# D-branes and gauge theory

Charged particle in $\mathbb{R}^{8,1} \times S^1$ ← $X^{10}$, radius $R$

$$S = \int d\tau \left( \frac{1}{2} \dot{X}^M \dot{X}_M - \frac{1}{2} m^2 - i q A_M \dot{X}^M \right)$$

In the gauge field by

$$A_{10}(X^M) = -\frac{\theta}{2\pi R} \qquad \theta = \text{const}$$

here $A_{10} = -\frac{i}{\Lambda} \frac{\partial}{\partial X^{10}} \Lambda \qquad \Lambda = \exp\left(-\frac{i\theta}{2\pi R} X^{10}\right)$

and $F = 0$

here, $z \equiv \int_0^{2\pi R} dX^{10} A_{10} = -\theta$ gauge inv't.

Nonrelativistic warmup:

$$H = \frac{1}{2m} (\vec{p} - q\vec{A})^2 + q\Phi$$

for $A_{10} = -\frac{\theta}{2\pi R}$,

$$E\psi = \frac{1}{2m} \left( -i\partial_{10} + \frac{q}{2\pi R} \theta \right)^2 \psi$$

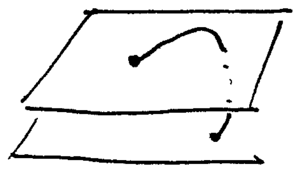
periodicity in $X^{10}$: $\psi_n(X^{10}) = e^{inX/R}$

$$\Rightarrow E_n = \frac{1}{2m} \left( \frac{n}{R} + \frac{q\theta}{2\pi R} \right)^2$$

Same works in rel. case, using

$$p^\mu_{\text{canon}} = p^\mu_{\text{kin}} - q A^\mu$$

Now strings on D-branes have CP factors



$|N, ij\rangle$

with charge +1 under $U(1)_i$

-1 " $U(1)_j$

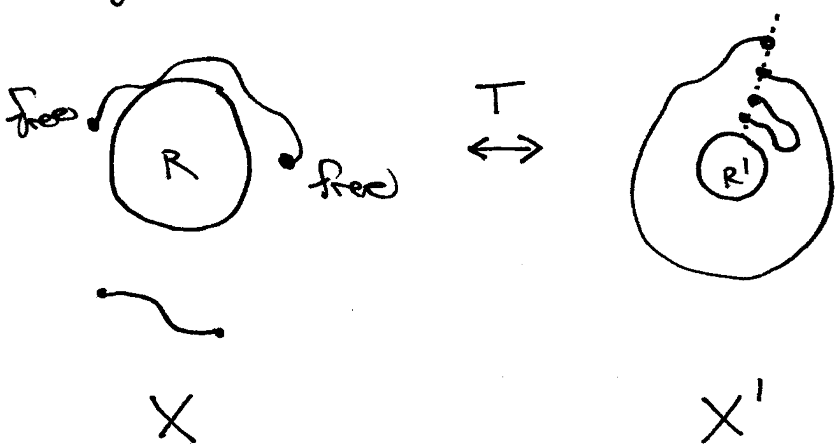
So, $$P_{10}^{(ij)} = P_{10}^{\text{kin.}} - \frac{\theta_i}{2\pi R} + \frac{\theta_j}{2\pi R}$$

$$\uparrow$$

$$\frac{2\pi n}{2\pi R}$$

So, $$M^2 = \left(\frac{2\pi n + \theta_j - \theta_i}{2\pi R}\right)^2 + \frac{1}{\alpha'}(N-1)$$

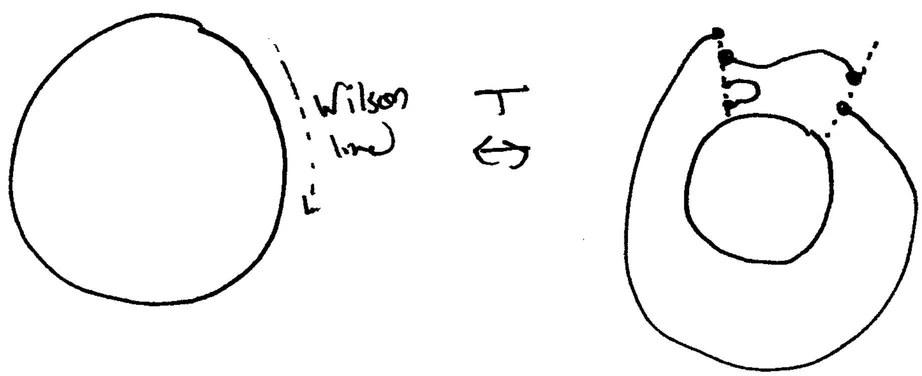
Without Wilson lines, T-duality acts on open strings as




$$X'(\pi) - X'(0) = 2\pi n R'$$

With Wilson lines,

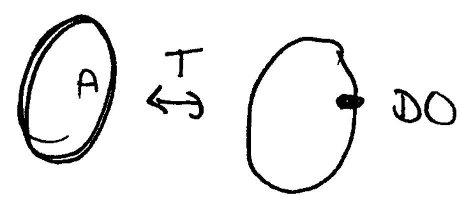
$$X'(\pi) - X'(0) = R'[2\pi n + \theta_j - \theta_i]$$



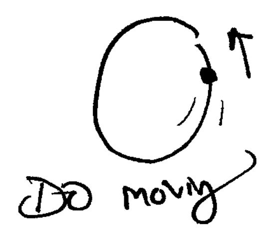
Consider a D1-brane on $S^1$

$$A = -\frac{\theta}{2\pi R}$$

For $\theta = \text{const}$, T-dual is



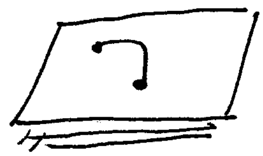

but for $\theta = \theta(t)$, T-dual is




Now $\partial_t A = F_{010}$ $\quad (F_{MN} = \partial_M A_N - \partial_N A_M)$

So <u>electric flux</u> is T-dual to <u>momentum</u>.

