Summary:

$$S \propto \mathrm{vol}(\Phi(\Sigma)) \Rightarrow S_{NG} = -T\int d^2\xi \sqrt{-h}$$

got rid of $\sqrt{\ }$ via $S_P = -\frac{T}{2}\int d^2\xi \sqrt{-\gamma}\,\gamma^{\alpha\beta} h_{\alpha\beta}$

e.o.m.: $\begin{cases} \nabla^2 X^\mu = 0 \\ T_{\alpha\beta} = 0 \end{cases}$ because $\frac{\delta S_P}{\delta \gamma^{\alpha\beta}} = 0.$

Symmetries: $\begin{cases} \text{Poincaré } (D) \\ \text{diff } (2d) \\ \text{Weyl} \end{cases}$

boundary conditions:

- $\partial^\sigma X^\mu(\tau,0) = \partial^\sigma X^\mu(\tau,\ell) = 0$ (Neumann)
  or
- periodic: $\phi(\tau,\ell) = \phi(\tau,0)$
  or
- Dirichlet $X^\mu(\tau,0) = X^\mu_0$, $X^\mu(\tau,\ell) = X^\mu_\ell$

(19.5)

$T_{\alpha\beta}$ and the $\gamma_{\alpha\beta}$ e.o.m.

$$\delta S_p = -\frac{T}{2}\int d^2\xi \left\{ \sqrt{-\gamma}\, \delta\gamma^{\alpha\beta} h_{\alpha\beta} + \gamma^{\gamma\delta} h_{\gamma\delta} \left(-\frac{1}{2}\right) \sqrt{-\gamma}\, \gamma_{\alpha\beta}\, \delta\gamma^{\alpha\beta} \right\}$$

used $\delta\sqrt{-\gamma} = \frac{1}{2}\sqrt{-\gamma}\,\gamma^{\alpha\beta}\,\delta\gamma_{\alpha\beta}$

$= -\frac{1}{2}\sqrt{-\gamma}\,\gamma_{\alpha\beta}\,\delta\gamma^{\alpha\beta}$

$$\Rightarrow \frac{\delta S_p}{\delta\gamma^{\alpha\beta}} = -\frac{T}{2}\sqrt{-\gamma}\left\{ h_{\alpha\beta} - \frac{1}{2}\gamma_{\alpha\beta}\,\gamma^{\gamma\delta} h_{\gamma\delta} \right\}$$

$$T_{\alpha\beta} \equiv \frac{C}{\sqrt{-\gamma}} \frac{\delta S}{\delta\gamma_{\alpha\beta}}$$

C usu -2 in GR
we use $C = +4\pi$

$$\Rightarrow T_{\alpha\beta} = (+4\pi)\left(-\frac{T}{2}\right)\left\{ \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2}\gamma_{\alpha\beta}\,\partial^\gamma X^\mu \partial_\gamma X_\mu \right.$$

$$= -2\pi T \left\{ \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2}\gamma_{\alpha\beta}\,\partial^\gamma X^\mu \partial_\gamma X_\mu \right\}$$

('19.75)

- This is the same $T_{\alpha\beta}$ (up to prefactor $\sim \mathcal{C}$) one gets by the canonical Noether method (cf 18.5)

$\xi'^{\alpha} = \xi^{\alpha} + \epsilon^{\alpha}$

$X^{\mu} \to X^{\mu} + \epsilon^{\alpha}\partial_{\alpha}X^{\mu}$

$\delta\mathcal{L} = \epsilon^{\beta}\partial_{\alpha}(\delta^{\alpha}_{\beta}\mathcal{L})$

Pf.
$$\frac{\partial\mathcal{L}}{\partial(\partial_{\alpha}X^{\mu})}\partial^{\alpha}X_{\mu} - \gamma^{\alpha\beta}\mathcal{L}$$

$$= -\frac{T}{2}\left[2\partial^{\beta}X^{\mu}\partial^{\alpha}X_{\mu} - \gamma^{\alpha\beta}\partial_{\alpha'}X^{\mu}\partial_{\beta'}X_{\mu}\gamma^{\alpha'\beta'}\right]$$

$$= -T\left[\partial^{\alpha}X^{\mu}\partial^{\beta}X_{\mu} - \frac{1}{2}\gamma^{\alpha\beta}\partial^{\gamma}X^{\mu}\partial_{\gamma}X_{\mu}\right]$$

- $T_{\alpha\beta}$ is traceless, $\gamma^{\alpha\beta}T_{\alpha\beta} = 0$.

The most painful way to write the string action is

$$S_{NG} = -T\int d^2\xi \sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}$$

in this form, ~~good~~ luck quantizing it.

We can use our redundant description and make clever gauge choices to solve more easily,

BUT

there are associated costs.

If we eliminate degrees of freedom by gauge choices, we <u>must</u> impose the 'lost' e.o.m. as <u>constraints</u>.

primary example: d.o.f. in $\gamma_{\alpha\beta}$

$$\begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} \quad 3 \text{ dof}$$

diff: 2 reparametrizations $(\sigma'(\sigma,\tau), \tau'(\sigma,\tau))$

Weyl: 1 rescaling.

