(z,\bar{z})]} \mathcal{O}_1(z_1,\bar{z}_1) \ldots \mathcal{O}_N(z_N,\bar{z}_N)$$

Connection:



specifies behavior of fields at $t \to -\infty \leftrightarrow z = 0$

What state $|\Psi_{\mathbb{1}}\rangle$ corresponds to the insertion of the unit operator (ie, nothing) at the origin?

We'll determine this by studying transformations under symmetry, in the operator picture.

skip to pg. 1 "more on the state-operator correspondence"

operator picture $[Q, \mathcal{O}] = \oint_{\delta} \mathcal{O}(\rho)\, d\sigma \quad \operatorname*{Res}_{z\to 0}\; j(z)\, \mathcal{O}'(0,0)$

where j is the cons. current for the symmetry in question.

and $Q = \oint \frac{dz}{2\pi i}\, j(z)$

skip me): too terse)

Take) $j^\mu_m = \partial X^\mu z^m$. $m \geq 0$

This is holomorphic $(\partial\bar\partial X^\mu = 0$ eom$)$

$\Rightarrow$ contour-independent

$\Rightarrow$ time-trans-invt.

$\Rightarrow$ gives a conserved charge)

$$Q_m = \alpha^\mu_m = \sqrt{\frac{2}{\alpha'}} \oint \frac{dz}{2\pi} z^m \partial X^\mu(z)$$

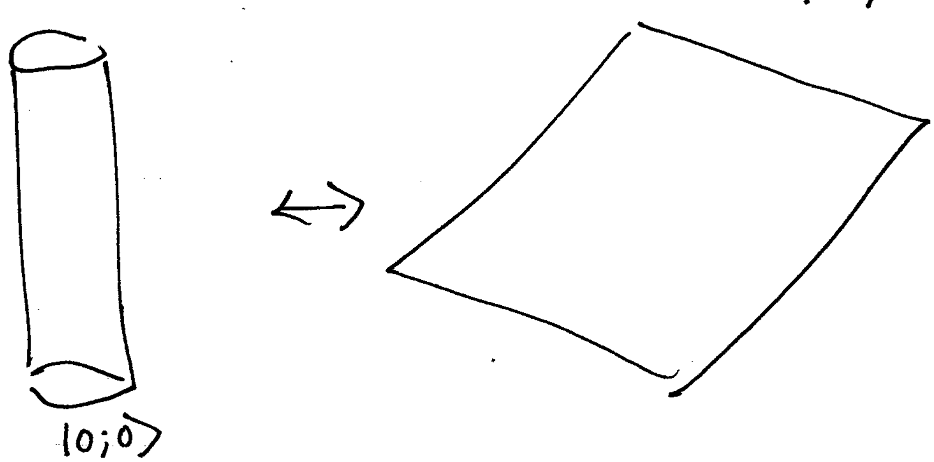
for $\mathcal{O} = \mathbb{1}$,

$$[Q_m, \mathcal{O}_{(op)}] = \operatorname*{Res}_{z\to 0} \partial X^\mu(z) z^m \underbrace{\mathcal{O}(0,0)}_{\mathbb{1}}$$

$$= \operatorname*{Res}_{z\to 0} \partial X^\mu(z) z^m = 0.$$

$$\Rightarrow \alpha^\mu_m \left| \begin{matrix}\text{state corresponding to} \\ \text{insertion of } \mathbb{1}(0,0)\end{matrix} \right\rangle = 0$$

$\Rightarrow$ state corresponding to insertion is $|0;0\rangle$ (used $m=0$ case to get $k=0$)



skip (more too dense)

Now act on this operator/state.

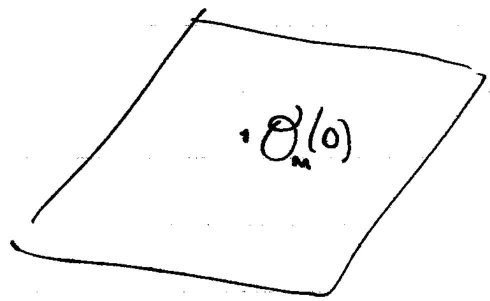
$$\alpha^\mu_{-m}|0;0\rangle \leftrightarrow \sqrt{\frac{2}{\alpha'}}\int \frac{dz}{2\pi} z^{-m} \underbrace{\partial X^\mu(z)}_{\frac{1}{(m-1)!}\partial^m X^\mu(0) z^{m-1}} \text{"} \cdot \mathbb{1}(0) \text{"}$$

state picture

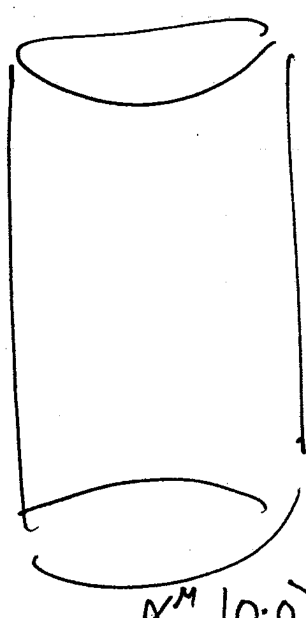
$(m>0)$

$$= \sqrt{\frac{2}{\alpha'}} \frac{i}{m-1!} \partial^m X^\mu(0)$$

so inserting $\mathcal{O}_m(0) = \sqrt{\frac{2}{\alpha'}} \frac{i}{m-1!} \partial^m X^\mu(0)$



corresponds to



$\alpha^\mu_{-m}|0;0\rangle$

More on the state-operator correspondence,

Consider the QM path integral

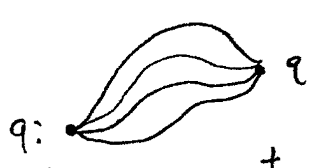
$$\langle q_f, T | q_i, 0\rangle = \int [dq]\, e^{\frac{i}{\hbar}\int_0^T dt\, L(q,\dot q)}$$

'cut open' at some time t, $0<t<T$:

$$\langle q_f, T | q_i, 0\rangle = \underbrace{\int dq(t)}_{\text{ordinary integral}} \int_{q,t}^{q_f,T} [dq]\, e^{\frac{i}{\hbar}\int_t^T L} \int_{q_i,0}^{q,t} [dq]\, e^{\frac{i}{\hbar}\int_0^t L}$$