For $N$ coincident D-branes



$U(N)$

Consider a Wilson line $W \in U(N)$

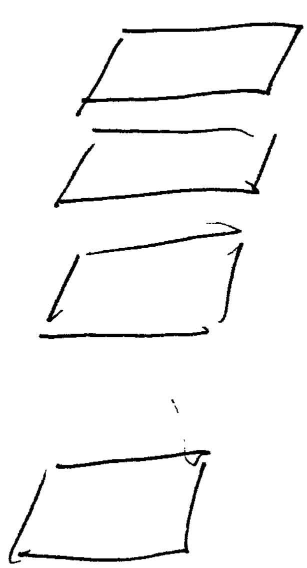
[NB since $UU^\dagger = U^\dagger U$, can write $U = A + iB$ with $[A, B] = 0$ ($A, B$ hermitian)

hence, diagonalizable by unitary transform. (ie gauge rotation)

$\Rightarrow$ wlog, $U(N) \to U(1)^N$

$$W = \exp[-i\, \mathrm{diag}[\theta_1, \ldots \theta_N]]$$
$$= \mathrm{diag}[e^{-i\theta_i}]$$

T-dualize:



# Worldvolume fields on D-branes

1) Isolated Dp-brane has, on its WV,

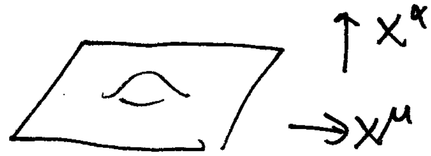
a Maxwell field in $(p+1)$d WV,

$$A_\mu(\xi^0, \ldots \xi^p)$$

a massless scalar for each $\perp$ dir,

$$\Phi^\alpha(\xi^0, \ldots \xi^p)$$

So a $\Phi^\alpha$ excitation is a 'bump' or ripple




2) $N$ coincident Dp-branes have

a YM field $\quad A^a_\mu \lambda^a_{ij}$ $\quad i,j = 1 \ldots N$, $a = 1 \ldots N^2$

$U(N)$ gauge group

transverse scalars $\quad \Phi^a_\alpha \lambda^a_{ij}$

in adjoint of $U(N)$

In flat $\mathbb{R}^{9,1}$ with $B=0$:

D9-branes

$D=10$ SYM

$$S=\int d^{10}x\ \mathrm{Tr}\left(-\frac{1}{4g_{YM}^2}F_{MN}F^{MN}\right)$$

dim. red on one direction (or on p directions)

$$A_M \to \underset{\substack{p+1\ d\\ WV}}{A_\mu}\ ,\ \underset{\substack{10-p-1\\ \text{transverse directions}}}{\Phi_\alpha}$$

$$S=\frac{1}{4g_{YM}^2}\int d^{p+1}x\ \mathrm{Tr}\Big(-F_{\mu\nu}F^{\mu\nu}-2D_\mu X^\alpha D^\mu X_\alpha$$
$$+[X_\alpha,X_\beta][X^\alpha,X^\beta]\Big)$$

$$D_\mu X^\alpha=\partial_\mu X^\alpha-i[A_\mu,X^\alpha]$$

The $[,]^2$ is a potential.

When $[X^a, X^\beta] \equiv 0$, we may diagonalize

$$X^\alpha_{N\times N} \to \mathrm{diag}(X_1^\alpha, \dots X_N^\alpha)$$

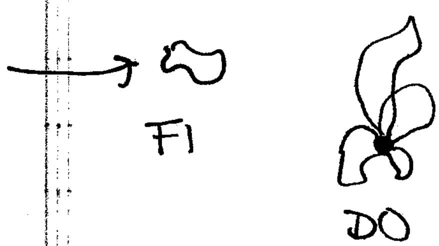
here
$N = \#$Dp-branes

eigenvalue $X_i^\alpha$ governs location of $i^{\text{th}}$ brane in $X^\alpha$ direction.

But if $[X^a, X^\beta] \neq 0$, we cannot simultaneously specify all the D-brane positions!

# How large is a D-brane?

'only one scale, $\sqrt{\alpha'}$': incorrect.




**soft scattering**, scale $\alpha'$

**Rough Idea:** gravitational BR of NS soliton is $\mathcal{O}(1)$ in string units

$$T_{NS5} = \frac{1}{g_s^2}\cdot\frac{1}{(2\pi)^5}\frac{1}{\alpha'^3}$$

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu} \propto g_s^2 T_{NS5} \sim \mathcal{O}(1) \text{ in } \alpha' \text{ units}$$

But D-branes are **lighter** (for $g_s \ll 1$),

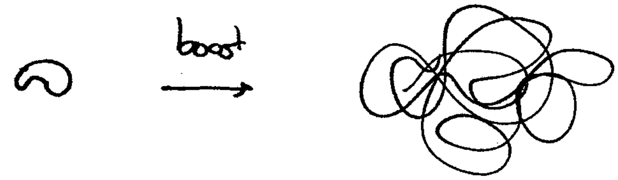
$$T_{D5} = g_s T_{NS5}$$

$\Rightarrow$ BR occurs at smaller distances?

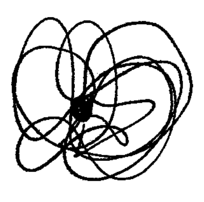
**Better idea** if all strings have size $\gtrsim l_s \equiv \sqrt{\alpha'}\,2\pi$ then use one D-brane to probe another!

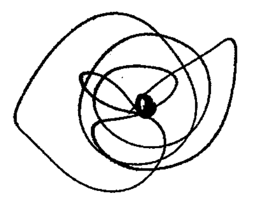
Problem: stringy halo.

Recall, a highly boosted string has many virtual d.o.f. visible)




So hard D0 scattering is namely

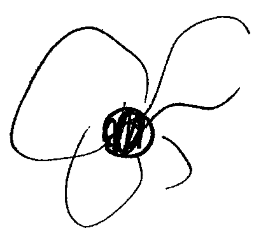




no good.

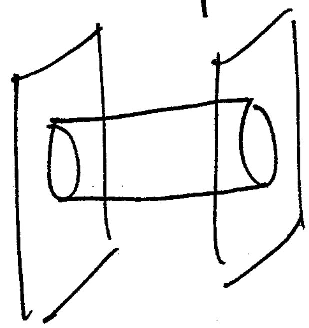
Need to slip inside the halo.