$\Rightarrow$ can get $\gamma_{\alpha\beta} \xrightarrow{\text{diff}} e^{2\omega(\sigma,\tau)}\eta_{\alpha\beta} \xrightarrow{\text{Weyl}} \eta_{\alpha\beta}$

but this removes 2 e.o.m. for $\gamma_{\alpha\beta}$

namely $\frac{\delta S}{\delta \gamma^{\alpha\beta}} = 0 \equiv T_{\alpha\beta} = 0$

why 2? $T_{01} = T_{10}$

$T_{00} = +T_{11}$

Since $T_{\alpha\beta} = \underbrace{\partial_\alpha X^\mu \partial_\beta X_\mu}_{} - \frac{1}{2}\gamma_{\alpha\beta}\gamma^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X_\mu$

is automatically traceless (and γ has signature $(1,1)$)

So
$$T_{00} = \dot{X}^2 - \tfrac{1}{2}(-1)(-\dot{X}^2 + X'^2) = \tfrac{1}{2}(\dot{X}^2 + X'^2)$$
$$T_{11} = X'^2 - \tfrac{1}{2}(+1)(-\dot{X}^2 + X'^2) = \tfrac{1}{2}(\dot{X}^2 + X'^2)$$

$\Rightarrow$ 2 eqns $\begin{cases} \dot{X}^2 + X'^2 = 0 \\ \dot{X}\cdot X' = 0 \end{cases}$

NB $\omega(\sigma,\tau)$ is not a physical field; we do not have an associated d.o.f.

<u>Upshot</u>: we may send $\gamma_{\alpha\beta} \to \eta_{\alpha\beta}$ provided we impose
$$\dot{X}^2 + X'^2 = 0$$
$$\dot{X}\cdot X' = 0$$
Virasoro constraints "V"

Two ways to do this.

(1) go ahead and quantize S_P in this gauge, then <u>impose</u> $\overline{V}$ on the Hilbert <u>space</u>.
i.e. project out 'states' that violate $\overline{V}$
[eg require that matrix element of V vanish between physical states]

(2) notice there is <u>still</u> some <u>additional</u> gauge freedom.
use it. make a non-covariant gauge choice and actually <u>solve</u> the constraints V.

As we will see, each method has drawbacks.

(1) • leads to negative-norm states
• lose Schrödinger eqn (i.e. H does not generate time-evolution)

covariant quantization

Lorentz symmetry manifest; positive probabilities not manifest

(2) • ~~leads to non~~ obscures Lorentz invariance, which must be checked (Lorentz anomaly possible)
• a bit unfamiliar.

light-cone gauge quantization

Lorentz symmetry non-manifest; positive probabilities manifest.

We'll first do option (2).

But first, let's solve the classical theory.

$$S_p = -\frac{T}{2}\int d^2\xi \sqrt{-\gamma}\,\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu$$

i) use diff × Weyl to send $\gamma_{\alpha\beta} \to \eta_{\alpha\beta}$
this eliminates metric (γ) dof + hence requires that we impose the γ e.o.m.'s as constraints,

$$T_{\alpha\beta} = 0 \Leftrightarrow \begin{array}{l} \dot{X}\cdot X' = 0 \\ \dot{X}^2 + X'^2 = 0 \end{array} \rightarrow (\dot{X} \pm X')^2 = 0.$$

ii) now $S_p = \overset{?}{+}\frac{T}{2}\int d^2\xi \left((\partial_\tau X^\mu)^2 - (\partial_\sigma X^\mu)^2\right)$

$$(\dot{X}^2 - X'^2)$$

$$\partial^\alpha X^\mu \partial_\alpha X_\mu$$

so X e.o.m.

$$\Rightarrow \partial_\alpha \partial^\alpha X^\mu = 0.$$

boundary condition

$$\partial^\sigma X^\mu \delta X_\mu \Big|_{\sigma=0}^{\sigma=l} = 0.$$

Let's first use Neuman b.c., $\partial^{\sigma} X^{\mu} = 0$ at $\sigma = 0, \ell$

to solve $^{*}(\partial_\tau X)^2 - (\partial_\sigma X)^2 = 0$ (~~will do other~~ b.c. in homework)

we have

$$X^\mu = X^\mu_L(\tau+\sigma) + X^\mu_R(\tau-\sigma)$$

b.c.: $\frac{\partial X^\mu}{\partial \sigma}(\tau, 0) = 0$

$$\Rightarrow X^{\mu\prime}_L - X^{\mu\prime}_R = 0 \qquad \prime : \frac{d}{d(\text{argument})}$$

$$\Rightarrow X^\mu_L(u) = X^\mu_R(u) + C$$

absorb in def. of X^μ_L

$$X^\mu = \left[f^\mu(\tau+\sigma) + f^\mu(\tau-\sigma)\right]\cdot\frac{1}{2} \quad \text{for some fn } f.$$

b.c. $\frac{\partial X^\mu}{\partial \sigma}(\tau, \ell) = 0$

$$\Rightarrow f^{\mu\prime}(\tau+\ell) - f^{\mu\prime}(\tau-\ell) = 0 \Rightarrow f^{\mu\prime} \text{ periodic, period } 2\ell.$$

convenient to set $\ell = \pi$: $\sigma \in [0, \pi]$.