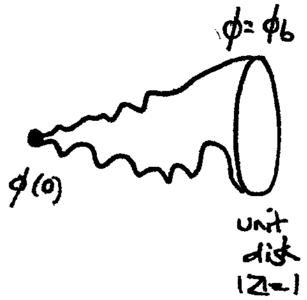
$$= \int \langle q_f, T | \underbrace{q,t\rangle\langle q,t}_{\text{could replace with any complete set of states}} | q_i, 0\rangle \, dq$$

upshot: $\displaystyle\int_{q_i,0}^{q,t} [dq]\, e^{\frac{i}{\hbar}\int_0^t L} \iff$ a state $\Psi[q(t)]$

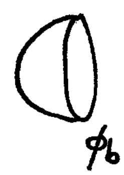


generalize to string case

$$\int_{\phi=\phi_b} [d\phi_i]\, e^{-S[\phi_i]} \qquad \equiv \Psi(\phi_b)$$



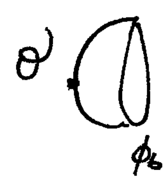
often drawn



is a <u>state</u>.

so is

$$\int_{\phi=\phi_b} [d\phi_i]\, e^{-S[\phi_i]}\, \mathcal{O}(0)$$



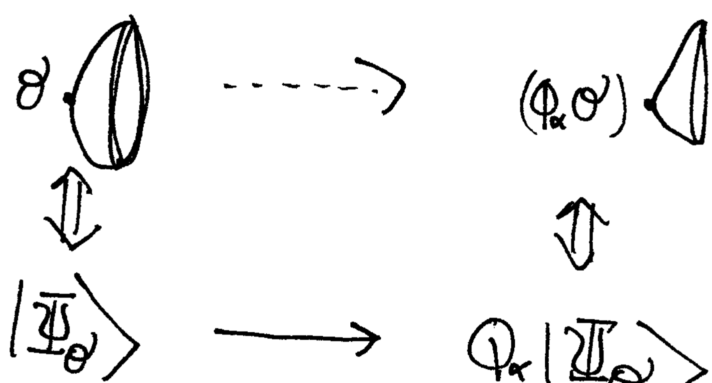
$$\equiv \Psi_{\mathcal{O}}(\phi_b)$$

Polchinski uses this method of cutting open path integral to (re)derive the prescription we proposed.

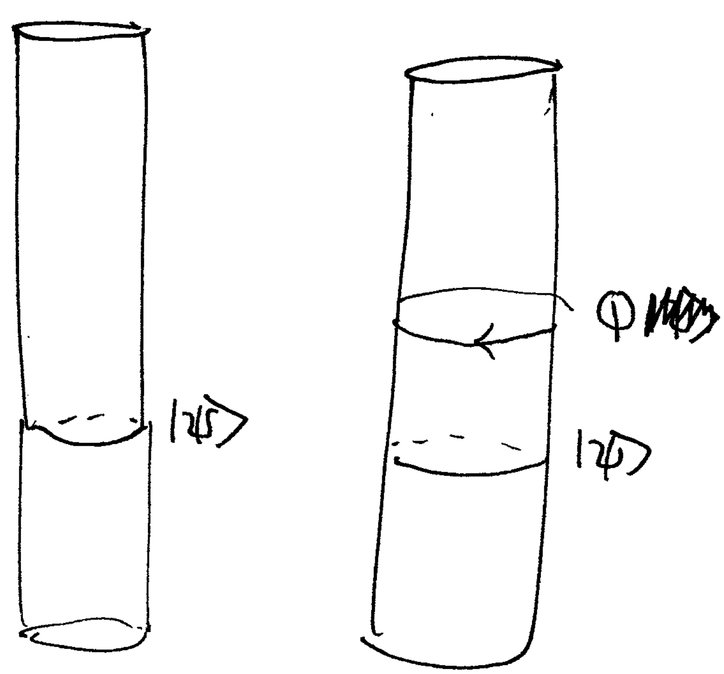
given some operators Φ_α acting on $\mathcal{H}$,

$$\Phi_\alpha|\Psi\rangle = |\Psi'\rangle \equiv |\Psi\rangle_{\Phi_\alpha}$$

we want to determine the corresponding action on local fields



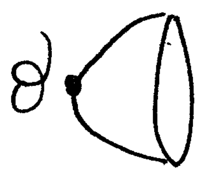
cylindrical quantization:



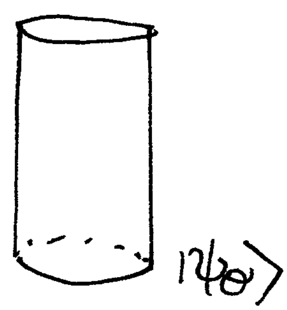
Action of an operator on a state is given by inserting operator ϕ_α in path integral.

$\Leftrightarrow$ inserting $\oint \frac{dz}{2\pi i} j_\alpha(z)$

So if $\mathcal{O}$ $\Leftrightarrow$ $|\psi_{\mathcal{O}}\rangle$



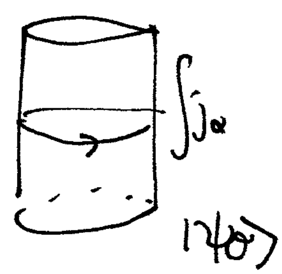
$\Leftrightarrow$



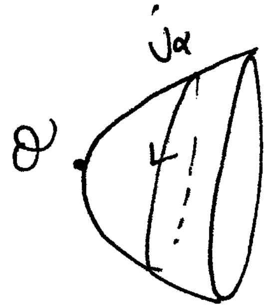
then

$\phi_\alpha|\psi_{\mathcal{O}}\rangle$

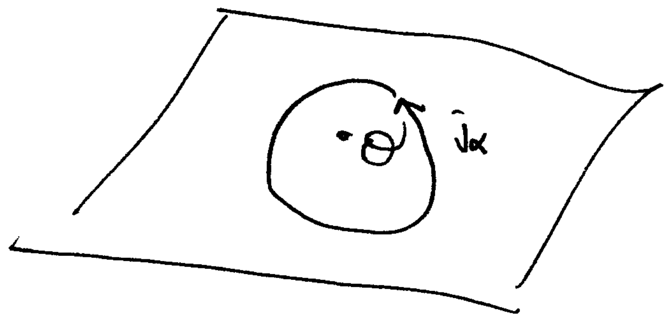
$\Updownarrow$



$\Leftarrow$



which we draw in the plane as



$$\phi_\alpha |\Psi_{\mathcal{O}}\rangle \equiv |\Psi_{\phi_\alpha \cdot \mathcal{O}}\rangle$$

where $$\phi_\alpha \cdot \mathcal{O} \equiv \oint \frac{dz}{2\pi i} j_\alpha(z)\, \mathcal{O}(0,0)$$

Let's consider an example.