So, take $v$ fixed, $M_{D0} = \frac{M_s}{g_s} \to \infty$ by $g_s \to 0$.

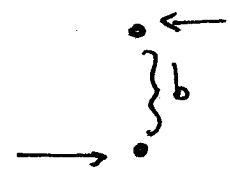


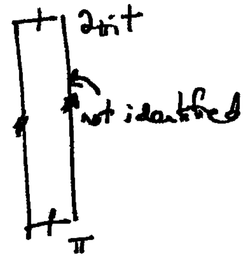
strings less important.

So let's compute

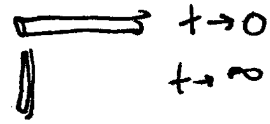


for




$0 \le \mathrm{Re}\, w \le \pi \qquad w \cong w + 2\pi i t$




$$A = \int_0^\infty \frac{dt}{2t} \sum_{i,k} \exp\left(-2\pi\alpha' t \left(k^2 + m_i^2\right)\right)$$

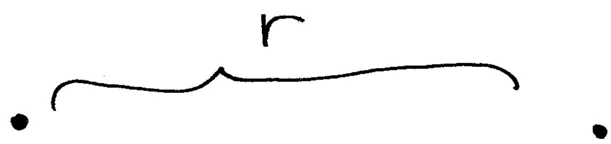
$$m_i^2 = \frac{b^2}{(2\pi\alpha')^2} + \frac{N}{\alpha'} \qquad (\text{for } v=0)$$

summing + expanding for small v, we get

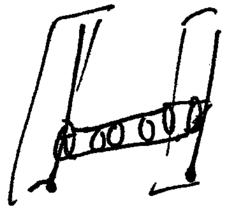
$$A = \int_0^\infty \frac{dt}{t^{1+\delta}}\, e^{-b^2 t/2\pi\alpha'} \underbrace{f(t,v)}_{\text{can expand this}}$$

$$\delta = \frac{1}{2}(\#\, NN \text{ directions})$$

** At small $\underline{b}$, integrand is fine at $t \to 0$
(i.e. no $\frac{1}{b^{\#}}$ terms).
Any singularity comes from $t \to \infty$ regime.
$\Leftrightarrow$ lightest open strings become <u>massless</u>



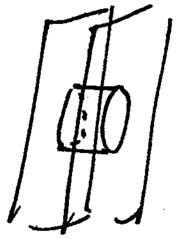

for $r \gg l_s$, dominant effect is exchange of massless closed strings (i.e., supergravity)



for $r \ll l_s$, lightest states have

$$m_W = M_s\left(r/l_s\right) \ll M_s$$

so WV QFT containing these light fields governs the dynamics.



(excited states $\Rightarrow$ $>2$ derivatives in QFT)

So as $r \to 0$,

governed by IR behavior of WV QFT!

Can compare

① DBI probe action in SUGRA bg. sourced by one brane

② LEEA for open string gauge thy (eg., metric on $\mathcal{M}$)

This justifies studying D0-brane dynamics, with $r \ll l_s$, using only the lightest strings between the branes!

(Pretty amazing!)

$$S = \int dt \, \frac{1}{2g_s} \mathrm{Tr} F_{\mu\nu} F^{\mu\nu} - i \, \mathrm{Tr} \, \bar{\psi} \Gamma^\mu D_\mu \psi$$

$$F_{0i} = \partial_0 \phi_i + [A_0, \phi_i]$$
$$F_{ij} = [\phi_i, \phi_j]$$

$$D_0 \psi = \partial_0 \psi + [A_0, \psi]$$
$$D_i \psi = [\phi_i, \psi]$$

$\psi, \phi$ fns of $t$ only (QFT on D0 WV)

To absorb $g_s$, rescale

$$t = g_s^{-1/3} t_{(11)}$$
$$A_0 = g_s^{1/3} A_0^{(11)} \quad \Rightarrow \quad S = \int dt_{11} \left(F^{(11)}\right)^2$$
$$\phi_i = g_s^{1/3} \phi_i^{(11)}$$

So characteristic time is $t_{11} = g_s^{-1/3} t_s$

mass is $g_s^{1/3} M_s$

now slow D0-branes ($v \ll 1$)

have $M_{D0} v^2 \sim g_s^{1/3} M_s$

$$\frac{v^2}{g_s} \sim g_s^{1/3}$$

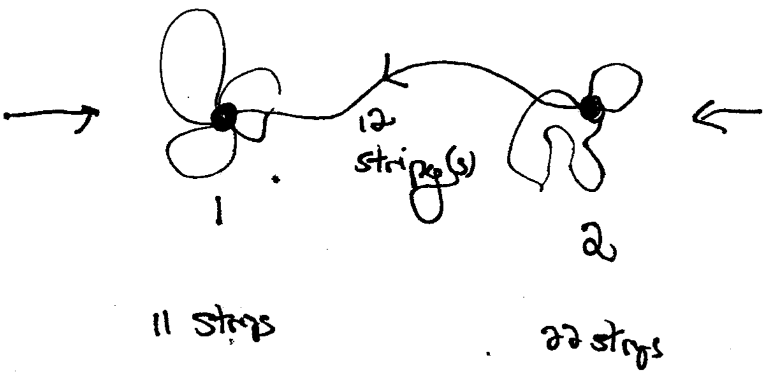
$$v \sim g_s^{2/3} \ll 1 \quad \checkmark$$

(Quick way: $\int \frac{dt\,(\dot{F})^2}{g_s}$ : $[g_s] = M^3$

so $g_s^{1/3} l_s$ is natural scale.)

All this is valid when $v, a \ll 1$.

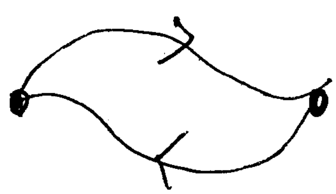
# D0-brane dynamics



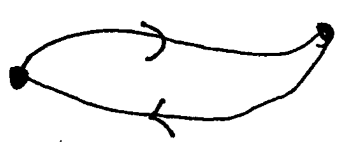

A single 12 string is not allowed, for then the endpoint charges are uncancelled (Gauss's law).