$$\Rightarrow f^\mu(u) = f^\mu_0 + f^\mu_1 u + \sum_{n=1}^{\infty}\left(A^\mu_n \cos nu + B^\mu_n \sin nu\right)$$

$$\Rightarrow X^\mu = f_0^\mu + f_1^\mu \tau + \sum_{n=1}^{\infty} \underbrace{A_n^\mu \cos[n(\tau+\sigma)] + B_n^\mu \sin[n(\tau+\sigma)] + A_n^\mu \cos[n(\tau-\sigma)] + B_n^\mu \sin[n(\tau-\sigma)]}$$

$$\sum_{n=1}^{\infty} A_n^\mu \cos n\tau \cos n\sigma + B_n^\mu \sin n\tau \cos n\sigma$$

use $\quad A_n^\mu \cos n\tau + B_n^\mu \sin n\tau \equiv \underbrace{\frac{-i\sqrt{2}}{\sqrt{n}\sqrt{\pi T}}}_{\text{chosen for later convenience}} \left(a_n^{\mu *} e^{in\tau} - a_n^\mu e^{-in\tau}\right),$

notation: $\alpha' = \frac{1}{2\pi T}$

$$X^\mu = x_0^\mu + \sqrt{2\alpha'}\,\alpha_0^\mu \tau - i\sqrt{2\alpha'} \sum_{n=1}^{\infty} \frac{\cos n\sigma}{\sqrt{n}} \left(a_n^{\mu *} e^{in\tau} - a_n^\mu e^{-in\tau}\right),$$

and define $\quad \alpha_n^\mu \equiv a_n^\mu \sqrt{n}$

$$\alpha_{-n}^\mu \equiv (a_n^{\mu *})\sqrt{n} \qquad n \geq 1$$

then

$$X^\mu = x_0^\mu + \sqrt{2\alpha'}\,\alpha_0^\mu \tau - i\sqrt{2\alpha'} \underbrace{\sum_{n=1}^{\infty} \frac{1}{n}\left(\alpha_{-n}^\mu e^{in\tau} - \alpha_n^\mu e^{-in\tau}\right)\cos n\sigma}_{+i\sqrt{2\alpha'} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos(n\sigma)}$$

23.35

worth summarizing.

We took S_P. Used metric (diff × Weyl) freedom to send $\gamma_{\alpha\beta} \to e^{2\omega}\delta_{\alpha\beta} \to \delta_{\alpha\beta}$ ("unit gauge").

In ~~unit~~ any gauge, eom is $\nabla^2 X^\mu = 0$.

in unit gauge $\nabla^2 X^\mu = 0 \to \partial_\tau^2 X^\mu - \partial_\sigma^2 X^\mu = 0$.

We will eventually have to impose the Virasoro constraints $T_{\alpha\beta} = 0$

$$\Leftrightarrow \quad (\dot{X} \pm X')^2 = 0.$$

but we haven't done so yet.

For open string with Neumann b.c., $\partial^\sigma X^\mu = 0$ at $\sigma = 0, \pi$

$$\boxed{X^\mu(\tau,\sigma) = X_0^\mu + \sqrt{2\alpha'}\,\alpha_0^\mu \tau + i\sqrt{2\alpha'} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos(n\sigma)}$$

classical, unit gauge, Neumann b.c., before imposing Virasoro constraint.

23.36

How to quantize this?

if awakened + gun pointed at head, would try:

from $\mathcal{L}$, find canonical momenta
then impose ~~[illegible]~~ canonical commutation relations

$$[x,p] = i$$

$$\mathcal{L}_{unit} = -\frac{T}{2}\int d\tau\, d\sigma \left[(\partial_\tau X)^2 - (\partial_\sigma X)^2\right].$$

coordinate $X^\mu(\tau,\sigma)$ has momentum $\Pi^\mu(\tau,\sigma)$

$$\Pi^\mu(\tau,\sigma) \equiv \frac{\delta \mathcal{L}}{\delta(\partial_\tau X^\mu)} = -T\,\partial_\tau X^\mu(\tau,\sigma)$$

so impose $\left[X^\mu(\tau,\sigma),\ \Pi^\nu(\tau',\sigma')\right] = i\,\eta^{\mu\nu}\,\delta^2(\tau-\tau',\sigma-\sigma')$

in terms of Fourier modes α_n^μ, this is

$$\left[\alpha_m^\mu,\ \alpha_n^\nu\right] = m\,\eta^{\mu\nu}\,\delta_{m,-n}$$

but this is terrible!

$$[\alpha^0_m, \alpha^0_{-m}] = +m\eta^{00} = -m$$

so $\| \alpha^0_{-m}|0\rangle \| \equiv \langle 0|\alpha^0_{+m}\ \alpha^0_{-m}|0\rangle$

(recall $\alpha^{\mu\,\dagger}_m = \alpha^\mu_{-m}$, with $\alpha^\dagger_m$ $m>0$, raising ops.

$$= -m\langle 0|0\rangle \quad < 0.$$

negative norms!!

negative probabilities!!

Let's find a way around this.

We'll see that being more careful about the metric gauge-fixing can help us.

This is a complete solution of the classical e.o.m.,
with the Neumann b.c.,
before imposing the constraints.

imposing the constraints requires that we choose
one of the two methods mentioned,
covariant
or
light-cone gauge

We'll do light-cone gauge.