Take $\mathcal{O} = \underline{\mathbb{1}}$, $\quad \phi_m = \alpha^\mu_m = \sqrt{\frac{2}{\alpha'}} \oint \frac{dz}{2\pi} z^m \partial X^\mu(z)$

Then $$\phi_m \cdot \underline{\mathbb{1}} = \oint \frac{dz}{2\pi i} \times \qquad \text{so } j_m = i\sqrt{\frac{2}{\alpha'}} z^m \partial X^\mu(z)$$

$$\times\ i\sqrt{\frac{2}{\alpha'}} z^m \partial X^\mu(z) \cdot \underline{\mathbb{1}}$$

$$= 0 \text{ for } m \geq 0$$

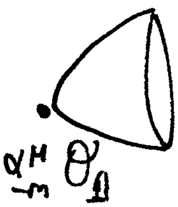
$\Rightarrow \quad \alpha^\mu_m \cdot \mathcal{O}_{\underline{\mathbb{1}}} = 0 \qquad m \geq 0$

$\Rightarrow \mathcal{O}_{\underline{\mathbb{1}}}$ is the ground state $|0,0\rangle$ $\left[\left(\lim_{k\to 0} |0,k\rangle\right)\right]$.

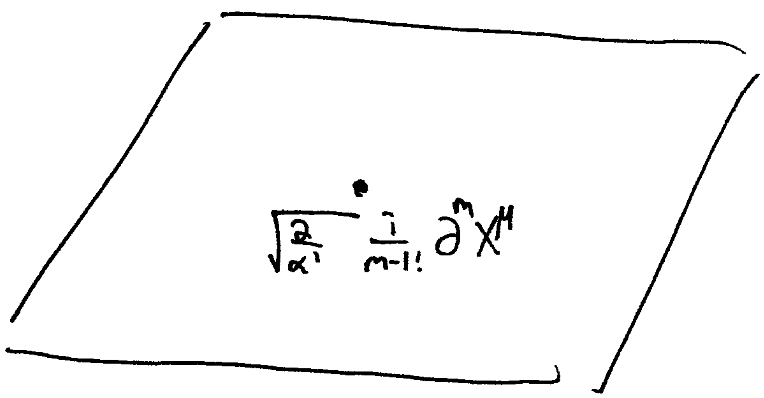
but for $m>0$,

$$\alpha^\mu_{-m}\,\theta_{\mathbb{1}} = \oint \frac{dz}{2\pi i}\, i\sqrt{\frac{2}{\alpha'}}\, z^{-m}\, \partial X^\mu(z)$$

$$= \sqrt{\frac{2}{\alpha'}}\,\frac{i}{m-1!}\,\partial^m X^\mu(0)$$

$$\Rightarrow \left|\Psi_{\alpha^\mu_{-m}\theta_{\mathbb{1}}}\right\rangle =$$ 

ie



This works generally: for any state $|\Psi_{\text{desired}}\rangle$

$$= \left(\alpha^\mu_{-m}\right)^{N_{\mu m}}\left(\tilde{\alpha}^\nu_{-n}\right)^{\tilde{N}_{\nu n}}\underbrace{|0,0\rangle}_{=|\Psi_{\mathbb{1}}\rangle}$$

The corresponding insertion is

$$\left[\sqrt{\frac{2}{\alpha'}}\,\frac{i}{m-1!}\,\partial^m X^\mu(0)\right]^{N_{\mu m}}\left[\sqrt{\frac{2}{\alpha'}}\,\frac{i}{n-1!}\,\bar{\partial}^n X^\nu(0)\right]^{\tilde{N}_{\nu n}}$$

Concerning the momentum k, we need

$$p^\mu |0,k\rangle = k^\mu |0,k\rangle$$

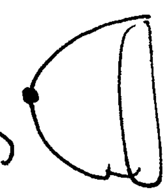
but $p^\mu = \sqrt{\frac{2}{\alpha'}}\, \alpha_0^\mu \quad \left(= \sqrt{\frac{2}{\alpha'}}\, \tilde{\alpha}_0^\mu\right)$

$$= \frac{2}{\alpha'} \int \frac{dz}{2\pi} \partial X^\mu(z) \qquad \left(j_p = \frac{2}{\alpha'} :\,\partial X^\mu(z)\right)$$

We want to find $\mathcal{O}$ corresponding to $|0,k\rangle$

$$\equiv \mathcal{O}_{|0,k\rangle} \qquad \left(\text{"} |\Psi_{\mathcal{O}_{|0,k\rangle}}\rangle = |0,k\rangle \text{"}\right)$$

or

$$|0,k\rangle = \mathcal{O}_{|0,k\rangle}$$ 

and we know that

$$|\Psi_{p^\mu \mathcal{O}_{|0,k\rangle}}\rangle = k^\mu |\Psi_{\mathcal{O}_{|0,k\rangle}}\rangle$$

$$\Rightarrow \quad p^\mu \cdot \mathcal{O}_{|0,k\rangle} \equiv \oint \frac{dz}{2\pi i} \left[\frac{2}{\alpha'} :\partial X^\mu(z)\right] \mathcal{O}_{|0,k\rangle}$$

must give $k^\mu |\Psi_{\mathcal{O}_{|0,k\rangle}}\rangle$.

Finally, consider not $|0;0\rangle \leftrightarrow \mathbb{1}$

but $|0;k\rangle$.