So,

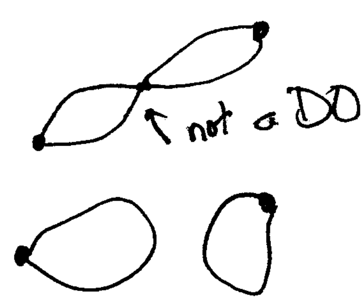
or



now



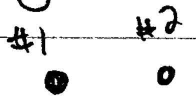
is unstable to



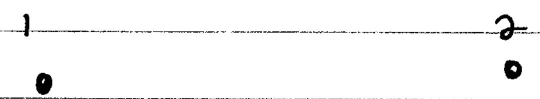
But also, the mass of the 12 string depends on the separation $|\vec{X}_2-\vec{X}_1| \equiv r$

$$m_{12}^2 = \left(\frac{r}{2\pi\alpha'}\right)^2$$

If I begin with

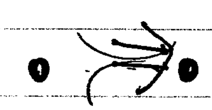



and adiabatically move #2,



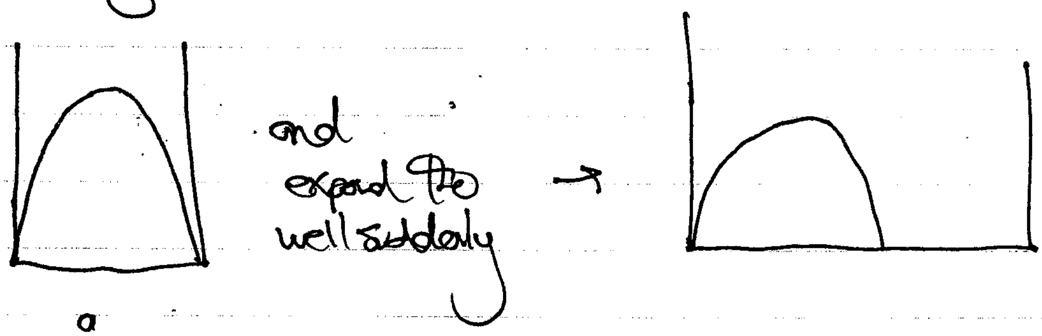

then nothing exciting occurs.

But if #2 moves rapidly/suddenly,



$$m_{12}(t) = \frac{1}{2\pi\alpha'} r(t)$$ changes abruptly.

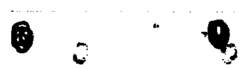
Recall from QM that if we begin with a square well ground state




The wavefunction $\psi(x)$ is momentarily unchanged, meaning the state is some nontrivial combination of the eigenstates of the new $H$.

Same occurs in QFT with a changing background, eg a rapidly expanding universe.

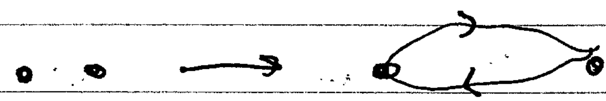
Here, the state of no 12 strings, separation $r_0$



gets transported into some state other than

separation $r_{new}$

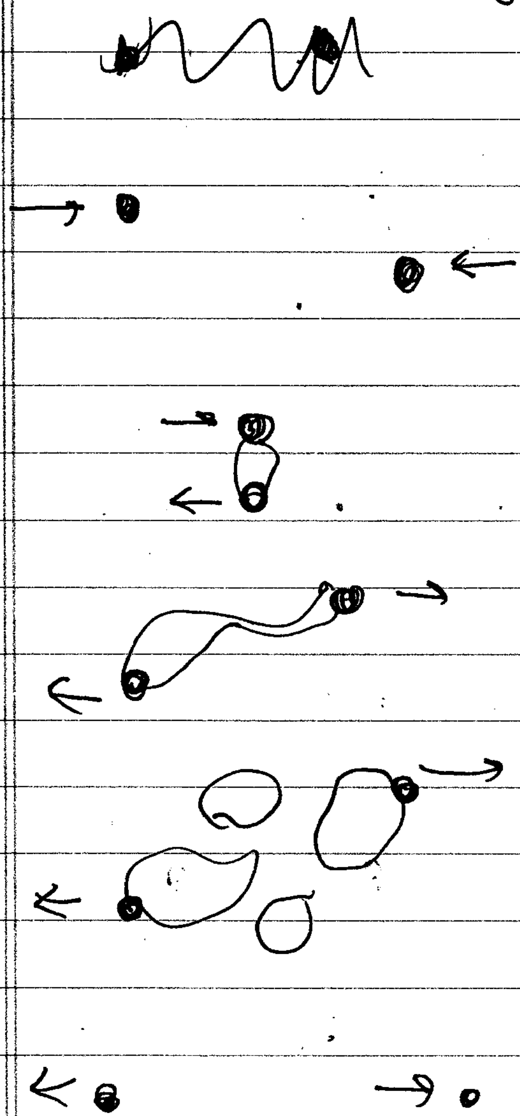
i.e. strings are present!

So 

(eg)

(one can compute occupation numbers)

In full, realizing the sudden change of separation via a near-miss scattering event,



meaning that D0-branes interact by producing stretched strings that gradually decay.

We've reduced D0-brane scattering to QM. Simplifying further to a (fairly good) toy model, we consider

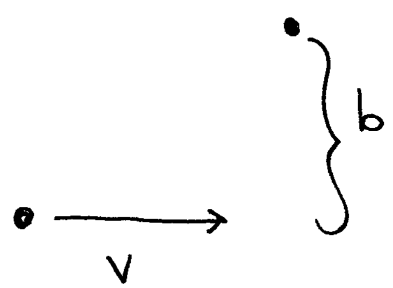
$$H = -\frac{1}{2} g_s \left(\nabla_x^2 + \nabla_w^2\right) + \frac{1}{2g_s} x^2 w^2 + \begin{pmatrix} 0 & x_1 - i x_2 \\ x_1 + i x_2 & 0 \end{pmatrix}$$

for $\Psi(x_1, x_2, w_1, w_2)$ a 2-component wavefunction

Here $x_1, x_2 \leftrightarrow$ positions

$\langle x_2(t) \rangle = b$
$\langle x_1(t) \rangle = vt$

while $w_1, w_2$ are massive stretched strings.