Have we used up all the diff $\times$ Weyl invariance in setting $\gamma_{\alpha\beta} \to \eta_{\alpha\beta}$?

Not quite.

<u>Recall</u>: if $\xi'^{\gamma} = \xi^{\gamma} + \epsilon^{\gamma}$

$$g'^{\alpha\beta} = g^{\gamma\delta} \frac{\partial \xi'^{\alpha}}{\partial \xi^{\gamma}} \frac{\partial \xi'^{\beta}}{\partial \xi^{\delta}}$$

$$= g^{\gamma\delta} \left(\delta^{\alpha}{}_{\gamma} + \epsilon^{\alpha}{}_{,\gamma} \right) \left(\delta^{\beta}{}_{\delta} + \epsilon^{\beta}{}_{,\delta} \right)$$

$$= g^{\alpha\beta} + g^{\gamma\beta} \partial_{\gamma} \epsilon^{\alpha} + g^{\alpha\delta} \partial_{\delta} \epsilon^{\beta} + \mathcal{O}(\epsilon^2)$$

$$= g^{\alpha\beta} + \partial^{\alpha} \epsilon^{\beta} + \partial^{\beta} \epsilon^{\alpha} + \mathcal{O}(\epsilon^2)$$

Now if $\delta g^{\alpha\beta} = \partial^{\alpha} \epsilon^{\beta} + \partial^{\beta} \epsilon^{\alpha}$

were to obey

$$\delta g^{\alpha\beta} = \Lambda(\xi) \eta^{\alpha\beta}$$

then $g'^{\alpha\beta} = e^{2\omega(\xi)} \eta^{\alpha\beta}$ for some $\omega(\xi)$

$\Rightarrow$ the above diffeo is a Weyl rescaling and can be undone by one.

$$\partial^\alpha \epsilon^\beta + \partial^\beta \epsilon^a = \Lambda \eta^{\alpha\beta}$$

$$= \eta^{\alpha\gamma}\partial_\gamma \epsilon^\beta + \eta^{\beta\gamma}\partial_\gamma \epsilon^\alpha = \Lambda \eta^{\alpha\beta}$$

if we define $\quad \xi^\pm = \xi^0 \pm \xi^1$

$$\epsilon^\pm = \epsilon^0 \pm \epsilon^1$$

$$\partial_0 = \frac{\partial}{\partial \xi^0} = 2(\partial_+ + \partial_-)$$

$$\partial_1 = \frac{\partial}{\partial \xi^1} \quad 2(\partial_+ - \partial_-)$$

$$\partial^0 = -2(\partial_+ + \partial_-)$$

$$\partial^1 = +2(\partial_+ - \partial_-)$$

furthermore, if $\quad \epsilon^+ = \epsilon^+(\xi^+)$ quite a restriction!

$$\epsilon^- = \epsilon^-(\xi^-)$$

then $\quad \partial^0 \epsilon^0 = -2(\partial_+ + \partial_-)[\epsilon^+ + \epsilon^-]\cdot \frac{1}{2} = -\partial_+ \epsilon^+ - \partial_- \epsilon^-$

$$\partial^1 \epsilon^1 = 2(\partial_+ - \partial_-)[\epsilon^+ - \epsilon^-]\cdot\frac{1}{2} = \partial_+ \epsilon^+ + \partial_- \epsilon^-$$

$$\partial^0 \epsilon^1 = -2(\partial_+ + \partial_-)[\epsilon^+ - \epsilon^-]\cdot\frac{1}{2} = -\partial_+ \epsilon^+ + \partial_- \epsilon^-$$

$$\partial^1 \epsilon^0 = 2(\partial_+ - \partial_-)[\epsilon^+ + \epsilon^-]\cdot\frac{1}{2} = \partial_+ \epsilon^+ - \partial_- \epsilon^-$$

$$\Rightarrow \quad \partial^0 \epsilon^1 + \partial^1 \epsilon^0 = 0$$

$$\partial^1 \epsilon^1 = -\partial^0 \epsilon^0$$

$$\Rightarrow \quad \partial^\alpha \epsilon^\beta + \partial^\beta \epsilon^\alpha = \Lambda \eta^{\alpha\beta} \quad !$$

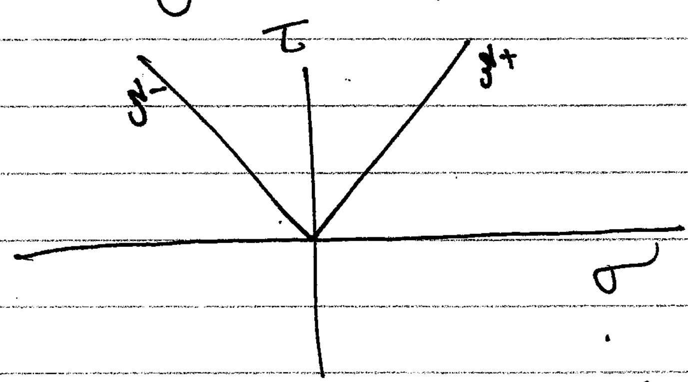
thus, reparametrizations

$$\xi^+ \to \xi^+ + \epsilon^+(\xi^+)$$
$$\xi^- \to \xi^- + \epsilon^-(\xi^-)$$

are actually also Weyl rescalings.