Now $p^\mu|0;k\rangle = k^\mu|0;k\rangle$

$$\Rightarrow \frac{\partial}{\partial x^\mu}|0;k\rangle = ik_\mu|0;k\rangle$$

so we take

$$|0;k\rangle \rightarrow e^{ik_\mu X^\mu(0,0)}$$

we'd better normal order this to avoid $\overbrace{XX}$

$$|0;k\rangle \rightarrow :e^{ikX(0,0)}:$$

$$\alpha^\mu_{-n}\tilde{\alpha}^\nu_{-n}|0,0;k\rangle \rightarrow \left(i\sqrt{\frac{2}{\alpha'}}\right)^2\left(\frac{1}{n-1!}\right)^2 :\partial^n X^\mu(0,0)\,\bar{\partial}^n X^\nu(0,0) \times e^{ikX(0,0)}:$$

rewrite:

$$\longrightarrow \underbrace{-\frac{2}{\alpha'}\left(\frac{1}{n-1!}\right)^2}_{\text{this gives normalization}} :\partial^n X^\mu(0,0)\,\bar{\partial}^n \cancel{X}^\nu(0,0)\, e^{ikX(0,0)}:$$

Note that our insertions are not diff invariant, because they're at (0,0).

So we define

$$V_0 = \underbrace{g_c}_{\substack{\text{constant}\\ \text{whose}\\ \text{definition}\\ \text{will emerge}\\ \text{soon}}} \int d^2z \, :e^{ikX(z\bar{z})}:$$

"vertex operator for $|0,k\rangle$"

now it's diff invariant.

It will be Weyl invariant if $:e^{ikX}:$ has $(h,\tilde{h}) = (1,1)$.

but $h = \frac{\alpha'}{4}k^2 = \tilde{h}$

$\Rightarrow$ Weyl invariant iff $k^2 = +\frac{4}{\alpha'} \Leftrightarrow m^2 = -\frac{4}{\alpha'}$.

vertex operator for string ground state $|0,k\rangle$ is Weyl invariant iff $m^2 = -\frac{4}{\alpha'}$ (= mass^2 in LCG!)

Next,

$$V_{-1,-1} = g_c \int d^2z \left(\frac{+2}{\alpha'}\right)\left(\frac{1}{|-1|!}\right)^2 :\partial X^\mu(z,\bar z)\bar\partial X^\nu(z,\bar z)e^{ikX(z,\bar z)}:$$

$$= \frac{2}{\alpha'} g_c \int d^2z :\partial X^\mu \bar\partial X^\nu e^{ikX}:$$

Weyl invariant if $(1,1)$

but now $h = 1 + \frac{\alpha'}{4}k^2 = \tilde h$

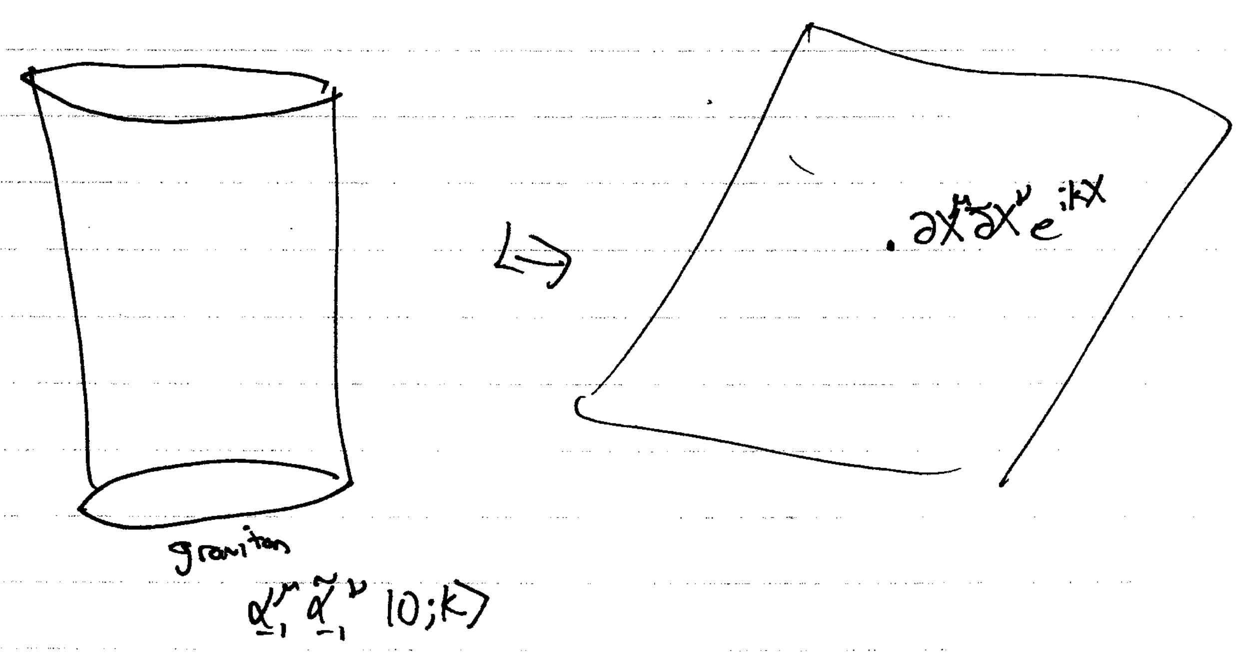
$\Rightarrow$ Weyl inv't iff $k^2 = 0 \Leftrightarrow m^2 = 0$

(agrees with LCG)

$\Rightarrow$ massless tensor.

(incl. graviton as symmetric traceless part).

So



8.1 0

BC CFT

Another important free CFT (besides $S = \int d^2z\, \partial X^\mu \bar{\partial} X_\mu$)

"bc CFT"

$$S = \frac{1}{2\pi}\int d^2z\, b\,\bar{\partial} c \qquad b, c \text{ anticommuting}$$

now if b has weights $(h_b, 0)$
and c " " $(h_c, 0)$

then provided $h_b + h_c = 1$, the theory is conformally inv't.

so write $h_b = \lambda$
$h_c = 1 - \lambda$.

eom: $\bar{\partial} c(z) = 0$ trivial

$\bar{\partial} b(z) = 0$ integrate by parts, or

$$0 = \bar{\partial} \frac{\delta \mathcal{L}}{\delta(\bar{\partial} c)}$$

Let's compute the propagator.

$$S = \frac{1}{2\pi}\int d^2z\, b\,\bar\partial c$$

$$= \frac{1}{2\pi}\int d^2z\, d^2z'\, b(z)\,\delta^2(z-z')\,\bar\partial\, c(z')$$

$$= \int d^2z\, d^2z'\, b(z)\,A(z,z')\,c(z') \qquad A(z,z') = \frac{1}{2\pi}\delta^2(z-z') \times \bar\partial$$