The 2 components correspond to $\psi_1, \psi_2$, Fermi superpartners of $w_1, w_2$.

The massive fields are $W_i$ mass $\sqrt{b^2+v^2t^2}$

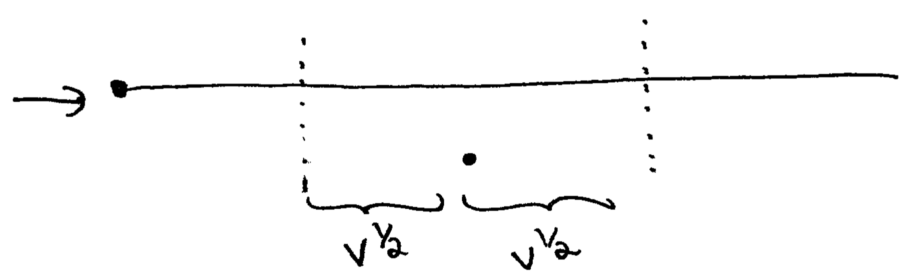
and $\psi_i$ mass $\pm\sqrt{b^2+v^2t^2}$

(plus excited strings atop this).

When $\frac{\dot{\omega}}{\omega^2} \ll 1$, can use adiabatic approx.
(Born-Oppenheimer)

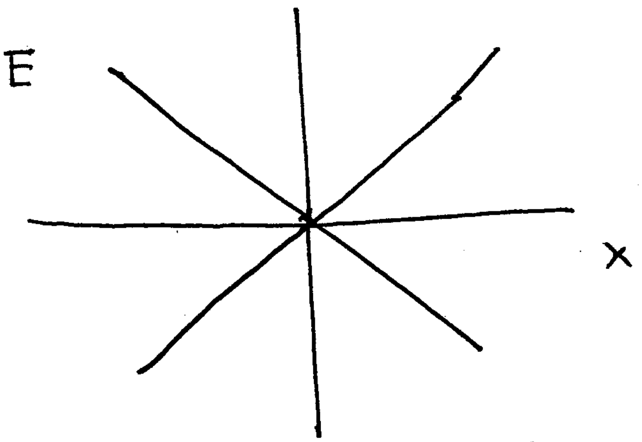
$$\frac{\dot{\omega}}{\omega^2} = \frac{tv^2}{\omega} \cdot \frac{1}{\omega^2} = \frac{v^2 \frac{x}{v}}{x^3} = \frac{v}{x^2}$$

so for $|x| > v^{1/2}$, ✓.



One can check that Bose d.o.f. $W_i$ are unchanged even for smaller $x$
(since $H_{bose}(t) = H_{bose}(-t)$.).

But the fermions $\psi_i$:

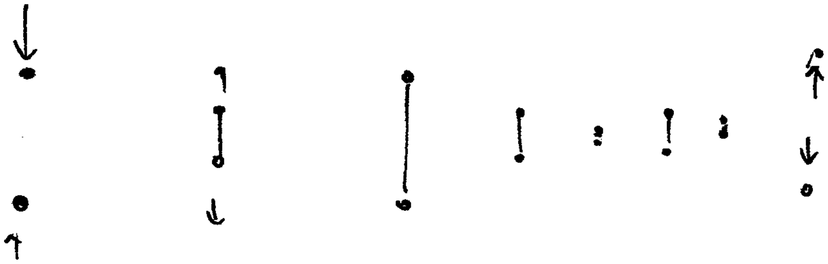



Result: straight fermionic strings nearly always created in near-miss scattering.

Max separation: $m_{D0} v^2 \approx \frac{L}{\alpha'}$

$$L \approx \frac{l_s v^2}{g_s}.$$

Lifetime of this resonance: $\frac{v l_s}{g_s}$



Radiation of RR photons is small, $P \propto a^2$ and $a \sim g_s$

Can work out resonance energies by WKB:

$$p = \sqrt{2m(E-\lambda x)}$$

$$\int p\,dx = n - \tfrac{1}{2} \approx n$$

$$\tfrac{1}{\lambda}\sqrt{2m}\,(E-\lambda x)^{3/2} \approx n$$

$$E_n \sim n^{2/3}\, m^{-1/3}\, \lambda^{2/3}$$

$$\lambda = \underset{(\text{ie. } \frac{1}{\alpha'})}{M_s^2} \qquad m = \frac{M_s}{g_s}$$

$$E_n \sim g_s^{1/3} M_s\, n^{2/3} \qquad \sim g_s^{1/3} M_s$$

$$\Rightarrow M_{D0} v^2 \sim g_s^{1/3} M_s$$

$$v^2 \sim g_s^{4/3} \qquad v \sim g_s^{2/3}$$

So <u>size</u> is $L \approx l_s v^2/g_s \sim \boxed{g_s^{1/3} l_s}$ .

Note, $l_{planck}^{[10D]}$ given by $g_s^2 \alpha'^4 = (l_p^{[10]})^8$

$$l_p^{[10]} \sim g_s^{1/4} l_s$$

$$l_p^{[4]} \sim g_s l_s$$

So $L_{resonance} \ll l_s$

but $L_{resonance} \gg l_p^{[10]} \gg l_p^{[4]}$.

# M-Theory

# Toroidal Compactification

$D = d+1 \qquad x^d$ periodic, $x^d = x^d + 2\pi R$

Metric

$$ds^2 = G^{(D)}_{MN}\, dx^M dx^N$$

$$= G_{\mu\nu}\, dx^\mu dx^\nu + G_{dd}(dx^d + A_\mu dx^\mu)^2$$

- invariant under $x^d$ translations $\qquad (G_{dd} = G_{dd}(x^\mu))$
- " " $d$-dim reparam.
- " " $x'^d = x^d + \lambda(x^\mu)$

... provided $A'_\mu = A_\mu - \partial_\mu \lambda$

so $dx^d + A_\mu dx^\mu \to dx'^d - \partial_\mu\lambda dx^\mu$
$+ A'_\mu dx^\mu + \partial_\mu \lambda dx^\mu$ ✓

Simple case, $G_{dd} = 1$.