We will soon see that this structure is (almost ~~entire~~ singlehandedly) responsible for the appearance of holomorphy in string theory!

These are light-cone reparametrizations:



These are a measure-zero subset of all reparametrizations

(~ holomorphic functions of $x+iy$ vs. functions of (x,y)).

So what?

This extra symmetry can help us choose an even better gauge.

$\gamma_{\alpha\beta} = \eta_{\alpha\beta}$ already using $\text{diff} \times \text{Weyl} / \{\text{diff} \cap \text{Weyl}\}$

use "$\text{diff} \cap \text{Weyl}$" to fix τ parametrization

Obvious choice: $\tau = X^0$

turns out not very useful!

Next guess: $\tau = \frac{1}{\sqrt{2}}(X^0 + X^1) \equiv X^+$

i.e. take lightcone coords in spacetime

$$\{X^0, X^1, X^i\} \rightarrow \{X^+, X^-, X^i\}$$

$$X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^1)$$

and fix $\tau = X^+$.

Strategy now

(1) use light-cone gauge $\tau = X^+$
to solve Virasoro constraints for the classical string

(2) extend this to quantization of the string.

3 28.8

Solution + light-cone gauge) quantization of open ~~closed~~ strings

Last time:

Saw that after fixing $\gamma_{\alpha\beta} \to \eta_{\alpha\beta}$ by diff × Weyl, an infinite gauge freedom remains: light-cone reparametrizations $\xi^{\pm} \to \xi^{\pm}(\xi^{\pm})$.

Decided to solve classical theory, + then solve Virasoro constraints $T_{\alpha\beta}=0$ by using this gauge freedom.

Solved classical theory:

$$X^{\mu}(\tau,\sigma) = x_0^{\mu} + \sqrt{2\alpha'}\,\alpha_0^{\mu}\tau + i\sqrt{2\alpha'} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n}\alpha_n^{\mu} e^{-in\tau} \cos(n\sigma)$$

This time:

impose constraints + quantize theory.

Classical String in Light-cone gauge

first recall lc coordinates: (for 4d)

$$V^\mu : \quad V^+ \equiv \frac{1}{\sqrt{2}}(V^0+V^1) \qquad \hat{\eta}_{\mu\nu} : \begin{array}{cccc} + & - & 2 & 3 \end{array}$$

$$V^- \equiv \frac{1}{\sqrt{2}}(V^0-V^1) \qquad \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow V^\mu W_\mu = -V^-W^+ - V^+W^- + \underbrace{\vec{V}\cdot\vec{W}}_{V^iW^i}$$

$$V_+ = -V^- \; ; \; V_- = -V^+ . \Rightarrow V^\mu W_\mu = V_+W^+ + V_-W^- + V^iW^i$$

$$V^\mu V_\mu = -2V^+V^- + V^iV^i$$

wavefunction: $$\psi(t,\vec{x}) = \exp\left(+\frac{i}{\hbar}P_\mu X^\mu\right)$$

$$= \exp\left(\frac{i}{\hbar}\left[+P_+X^+ + P_-X^- + P_iX^i\right]\right)$$

taking X^+ as lc time,

$$i\hbar \frac{\partial}{\partial x^+}\psi = E\psi$$

$$\boxed{E = -P_+ = +P^-}$$

Illustrate with point particle

$$S = \frac{1}{2}\int d\tau \left(e^{-1}\dot{X}^\mu \dot{X}_\mu - e m^2 \right)$$

set $X^+(\tau) = \tau$ $\quad \cdot : \frac{\partial}{\partial \tau}$

$\Rightarrow \dot{X}^+ = 1$

$$\dot{X}^\mu \dot{X}_\mu = -2\dot{X}^+\dot{X}^- + \dot{X}^i\dot{X}^i$$
$$= -2\dot{X}^- + \dot{X}^i\dot{X}^i$$

$$S = \frac{1}{2}\int d\tau \left(-2e^{-1}\dot{X}^- + e^{-1}\dot{X}^i\dot{X}^i - e m^2 \right)$$

Canon. momenta $\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu}$: $\quad p_- = -e^{-1}$

$$p_i = e^{-1}\dot{X}^i$$

$\left(p_+ : \text{irrelevant, } \dot{X}^+ = \text{const} \right)$

$p_e = 0 \quad (\text{no } \dot{e})$

$\Rightarrow$ quantize by imposing con

$$[X^i, p_j] = i\delta^i{}_j$$

$$[X^-, p_-] = i$$

and, states can be written $|k_-, k^i\rangle$.

now for the ~~[illegible]~~ string.

~~[illegible]~~ set $X^+ = 2\alpha' p^+ \tau$. (covariant normalization for open string)

The constraint we must obey is

$$(\dot{X}^\mu \pm X'^\mu)(\dot{X}_\mu \pm X'_\mu) = 0$$

$$\Rightarrow \quad -2(\dot{X}^+ \pm X'^+)(\dot{X}^- \pm X'^-) + (\dot{X}^i \pm X'^i)^2 = 0$$

now $\cdot \equiv \frac{\partial}{\partial \tau} \Rightarrow$

$$\dot{X}^+ = 2\alpha' p^+$$
$$X'^+ = 0$$

$\Rightarrow$ constraints equivalent to

$$\dot{X}^- \pm X'^- = \left(\frac{1}{2\alpha' p^+}\right)(\dot{X}^i \pm X'^i)^2$$

$\Rightarrow$ specify $X^i(\tau, \sigma)$ and the zero modes p^+, X_0^-

and evolution (obeying constraints!!) is specified.