$\Rightarrow$ propagator is A^{-1} by usual gaussian integral

(analogous to $\langle x_i x_j\rangle = \int d^dx\, x_i x_j\, e^{-\frac{1}{2}\vec{x}^T A \vec{x}}$

$\langle \theta_i \theta_j\rangle = \int d\theta_1 \cdots d\theta_d\, \theta_i\theta_j \times e^{-\frac{1}{2}\vec\theta^T A \vec\theta}$

(nb. $\int d\theta_1 \cdots d\theta_n \exp\left(-\frac{1}{2}\theta^T A \theta\right) = \sqrt{\det A}$; opposite power of bosonic case.

propagator

$\Rightarrow K(z,z') = A^{-1}(z,z')$

$\Rightarrow \frac{1}{2\pi}\,\bar\partial K(z,z') = \delta^2(z-z')$

but $\bar\partial \frac{1}{z} = 2\pi\,\delta^2(z-z')$

$\Rightarrow K(z,z') = \frac{1}{z-z'}$

That is, $K(z,\bar{z}')$ is a Green's function of the operator $\frac{1}{2\pi}\bar{\partial}$.

$$\Rightarrow \langle\, b(z)\,c(w) - :b(z)\,c(w):\,\rangle$$

$$= \langle b(z)\,c(w)\rangle = \frac{1}{z-w}$$

$\Rightarrow$ OPE $\quad b(z)\,c(w) \sim \frac{1}{z-w}$

$$c(z)\,b(w) \sim -b(w)\,c(z) \sim \frac{-1}{w-z} = \frac{1}{z-w}$$

$b(z)\,b(w)$, $c(z)\,c(w)$ nonsingular (and, no constant by antisymmetry.)

Let's compute $T(z)$.

$$z \to z + \epsilon v^z \qquad \delta b = -\epsilon v\,\partial b - \epsilon v^*\bar{\partial} b - \epsilon\lambda b\frac{\partial v}{\partial z}$$

$$\bar{z} \to \bar{z} + \epsilon v^{z*} \qquad \delta c = -\epsilon v\,\partial c - \epsilon v^*\bar{\partial} c - \epsilon(1-\lambda)c\frac{\partial v}{\partial z}$$

$$\delta S = \frac{1}{2\pi}\int d^2z\left[b\,\bar{\partial}\left(-\epsilon v\,\partial c - \epsilon v^*\bar{\partial} c \underset{-\epsilon(1-\lambda)c\frac{\partial v}{\partial z}}{}\right) + \left(-\epsilon v\,\partial b - \epsilon v^*\bar{\partial} b \underset{-\epsilon\lambda b\frac{\partial v}{\partial z}}{}\right)\times \bar{\partial} c\right]$$

focus on hol. part

$$-\epsilon b\bar{\partial}(v\,\partial c) - \epsilon v\,\partial b\,\bar{\partial} c - \epsilon\lambda b\frac{\partial v}{\partial z}\bar{\partial} c - \epsilon(1-\lambda)b\bar{\partial}\left(c\frac{\partial v}{\partial z}\right)$$

$$= -\epsilon\left[b\bar{\partial}(v\,\partial c) + v\,\partial b\,\bar{\partial} c \;\cancel{}\; (1-\lambda)(\bar{\partial} b)\,c\,\partial v + \lambda b\,\partial v\,\bar{\partial} c\right]$$

Properly:

$$\delta S_{hol} = -\frac{\epsilon}{2\pi}\int d^2z \left\{ \bar\partial\left[\partial b\cdot v\cdot c\right] - \partial b\cdot\bar\partial v\cdot c - \lambda\bar\partial\left[\partial(bc)v\right] - \lambda bc\bar\partial\partial v \right.$$

$$= -\frac{\epsilon}{2\pi}\int d^2z \left\{ \bar\partial v\left[-\partial b\cdot c + \lambda\partial(bc)\right] + \bar\partial\left[\partial b\cdot c\cdot v - \lambda\partial(bc)v\right] \right.$$

$$= -\frac{\epsilon}{2\pi}\int d^2z \left\{ v\,\bar\partial\Big[\underbrace{\partial b\cdot c - \lambda\,\partial(bc)}_{T(z)}\Big] \right.$$

$$\Rightarrow \quad T(z) = \,:\partial b\cdot c: - \lambda\partial:bc:$$

$$\delta S_{antihol} = -\frac{\epsilon}{2\pi}\int d^2z \left\{ \bar\partial\left(b\,\bar\partial v\cdot v^*\right) \right\}$$

$$\Rightarrow \tilde{T}(\bar z) = 0.$$

Next, check:

$$T(z)\,b(w) \sim \,:\partial b(z)\,c(z):\,b(w) - \lambda\,\partial_z :b(z)c(z):\,b(w)$$

$$\frac{1}{z-w} \leftrightarrow -\frac{\alpha'}{2}\ln|z-w|^2$$

NB

$$:\partial b(z):\frac{1}{z-w} - \lambda\,\partial_z\left(:b(z):\frac{1}{z-w}\right)$$

$$= \frac{\lambda}{(z-w)^2}b(z) + \frac{\partial b(z)}{z-w} - \frac{\lambda}{z-w}\partial b(z)$$

$$= \frac{\lambda}{(z-w)^2}b(w) + \cancel{\frac{\lambda\partial b(w)}{z-w}} + \frac{\partial b(w)}{z-w} - \cancel{\frac{\lambda}{z-w}\partial b(w)} + \text{reg}$$

$$\Rightarrow h = \lambda. \checkmark$$

$$T(z)\,c(w) \sim \,:\partial b(z)\,c(z):\,c(w) - \lambda\,\partial_z :b(z)c(z):\,c(w)$$

$$\sim -:c(z)\,\underbrace{\partial b(z):c(w)}_{-\frac{1}{(z-w)^2}} + \lambda\,\partial_z :c(z)\,\underbrace{b(z):c(w)}_{\frac{1}{z-w}}$$

$$\sim \frac{c(z)}{(z-w)^2} + \frac{\lambda\partial c(z)}{z-w} - \frac{\lambda}{(z-w)^2}c(z)$$

$$\sim \frac{c(w)}{(z-w)^2} + \frac{\partial c(w)}{z-w} + \cancel{\frac{\lambda\partial c(z)}{z-w}} - \frac{\lambda c(w)}{(z-w)^2} - \cancel{\frac{\lambda\partial c(w)}{z-w}}$$

$$h = 1-\lambda \checkmark$$

The bc CFT will arise as the ghost sector upon path integral quantization.