(massless) Scalar $\phi(x^M) = \sum_{n=-\infty}^{\infty} \phi_n(x^\mu) \exp(i n x^d / R)$

then $\partial^M \partial_M \phi = 0$

$$\Rightarrow \partial^\mu \partial_\mu \phi_n(x^\mu) e^{inx^d/R} + \left(\frac{in}{R}\right)^2 \phi_n(x^\mu) e^{inx^d/R}$$

or $\partial^\mu \partial_\mu \phi_n(x^\mu) = \left(\frac{n}{R}\right)^2 \phi_n(x^\mu)$

$$m^2 = \left(\frac{n}{R}\right)^2.$$

Also we'll need:

$$R^{(D)} = R^{(d)} - 2e^{-\sigma}\nabla^2 e^{\sigma} - \frac{1}{4}e^{2\sigma}F_{\mu\nu}F^{\mu\nu}$$

$$(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu)$$

Bosonic action of IIA:

$$S_{10} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}\left\{e^{-2\Phi}\left(R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}|H_3|^2\right)\right\}$$

$$-\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-G}\left(|F_2|^2+|\tilde{F}_4|^2\right)$$

$$-\frac{1}{4\kappa_{10}^2}\int B_2\wedge F_4\wedge F_4$$

where
$$\begin{cases} H_3 = dB_2 \\ F_2 = dA_1 \\ \tilde{F}_4 = dA_3 - A_1\wedge F_3 \end{cases} \qquad F_4 = dA_3$$

Bosonic action of Cremmer-Julia 11D SUGRA:

$$S_{11} = \frac{1}{2\kappa^2}\int d^{11}x\sqrt{-G}\left(R^{(11)}-\frac{1}{2}|F_4^{(11)}|^2\right)-\frac{1}{12\kappa^2}\int A_3^{(11)}\wedge F_4^{(11)}\wedge F_4^{(11)}$$

Now take

$$ds^2 = G^{(11)}_{MN} dx^M dx^N = G^{(10)}_{\mu\nu} dx^\mu dx^\nu + e^{2\sigma(x^\mu)}\left[dx^{10} + A_\nu(x^\mu) dx^\nu\right]^2$$

Then $$R^{(11)} = R^{(10)} - 2e^{-\sigma}\nabla^2 e^{\sigma} - \frac{1}{4}e^{2\sigma} F_{\mu\nu}F^{\mu\nu}$$

$$S_{11}^{\text{from } R^{(11)}} = \underbrace{\frac{2\pi R_{10}}{2\kappa^2}}_{\int dx^{10}} \int d^{10}x \sqrt{-G^{(10)}}\, e^{\sigma}\left(R^{(10)} - 2e^{-\sigma}\nabla^2 e^{\sigma} - \frac{1}{4}e^{2\sigma}F_{\mu\nu}F^{\mu\nu}\right)$$

Joe's convention: $|F_p|^2 = \frac{1}{p!} F_{M_1 \cdots M_p} F^{M_1 \cdots M_p}$

define $\kappa_{10}^2 = \frac{\kappa^2}{2\pi R_{10}}$

$$S_{11}^{\text{from } |F_4|^2} = \frac{2\pi R_{10}}{2\kappa^2}\int d^{10}x \sqrt{-G^{(10)}}\, e^{\sigma}\left(-\frac{1}{2}\right)\left[\frac{1}{4!} F_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma} + \frac{1}{4!}F_{10\,\nu\rho\sigma}F^{10\,\nu\rho\sigma}\cdot 4\right]$$

$$\Big[\text{now } F^{10}{}_{\nu\rho\sigma} = G^{10\,10}F_{10\,\nu\rho\sigma} + G^{10\mu}F_{\mu\nu\rho\sigma}$$
$$= e^{-2\sigma}F_{10\,\nu\rho\sigma} +$$
$$F^{\mu}{}_{\nu\rho\sigma} = G^{(10)\mu\alpha}F_{\alpha\nu\rho\sigma} + G^{10\mu}F_{10\,\nu\rho\sigma}\Big]$$

use this to raise/lower in 10D.

Remember $G^{(10)}_{\mu\nu} \neq G_{\mu\nu}$.

$$F_{MNPQ}F^{MNPQ} = G_{\mu\nu}F^{\mu}{}_{\tau\rho\sigma}F^{\nu\tau\rho\sigma}$$

$$+ G_{1010}F^{10}{}_{\tau\rho\sigma}F^{10\tau\rho\sigma}\cdot 4$$

$$+ 2G_{10\mu}F^{\mu}{}_{\tau\rho\sigma}F^{10\tau\rho\sigma}\cdot 4$$

Joe's convention: $|F_p|^2 \equiv \frac{1}{p!}F_{\cdots}F^{\cdots}$

but
$$\begin{cases} G_{\mu\nu} = G^{(10)}_{\mu\nu} + e^{2\sigma}A_\mu A_\nu \\ G_{1010} = e^{2\sigma} \\ G_{10\mu} = e^{2\sigma}A_\mu \end{cases}$$

we get
$$\frac{2\pi R_{10}}{2\kappa_1^2}\int d^{10}x\sqrt{-G^{(10)}}\,e^{\sigma}\left(-\frac{1}{2}\right)\Big[\frac{1}{4!}G^{(10)}_{\mu\nu}F^{\mu}{}_{\tau\rho\sigma}F^{\nu\tau\rho\sigma}$$

$$+\frac{1}{4!}e^{2\sigma}A_\mu A_\nu F^{\mu}{}_{\tau\rho\sigma}F^{\nu\tau\rho\sigma}$$

$$+\frac{1}{4!}\cdot 2e^{2\sigma}A_\mu F^{\mu}{}_{\tau\rho\sigma}F^{10\tau\rho\sigma}\cdot 4$$

$$+\frac{1}{4!}e^{2\sigma}F^{10}{}_{\tau\rho\sigma}F^{10\tau\rho\sigma}\cdot 4\Big]$$

$$= -\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-G^{(10)}}\left\{e^{\sigma}|F_4|^2 + \cdots\right\}$$