We have

$$X^i(\tau,\sigma) = x_0^i + \sqrt{2\alpha'}\,\alpha_0^i\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^i e^{-in\tau}\cos n\sigma$$

$$X^+ = 2\alpha' p^+\tau \equiv \sqrt{2\alpha'}\,\alpha_0^+\tau \qquad (\alpha_0^\mu \equiv \sqrt{2\alpha'}\,p^\mu)$$

$$(\text{so } x_0^+ = \alpha_n^\oplus = \alpha_{-n}^\oplus = 0 \quad \forall n>0)$$

$$X^- = x_0^- + \sqrt{2\alpha'}\,\alpha_0^-\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^- e^{-in\tau}\cos n\sigma$$

So
$$\dot{X}^\alpha \pm X'^\alpha = \sqrt{2\alpha'}\sum_{n=-\infty}^{\infty}\alpha_n^\alpha e^{-in(\tau\pm\sigma)}$$
where $\alpha \in \{-, i\}$.

$\Rightarrow$ by virasoro constraints
$$\sqrt{2\alpha'}\sum_{n=-\infty}^{\infty}\alpha_n^- e^{-in(\tau\pm\sigma)} = \left(\frac{1}{\cancel{4}2\alpha' p^+}\right)2\alpha'\sum_{n',n''}\alpha_{n'}^i\alpha_{n''}^i e^{-i(n'+n'')(\tau\pm\sigma)}$$

$$= \cancel{\frac{1}{2p^+}\sum_{m,p}\alpha_p^i\alpha_{m-p}^i e^{-i(m)(\tau\pm\sigma)}}$$

$$= \frac{1}{2p^+}\sum_{n\in\mathbb{Z}}\left[\sum_{p\in\mathbb{Z}}\alpha_p^i\alpha_{n-p}^i\right]e^{-in(\tau\pm\sigma)}$$

So $$\sqrt{2\alpha'}\,\alpha_n^- = \frac{1}{2p^+}\sum_{p\in\mathbb{Z}} \alpha_p^i \alpha_{n-p}^i$$

define $$L_n^\perp \equiv \frac{1}{2}\sum_{p\in\mathbb{Z}} \alpha_{n-p}^i \alpha_p^i$$

$$\Rightarrow \dot{X}^- \pm X^{-\prime} = \frac{1}{p^+}\sum_{n\in\mathbb{Z}} L_n^\perp e^{-in(\tau\pm\sigma)}$$

$L_n^\perp$ are modes of the $\underset{\text{constraint}}{\overset{\text{Virasoro}}{}}$.

↑ "transverse Virasoro modes"

Recall $$\sqrt{2\alpha'}\,\alpha_0^- \equiv 2\alpha' p^-$$

and $$\sqrt{2\alpha'}\,\alpha_0^- = \frac{1}{p^+} L_0^\perp$$

$$\Rightarrow 2\alpha' p^+ p^- = L_0^\perp = \frac{1}{2}\sum_p \alpha_{-p}^i \alpha_{+p}^i$$

but $$p^2 = -2p^+p^- + p^i p^i = -M^2$$

So $M^2 = 2p^+p^- - p^ip^i$

$= \frac{1}{\alpha'} L_0^\perp - p^ip^i$

but $\frac{1}{\alpha'} L_0^\perp = \frac{1}{2\alpha'} \alpha_0^i \alpha_0^i + \frac{1}{2\alpha'} \sum_{n \neq 0} \alpha_n^i \alpha_{-n}^i \cdot 2$ ↙ for $n>0$ ($n<0$)

and using $\alpha_0^i = \sqrt{2\alpha'}\, p^i$ and $\alpha_n^i = a_n^i \sqrt{n}$, $\alpha_{-n}^i = a_n^{i*} \sqrt{n}$

$\frac{1}{\alpha'} L_0^\perp = \left(\frac{1}{2\alpha'}\right)(2\alpha')\, p^ip^i + \frac{1}{2\alpha'} \sum_{n=1}^{\infty} n\, a_n^{i*} a_n^i \cdot 2$

$= p^ip^i + \frac{1}{\alpha'} \sum_{n=1}^{\infty} n\, a_n^{i*} a_n^i$

$$\Rightarrow \boxed{M^2 = \frac{1}{\alpha'} \sum_{n=1}^{\infty} n\, a_n^{i*} a_n^i}$$

but note $a_n^i \in \mathbb{C}$, not quantized yet of course.

Open String

Quantization in light-cone gauge

$$X^+ = 2\alpha' p^+ \tau$$

$$X^- = x_0^- + \sqrt{2\alpha'}\,\alpha_0^-\,\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^-\,e^{-in\tau}\cos(n\sigma)$$

$$\left(\sqrt{2\alpha'}\,\alpha_n^- = \frac{1}{2p^+}\sum_{p\in\mathbb{Z}}\alpha_p^i\,\alpha_{n-p}^i\right)$$

$$X^i = x_0^i + \sqrt{2\alpha'}\,\alpha_0^i\,\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^i\,e^{-in\tau}\cos(n\sigma)$$

(unit gauge, constraints solved)

We saw that $\left[X^\mu(\sigma), \Pi^\mu(\sigma')\right] = i\,\delta(\sigma-\sigma')$

$$\Rightarrow \quad \left[\alpha_m^\mu, \alpha_n^\nu\right] = i\eta^{\mu\nu}\,\delta_{m,-n}$$

Fourier coeffs become raising/lowering operators.