We'll use a close relative to serve as the fermion sector in the superstring SCFT.

HW asks you to compute c_{bc}, $\tilde{c}_{bc}$.

This isn't the last fact we need about bc.

Suppose $\delta b = -i\epsilon b$

$\delta c = i\epsilon c$

Then $\delta S = \frac{1}{2\pi}\int d^2z \left(-i\epsilon b\bar{\partial} c + b(i\epsilon)\bar{\partial} c\right) = 0$

but with $\epsilon = \epsilon(z)$ $= \frac{-i}{2\pi}\int d^2z \left[\underbrace{\epsilon(z) b\bar{\partial} c} - b\bar{\partial}[\epsilon(z) c]\right]$

$= \frac{-i}{2\pi}\int d^2z\, \epsilon(z)\bar{\partial}(bc)$

$\Rightarrow j = -:bc:$ (conventional sign)

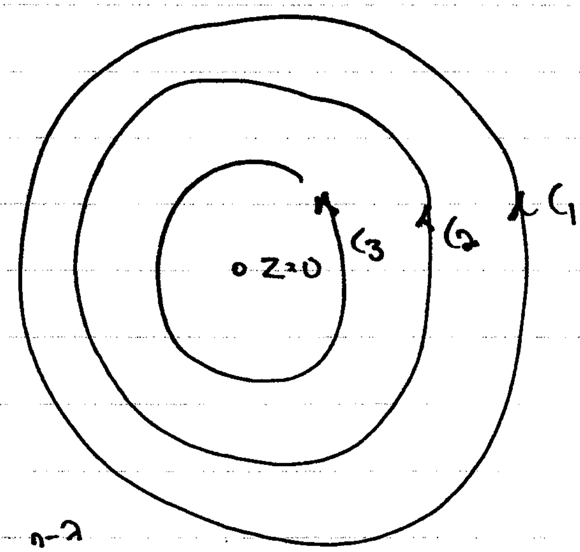
"ghost number current"

Suppose $\lambda \in \mathbb{Z}$.

Then $b(z) = \sum_{m=-\infty}^{\infty} \frac{b_m}{z^{m+\lambda}}$ has weight λ

$c(z) = \sum_{m=-\infty}^{\infty} \frac{c_m}{z^{m+(1-\lambda)}}$ has weight $1-\lambda$.

and $b(z)\,c(w) \sim \frac{1}{z-w}$



$$b_m = \oint z^{m+\lambda-1}\, b(z)\, \frac{1}{2\pi i}$$

$$c_n = \oint z^{n+(1-\lambda)-1}\, c(z) \cdot \frac{1}{2\pi i}$$

$$b_m c_n + c_n b_m = \left(\frac{1}{2\pi i}\right)^2 \oint_{C_1} dz\, z^{m+\lambda-1}\, b(z) \oint_{C_2} dw\, w^{n-\lambda}\, c(w)$$

$$+ \left(\frac{1}{2\pi i}\right)^2 \oint_{C_2} dw\, w^{n-\lambda}\, c(w) \oint_{C_3} dz\, z^{m+\lambda-1}\, b(z)$$

$$= \oint_{C_2} dw \operatorname*{Res}_{z \to w} \left[z^{m+\lambda-1} w^{n-\lambda}\, b(z)\, c(w) \right] \frac{\cdot 2\pi i}{(2\pi i)^2}$$

where I used $c(w)\,b(z) = -b(z)\,c(w)$

(8)

$$= \frac{2\pi i}{(2\pi i)^2} \oint_{C_2} dw \; \underbrace{\text{Res} \frac{1}{z-w} (z-w+w)^{m+\lambda-1}}_{\text{Res}\left(\frac{1}{z-w}\right) \cdot w^{m+n-1} \big(1+\underbrace{\tfrac{z-w}{w}}_{\sim 1}\big)^{m+\lambda-1}} w^{n-\lambda}$$

$$= \frac{2\pi i}{(2\pi i)^2} \oint_{C_2} dw \; w^{m+n-1} = \frac{\cancel{(2\pi i)^2}}{\cancel{(2\pi i)^2}} \delta_{m+n,0}$$

$$\Rightarrow \{b_m, c_n\} = \delta_{m,-n}$$

~~looks like $(2\pi i)^2$ error.~~ fixed.

$$\begin{array}{cc} \vdots & \vdots \\ b_2 & c_2 \\ b_1 & c_1 \\ b_0 & c_0 \\ b_{-1} & c_{-1} \\ b_{-2} & c_{-2} \\ \vdots & \vdots \end{array}$$

take ground <u>states</u> to be those annihilated by b_n, c_n for $n>0$.

but $b_0 c_0 + c_0 b_0 = 1$

ground states give representation of b_0, c_0 algebra.

Single ground state insufficient:

say $b_0|\psi\rangle = B|\psi\rangle$

$c_0|\psi\rangle = C|\psi\rangle$

obviously fails to rep. algebra

try two: $|\psi\rangle, |\chi\rangle$

with $b_0|\psi\rangle = |\chi\rangle$ $\Rightarrow$ $c_0 b_0|\psi\rangle = |\psi\rangle$

$c_0|\chi\rangle = |\psi\rangle$ $\quad$ $b_0 c_0|\chi\rangle = |\chi\rangle$

so must also impose)

$c_0 b_0|\chi\rangle = 0 \qquad b_0|\chi\rangle = 0$

$b_0 c_0|\psi\rangle = 0 \qquad c_0|\psi\rangle = 0$

Convention: write $|\chi\rangle = |\downarrow\rangle$

$|\psi\rangle = |\uparrow\rangle$

so:

$b_0|\downarrow\rangle = 0; \; c_0|\uparrow\rangle = 0$

$b_0|\uparrow\rangle = |\downarrow\rangle; \; c_0|\downarrow\rangle = |\uparrow\rangle$

so b_0 is, pictorially, a lowering op. (better justification eventually...)