When one is careful about raising + lowering indices, this assembles into

$$S_{10}^{\text{from } |F_4|^2} = -\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-G^{(10)}}\left(e^{\sigma}|\tilde{F}_4|^2 + e^{-\sigma}|F_3|^2\right)$$

where:
$$A^{(10)}_{\mu\nu\rho} = A^{(11)}_{\mu\nu\rho}$$
$$B^{(10)}_{\mu\nu} = A^{(11)}_{10\,\mu\nu}$$
$$F^{(10)}_{4\,\mu\nu\rho\sigma} = F^{(11)}_{4\,\mu\nu\rho\sigma}$$
$$F^{(10)}_{3\,\mu\nu\rho} = F^{(11)}_{4\,10\,\mu\nu\rho}$$

or,

$$\left\{\begin{array}{lll} A_3^{(11)} \to & A_3^{(10)},\ B_2^{(10)} & \text{with} \quad F_4^{(10)} = dA_3^{(10)} \\ F_4^{(11)} \to & F_4^{(10)},\ F_3^{(10)} & \phantom{\text{with}} \quad F_3^{(10)} = dB_2^{(10)}. \end{array}\right\}$$

and, $\tilde{F}_4 = dA_3 - A_1 \wedge F_3$

$A_1$: $A_\mu dx^\mu$ from $G_{\mu 10}$.

Finally,

$$S_{10}^{\text{from AFF}} = -\frac{1}{12\kappa_{11}^2}\int A_3^{(11)} \wedge F_4^{(11)} \wedge F_4^{(11)}$$

if $A_3 \supset dx^{10}$,

$A_{\mu\nu 10}$ aka $B_{\mu\nu}$ $\quad B_2^{(10)} \wedge F_4^{(10)} \wedge F_4^{(10)}$

if $F_4 \supset dx^{10}$,

$F^{(11)}_{\mu\nu\rho 10}$ aka $F^{(10)}_{\mu\nu\rho}$ $\quad A_3 \wedge F_3 \wedge F_4$

related by parts $\int$, $\int B_2 \wedge dA_3 = \int A_3 \wedge dB_2$

$$\equiv \int A_3 \wedge F_3 .$$

three terms to choose, so get $\times 3$

$$S_{10}^{\text{from AFF}} = -\frac{1}{4\kappa_{11}^2}\int B_2 \wedge F_4 \wedge F_4$$

can rescale fields to make this $\kappa_{10}$ as other terms are quadratic in $(d)A_{(2,3)}$.

So from $G_{MN} \to \begin{cases} \text{metric} & G_{\mu\nu} \\ \text{1-form} & A_{\mu} \\ \text{scalar} & e^{\sigma} \end{cases}$

$A_{MNP} \to \begin{cases} \text{3-form} & A_{\mu\nu\rho} \\ \text{2-form} & B_{\mu\nu} \end{cases}$ with $\begin{matrix} F_4 \\ F_3 \end{matrix}$ field strengths.

After appropriate rescalings (ie redefinitions) we arrive at $S_{IIA}$!

How could we guess this from IIA in 10D?

Consider a D0-brane.

Mass $M = \dfrac{1}{g_s\sqrt{\alpha'}}$

N D0-branes mutually BPS, so

$$M = \frac{N}{g_s\sqrt{\alpha'}}.$$

As $g_s \to \infty$, spacing in spectrum gets small.

cf. $M_{KK}^2 = \left(\dfrac{n}{\hat{R}_{10}}\right)^2$.

for $\underset{\underset{R_{10}e^{\sigma}}{\shortparallel}}{\hat{R}_{10}} \equiv g_s\sqrt{\alpha'}$, the spectra match.

So, $g_s = \frac{R_{10}}{\sqrt{\alpha'}}$

and $K_{10}^2 = \frac{1}{2}(2\pi)^7 g_s^2 \alpha'^4$

so $K_{11}^2 = 2\pi R_{10} K_{10}^2$

$= \frac{1}{2}(2\pi)^8 g_s^3 \alpha'^{9/2}$

define $M_{11} = \frac{1}{\sqrt{\alpha'}} \frac{1}{g_s^{1/3}}$

so $2K_{11}^2 = (2\pi)^8 \left(\frac{1}{M_{11}}\right)^9$

while correspondingly, $l_{11} = \frac{1}{M_{11}} = g_s^{1/3} \sqrt{\alpha'}$

recall also $\frac{1}{l_4^2} = \frac{1}{K_{10}^2} \int \begin{pmatrix} \text{vol. form} \\ \text{internal} \\ \text{space} \end{pmatrix} = \frac{V}{K_{10}^2}$

$l_4^2 = \frac{1}{2}(2\pi)^7 g_s^2 \alpha' \left(\frac{\alpha'^3}{V}\right)$

so $l_4 \propto g_s \sqrt{\alpha'}$

Therefore at small $g_s$, fixed $V$,

$l_4 \ll l_{11} \ll l_s \equiv \sqrt{\alpha'}$

$g_s \qquad g_s^{1/3} \qquad g_s^0.$

Now let's think about charged objects.

| IIA: | | couples $E$ to | couples $B$ to |
|---|---|---|---|
| | D0 | $A_1$ | $(A_7)$ |
| | D2 | $A_3$ | $(A_5)$ |
| | D4 | $(A_5)$ | $A_3$ |
| | D6 | $(A_7)$ | $A_1$ |

NB. if we use $*F_{(E)} = F_{(M)}$

then $A_p^{(E)} \Leftrightarrow A_{8-p}^{(M)}$.

D0 couples to $A_1$. $A_1 \Leftrightarrow G_{10\mu}$ ("KK $\gamma$").

$D0 \Leftrightarrow P_{10}$

metric couples to momentum.

| Also have | $E$ | $B$ |
|---|---|---|
| F1 | $B_2$ | |
| NS5 | | $B_2$ |

Where do these D-branes (and NS-branes) come from in 11D?