So our full prescription is:

$$[x^-, p^+] = i\eta^{-+} = -i$$

$$[X^i(\sigma), \Pi^j(\sigma')] = i\delta^{ij}\delta(\sigma-\sigma')$$

$$\Rightarrow \quad [x^i, p^j] = i\delta^{ij}$$

$$[\alpha^i_m, \alpha^j_n] = m\,\delta^{ij}\delta_{m,-n}$$

(nb. X^- part made of α^-_n is expressed in terms of α^i_m and X^+ is gauged away.)

so $\quad [\alpha^i_m, \alpha^i_{-m}] = m \quad \Rightarrow \quad \left[\underbrace{\frac{\alpha^i_m}{\sqrt{m}}}_{a_m}, \underbrace{\frac{\alpha^i_{-m}}{\sqrt{m}}}_{a^+_m}\right] = 1$

full set of commuting operators: $\quad p^+, p^i$

$\alpha^i_m \quad m<0$

$i = 2, \ldots, D-1$

no-string-state $|\text{vacuum}\rangle$

string ground state $|0;k\rangle$

$$p^+|0;k\rangle = k^+|0;k\rangle$$
$$p^i|0;k\rangle = k^i|0;k\rangle$$
$$\alpha_m^i|0;k\rangle = 0 \quad \text{for } m>0.$$

Raising operators α_{-m}^i ~~with~~ $n>0$

$\Rightarrow$ general state is

$$\boxed{\prod_{i=2}^{D-1}\prod_{n=1}^{\infty}\frac{(\alpha_{-n}^i)^{N_{in}}}{\sqrt{n^{N_{in}}N_{in}!}}|0;k\rangle \equiv |N,k\rangle}$$

free particle + infinite tower of oscillators

$$N = \sum_{i=2}^{D-1}\sum_{n\geq 1} n\,N_{in}$$

Recall $H = p^-$

and $2p^+p^- - p^i p^i = M^2$

and $p^- = \frac{1}{2p^+\alpha'} L_0^\perp = \cancel{\frac{1}{2p^+\alpha'}} = \frac{1}{4p^+\alpha'} \sum_{p\in\mathbb{Z}} \alpha_{-p}^i \alpha_p^i$

(sum on i)

and $\alpha_0^\mu = \sqrt{2\alpha'}\, p^\mu$

$$\Rightarrow p^- = \frac{1}{4p^+\alpha'}\left(\sum_{p=1}^{\infty} \alpha_{-p}^i \alpha_p^i + \sum_{p=1}^{\infty} \alpha_p^i \alpha_{-p}^i + \underbrace{\alpha_0^i \alpha_0^i}_{2\alpha' p^i p^i} \right)$$

$$= \frac{p^i p^i}{2p^+} + \frac{1}{2p^+\alpha'}\left(\sum_{p=1}^{\infty} \alpha_{-p}^i \alpha_p^i + A \right)$$

ambiguity!

$$\Rightarrow M^2 = \frac{1}{\alpha'}\left(\sum_{p=1}^{\infty} \alpha_{-p}^i \alpha_p^i + A \right)$$

$$\boxed{M^2 = \frac{1}{\alpha'}(N + A)}$$

What is A?

Various methods exist for computing A. Conformal field theory in a few weeks.

Simplest way:

ground state $|0;k\rangle$ $\qquad M^2 = \frac{1}{\alpha'} A$

(tachyon if $A<0$)

1st excited state $\alpha^i_{-1}|0;k\rangle$ $\qquad M^2 = \frac{1}{\alpha'}(1+A)$

This is a vector particle with D-2 states.

Note: massive particle in rest frame

$$p^\mu = (m, \underbrace{0, \ldots 0}_{SO(D-1)})$$

forms - rep of $SO(D-1)$.

whereas massless particle,

$$p^\mu = (E, E, \underbrace{0 \ldots 0}_{SO(D-2)})$$

is a rep of $SO(D-2)$

cf 4d: massive vector (spin 1) 3 states $SO(3)$
massless vector ~~spin~~, 2 states $SO(2)$

We simply cannot form a vector rep of $SO(D-1)$, because we have $D-2$ states

$\Rightarrow$ can only transform under $SO(D-2)$

$\Rightarrow$ must be massless!

$$\Rightarrow \quad 1 + A = 0 \quad \Rightarrow \quad \boxed{A = -1.}$$

(homework develops this further.)