$\Rightarrow$ call $|\downarrow\rangle$ the <u>ghost vacuum</u>.

general state:

$$\prod_{i=1}^{\infty} b_{-i}^{nb_i} c_{-i}^{nc_i} c_0^{nc_0} |\downarrow\rangle \qquad \text{where } nb_i, nc_i \in \{0,1\}.$$

$$T(z) = :\partial b\, c: - \lambda \partial :bc:$$

$$= :\sum_{m=-\infty}^{\infty} \frac{-(m+\lambda) b_m}{z^{m+\lambda+1}} \sum_{m'=-\infty}^{\infty} \frac{c_{m'}}{z^{m'+(1-\lambda)}} : - \lambda \partial : \sum_{m,m'} \frac{b_m c_{m'}}{z^{m+m'+1}} :$$

$$= \sum_{m,m'} : \frac{b_m c_{m'}}{z^{m+m'+2}} \left\{ -(m+\lambda) - \lambda(-1)(m+m'+1) \right\}$$

$$-m \cancel{-\lambda} + \lambda m + \lambda m' + \cancel{\lambda}$$

$m+m' \equiv n$

$$= \sum_{m,n} \frac{b_m c_{n-m}}{z^{n+2}} \left\{ \lambda n - m \right\}$$

(swap $m \leftrightarrow n$)

$$\Rightarrow L_m = \sum_{n=-\infty}^{\infty} :b_n c_{m-n}: (m\lambda - n) + \delta_{m,0}\, a^{ghost}$$

again, $2L_0 = L_1 L_{-1} - L_{-1} L_1$.

$$L_1 = \sum_n b_n c_{1-n} (\lambda - n)$$

$$L_{-1} = \sum_n b_n c_{-1-n} (-\lambda - n)$$

$L_1 |\downarrow\rangle \neq 0$ for $n=$

$L_{-1}|\downarrow\rangle \neq 0$ for $n=-1$: $L_{-1}|\downarrow\rangle = b_{-1} c_0 (-\lambda+1)|\downarrow\rangle$

$$= b_{-1}|\uparrow\rangle (1-\lambda)$$

$$L_1 L_{-1} |\downarrow\rangle = \sum_n b_n c_{1-n} (\lambda - n)(1-\lambda) b_{-1} |\uparrow\rangle$$

$$= b_0 c_1 (\lambda - 1)(1-\lambda) b_{-1} |\uparrow\rangle$$

$$= b_0 (\lambda)(1-\lambda)(b_{-1} c_1 + 1)|\uparrow\rangle$$

since $b_{-1} c_1 + c_1 b_{-1} = 1$

$$= \lambda(1-\lambda) b_0 |\uparrow\rangle = \lambda(1-\lambda)|\downarrow\rangle$$

$$\Rightarrow 2L_0 |\downarrow\rangle = \lambda(1-\lambda)|\downarrow\rangle$$

$$\Rightarrow \boxed{a^{ghost} = \frac{\lambda(1-\lambda)}{2}}$$

Once we know the proper value of λ for the ghosts in the gauge-fixed string, we'll have

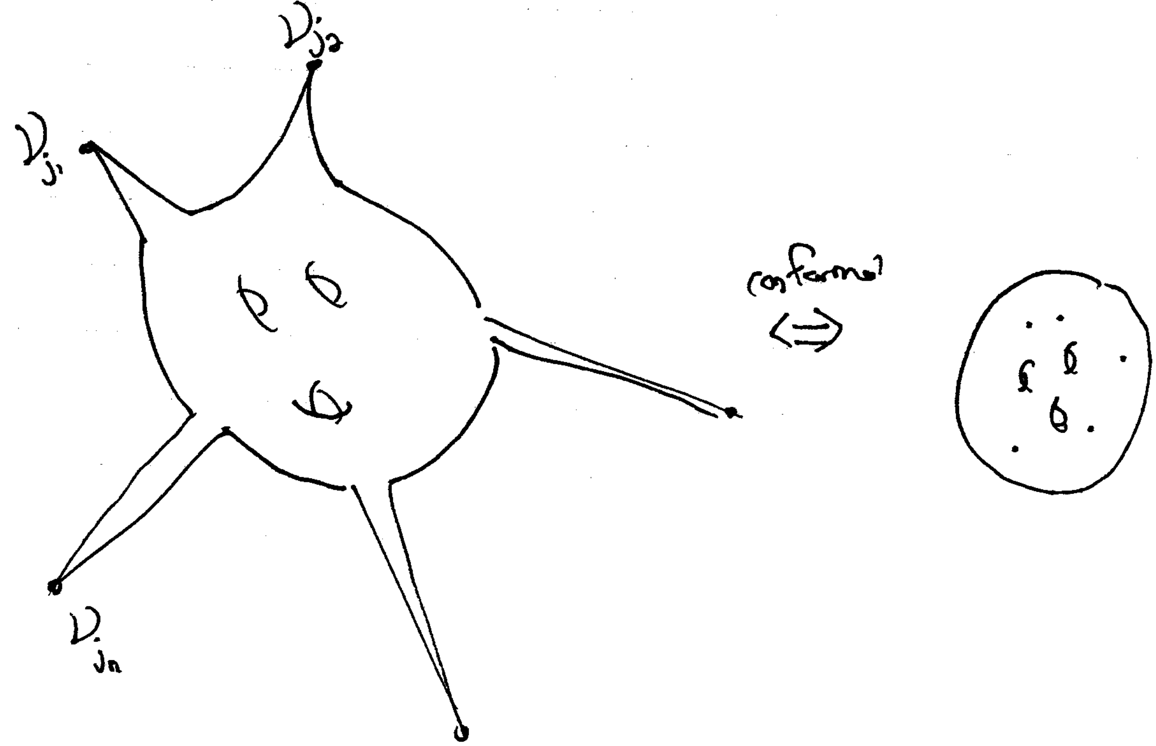
$$A = a^X + a^{ghost}$$
$$= 0 + \tfrac{1}{2}\lambda(1-\lambda).$$

Faddeev-Popov procedure for path integral quantization

We want to compute

$$S_{j_1 \cdots j_n}(k_1, .. k_n)$$

$$= \sum_{\substack{\text{suitable}\\ \text{compact}\\ \text{topologies}}} \frac{[DX\, Dg]}{V_{\text{diff} \times \text{Weyl}}} e^{-S_p - \lambda \chi} \prod_{i=1}^{n} d^2\sigma_i \sqrt{g(\sigma_i)}\, \mathcal{V}_{j_i}(k_i, \sigma_i)$$



$$[Dg] = D\left\{\begin{smallmatrix}\text{gauge inequivalent}\\ \text{metrics}\end{smallmatrix}\right\} \times D\left\{\begin{smallmatrix}\text{gauge}\\ \text{equivalences}\end{smallmatrix}\right\}$$

"$\times$" to be explained fully today

but $\int D\left\{\begin{smallmatrix}\text{gauge}\\ \text{equivalences}\end{smallmatrix}\right\} = \begin{smallmatrix}\text{volume}\\ \text{of}\\ \text{gauge gp}\end{smallmatrix} = V_{\text{diff}\times\text{Weyl}}$.