Postulate:

| | E | B |
|---|---|---|
| M2 | $A_3$ | |
| M5 | | $A_3$ |

these are the only possible BPS extended objects.

$$\left( \text{so } \int_{S^4 \text{ around M5-brane}} dA_3 = 1 \right)$$

in general, $dF_p = (*j)_{p+1}$

in D dim, $(*j)_{p+1} \to j$ is a $D-p-1$ form corresponding to a $(D-p-2)$ brane

g. D0 $\leftrightarrow$ 1 form $j$ $p=8$

and the magnetic potential is then a

$p-1$ form for a $D-(p-1)-3$ brane

or $k$ form for a $D-k-3$ brane.

so, 10D: $k$ potential form $\leftrightarrow$ $7-k$ brane

11D: $k$ form $\leftrightarrow$ $8-k$ brane

Use $\mu\nu\ldots$ or $10$ to denote alignment of M2/M5

$(M2)_{\mu\nu}$: D2

check:
$$\begin{array}{ccc} M2 & \overset{E}{\Leftrightarrow} & A_3 \\ \downarrow & & \downarrow \\ D2 & \overset{E}{\Leftrightarrow} & A_3 \end{array} \checkmark$$

$(M2)_{\mu 10}$: F1

check:
$$\begin{array}{ccc} M2 & \overset{E}{\Leftrightarrow} & A_3 \\ \downarrow S^1 & & \downarrow S^1 \\ F1 & \overset{E}{\Leftrightarrow} & B_2 \end{array} \checkmark$$

$(M5)_{\mu\nu\rho\sigma\tau}$ NS5

$(dF = *j_m)$

$$\begin{array}{ccc} M5 & \overset{B}{\Leftrightarrow} & A_3 \\ \downarrow & & \downarrow S^1 \\ NS5 & \overset{B}{\Leftrightarrow} & B_2 \end{array} \checkmark$$

NOTE: magnetic coupling have (brane on $S^1$ $\Leftrightarrow$ field not on $S^1$)

$(M5)_{\mu\nu\rho\sigma 10}$ D4

[if $\tilde{A}_6$ has leg on $X^{10}$, $A_3$ does not, $dA_3 = *d\tilde{A}_6$]

$$\begin{array}{ccc} M5 & \overset{B}{\Leftrightarrow} & A_3 \\ \downarrow & & \downarrow \\ D4 & \longleftrightarrow & A_3 \end{array} \checkmark$$

Where is the D0-brane?

well, D6 $\overset{B}{\Leftrightarrow}$ $A_1$ is dual to D0-brane

D0 carries KK E charge $\int A^{KK}_\mu dx^\mu$

so D6 must carry KK B charge, i.e. be a KK monopole!

NB monopole for 1-form potential is a point in 4D, or a 6-manifold in 10D.

Tensions:

$$T_{F1} = \frac{1}{2\pi\alpha'} \qquad \text{with} \qquad l_s = \sqrt{\alpha'}$$

$$T_{M2} = \frac{2\pi}{(2\pi l_{11})^3} \qquad\qquad l_{11} = g_s^{1/3} l_s$$

Then $2\pi \cdot T_{M2} \cdot R_{11} = l_s g_s \cdot \frac{1}{(2\pi)^2} \frac{1}{g_s l_s^3} \cdot 2\pi$

$$= \frac{2\pi}{(2\pi)} \frac{1}{(2\pi) l_s^2} = T_{F1} \checkmark$$

What is the KK monopole?

$$ds_{10}^2 = ds_{\mathbb{R}^6}^2 - dt^2 + ds_{TN}^2$$

$$ds_{TN}^2 = \left(1 + \frac{R}{2r}\right)\left(dr^2 + r^2 d\Omega_2^2\right) +$$

$$\frac{1}{\left(1+\frac{R}{r}\right)}\left(dy + R\sin^2(\theta/2)\,d\phi\right)^2$$

with $\vec{B} = \vec{\nabla}\times\vec{A} \quad A_\phi = R\sin^2(\theta/2)$

for $R = g_s l_s$ $\left(\overset{\text{i.e}}{R = R_{11}}\right)$

we can compute $\int_{\text{trasverse}} T_{00} = T_{D6}$.

# Matrix Theory

We've seen that low-v collisions of D0-branes know about 11D. Can we go further?

Consider a state in IIA with

$p_{10}$, $q$ mass $m$
↳ momentum in $\mathbb{R}^{9,1}$

Then $E = (p_{10}^2 + q^2 + m^2)^{1/2}$.

When $p_{10} \gg q, m$ we have

$$E \approx p_{10} + \frac{q^2+m^2}{2p_{10}}$$

$$= \frac{n}{R_{10}} + \frac{R_{10}}{2n}(q^2+m^2)$$

$n$ = # D0-branes
ie D0 charge
⟺ momentum in 10-direction

So if we start with a reasonable state
eg some D0's moving around, some excited strings, ..

... and boost in $R_{10}$ direction,

Then $E - \frac{n}{R_{10}} \approx \frac{R_{10}}{2n}(q^2+m^2) \sim \frac{R_{10}}{2n} \ll 1$

Thus in the new frame, if we constructed something ab initio, we can only use D0-branes (with $q \ll p_{10}$)

i.e. nonrelativistic D0-branes,

and massless strings on/between them.

(Massive strings would give $E \propto O(1)$.)

Another way to understand this:

if we boost (enormously) in one direction ($x_{10}$)

then the dynamics in the $\perp$ dirs. is Galilean (NR) rather than Lorentzian,

g. $E = \underbrace{\frac{n}{R_{10}}}_{p_{10}} + \frac{q^2}{2p_{10}} + \frac{m^2}{2p_{10}}$

$p_{10}$ serves as a mass for otherwise-massless particles.

Boost choice breaks Lorentz inv. since we're selectively increasing the momentum in 1 Direction.

Residual inv: Galilean.

The boost renders the frequencies of excitations other than D0-branes very large (since time is dilated).

So $\int$ out those other fast modes, leaving only D0-branes + their straight strings.

Upshot: IIA with huge boost in $X_{10}$
$\Leftrightarrow$ system of D0-branes + straight strings with non relativistic dispersion in 10D.

D0-brane QM.

Finally: 11D Lorentz invariance $\Rightarrow$ can describe any M-theory configuration in this way!

QM of D0-branes $\Leftrightarrow$ M-theory?