Idea: fix gauge carefully.

Simple warmup:

$$Z \equiv \int dx\,dy\, e^{-S(x)}$$

$$= \int dx\,dy\,\delta(y)\, e^{-S(x)} \cdot \text{vol}(Y)$$

$$= \text{vol}(Y) \int dx\,dy\,\delta(y - f(x))\, e^{-S(x)}$$

~~AAA~~

but if we don't know $f(x)$ – we just know

$$F(x,y) = 0 \quad \text{is solved (uniquely) by} \quad y = f(x)$$

then $\delta(y - f(x)) = \det\left|\frac{\partial F}{\partial y}\right| \delta(F(x,y))$

F = gauge-fixing function

$$Z = \text{vol}(Y) \int dx\,dy\,\delta(F(x,y)) \det\left|\frac{\partial F}{\partial y}\right| e^{-S(x)}$$

In LCG and in CFT, we have fixed the diff × Weyl gauge symmetry.

The best way to set up a path integral for a gauge theory is the FP method.

Call the gauge group G and its elements g.

We'd naively want to compute

$$\int [D\phi]\, e^{+i/\hbar\, S[\phi,\hat{g}]} \quad \text{with } \hat{g} \text{ some fixed gauge}$$

$$\underset{\text{naively}}{=} \int [D\phi][Dg]\, e^{i/\hbar\, S[\phi,g]}\, \delta(g-\hat{g})$$

but we should be careful.

if $f: \mathbb{R}^n \to \mathbb{R}^n$ $\quad \vec{f}(\vec{x})$

then $\int d^n x\, \delta(\vec{f}(\vec{x})) \neq 1$ in general

rather $\int d^n x\, \delta(\vec{f}(\vec{x}))\, \det\left(\frac{\partial f_i}{\partial x_j}\right) = 1$

Similarly, supposing our gauge choice $g=\hat{g}$
is expressed as $F(g)=0$

then $1 = \int [Dg]\, \delta(F(g)) \det\left(\frac{\delta F(g)}{\delta g}\right)$

and inserting this into $\int [D\phi]\, e^{i/\hbar S[\phi,g]} \equiv Z[g]$

leads to

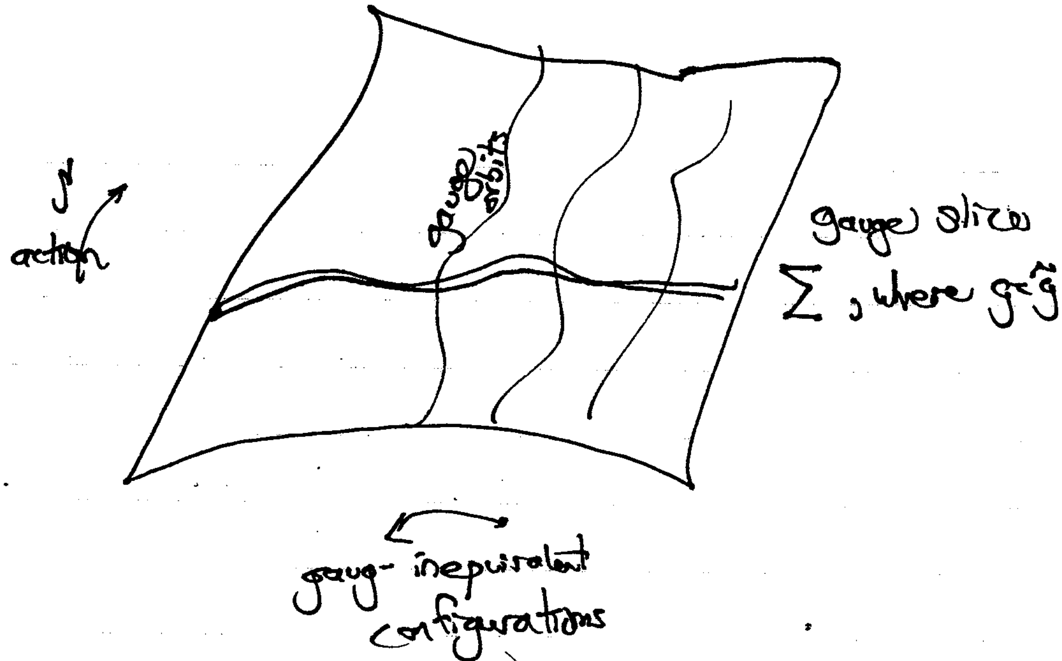
$$Z[\hat{g}] = \int [D\phi\, Dg]\, \delta(F(g))\, e^{i/\hbar S[\phi,g]} \det\left(\frac{\delta F(g)}{\delta g}\right)$$

given $F(g)$, one can often compute the det.
This is the FP procedure.

So we <u>define</u> Δ_{FP} by

$$1 = \int [D\zeta]\, \delta(g^{\zeta} - \hat{g})\, \Delta_{FP}(g^{\zeta})$$

here $\zeta : g \to g^{\zeta}$ is a diff $\times$ Weyl element
and $\hat{g}$ is our fiducial metric.



Our def. of $\Delta_{FP}(g^\zeta)$ is valid when $g^\zeta = \hat{g}$, i.e. on gauge slice. That's all we'll need.

So insert 1_{FP} into

$$Z = \int \frac{[DX\, Dg]}{V_{diff \times Weyl}} e^{-S[X,g]}$$

(I've gone to Euclidean signature)

$$Z = \int \frac{[DX\, Dg\, D\zeta]}{V_{diff \times Weyl}} \Delta_{FP}(g^\zeta)\, \delta(g^\zeta - \hat{g})\, e^{-S[X,g]}$$