Spectrum can be Lorentz-invariant only if $A = -1$. Still have to check higher levels. There could be an Lorentz anomaly!

note $|0;k\rangle$ is a tachyon,

$$M^2 = -\frac{1}{\alpha'}.$$

Closed String quantized in light-cone gauge

$$S = -\frac{T}{2}\int d\tau\, d\sigma \left[(\partial_\tau X)^2 - (\partial_\sigma X)^2\right]$$

eom $\quad \partial_\tau^2 X^\mu - \partial_\sigma^2 X^\mu = 0$

bc: now choose periodic bc,

$$X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi).$$

Solve eom:

$$X^\mu = X^\mu_L(\tau+\sigma) + X^\mu_R(\tau-\sigma)$$

bc:

$$X^\mu_L(\tau+\sigma) + X^\mu_R(\tau-\sigma) = X^\mu_L(\tau+\sigma+2\pi) + X^\mu_R(\tau-\sigma-2\pi)$$

$$\Rightarrow \quad X^\mu_L(\tau+\sigma) - X^\mu_L(\tau+\sigma+2\pi) = X^\mu_R(\tau-\sigma-2\pi) - X^\mu_R(\tau-\sigma)$$

but $\tau+\sigma$, $\tau-\sigma$ are indep

$$\Rightarrow \quad X_L^{\mu\,\prime}(\tau+\sigma+2\pi) - X_L^{\mu\,\prime}(\tau+\sigma) = \frac{\partial}{\partial(\tau+\sigma)} \text{RHS} = 0$$

$$/: \frac{d}{d(\tau+\sigma)}$$

also $$X_R^{\mu\,\prime}(\tau-\sigma+2\pi) - X_R^{\mu\,\prime}(\tau-\sigma) = 0$$

$$/: \frac{d}{d(\tau-\sigma)}$$

$\Rightarrow$ can write

$$X_L^{\mu\,\prime}(\tau+\sigma) = \sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \tilde{\alpha}_n^\mu e^{-in(\tau+\sigma)}$$

$$X_R^{\mu\,\prime}(\tau-\sigma) = \sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \alpha_n^\mu e^{-in(\tau-\sigma)}$$

$\tilde{\alpha} \neq \bar{\alpha}$ or α^+ or anything.

These are <u>independent</u> modes.

integrate to get

$$X_L^\mu = \frac{1}{2} x_0^{L\mu} + \sqrt{\frac{\alpha'}{2}}\, \tilde{\alpha}_0^\mu (\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\tilde{\alpha}_n^\mu}{n} e^{-in(\tau+\sigma)}$$

$$X_R^\mu = \frac{1}{2} x_0^{R\mu} + \sqrt{\frac{\alpha'}{2}} \cdot \alpha_0^\mu (\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} e^{-in(\tau-\sigma)}$$

but taking the differences after 2π shift

$$X_L^\mu(\tau+\sigma) - X_L^\mu(\tau+\sigma+2\pi) = \sqrt{\frac{\alpha'}{2}}\, \tilde{\alpha}_0^\mu \cdot (-2\pi)$$

$$X_R^\mu(\tau-\sigma) - X_R^\mu(\tau-\sigma-2\pi) = \sqrt{\frac{\alpha'}{2}}\, \alpha_0^\mu (2\pi)$$

$\Rightarrow$ referring back to periodicity condition,

$$\boxed{\alpha_0^\mu = \tilde{\alpha}_0^\mu}$$

More sophisticated approach:

use σ-translations $\quad \sigma \to \sigma + \epsilon$

$$\delta X^i = \epsilon \partial_\sigma X^i$$

find associated conserved current + charge

demanding invariance under σ shifts then gives $\alpha_0^\mu = \tilde{\alpha}_0^\mu$!

(exercise 13.4 in Zwiebach)

Now we'll impose LCG.

$$X^+ = \alpha' p^+ \tau \qquad \left[\begin{array}{l}\text{differs from open string} \\ \text{convention} \\ \qquad X^+_{\text{open}} = 2\alpha' p^+ \tau\end{array}\right]$$

$\Rightarrow$ solve II by

$$\dot{X}^- \pm X'^- = \frac{1}{2p^+\alpha'} \left(\dot{X}^i \pm X'^i\right)^2$$

~~defined~~ Then

$$\left(\dot{X}^i + X'^i\right)^2 = 4\alpha' \sum_{n\in\mathbb{Z}} \left(\frac{1}{2}\sum_{p\in\mathbb{Z}} \tilde{\alpha}^i_p \tilde{\alpha}^i_{n-p}\right) e^{-in(\tau+\sigma)}$$

$$\equiv 4\alpha' \sum_{n\in\mathbb{Z}} \tilde{L}^\perp_n e^{-in(\tau+\sigma)}$$

and

$$\left(\dot{X}^i - X'^i\right)^2 = 4\alpha' \sum_{n\in\mathbb{Z}} \left(\frac{1}{2}\sum_{p\in\mathbb{Z}} \alpha^i_p \alpha^i_{n-p}\right) e^{-in(\tau-\sigma)}$$

$$\equiv 4\alpha' \sum_{n\in\mathbb{Z}} L^\perp_n e^{-in(\tau+\sigma)}$$

but using moding for X^-, as before, we get

$$\dot{X}^- + X'^- = \sqrt{2\alpha'} \sum_{n\in\mathbb{Z}} \tilde{\alpha}^-_n e^{-in(\tau+\sigma)}$$

$$\dot{X}^- - X'^- = \sqrt{2\alpha'} \sum_{n\in\mathbb{Z}} \alpha^-_n e^{in(\tau-\sigma)}$$

$$\Rightarrow \quad \sqrt{2\alpha'}\,\tilde{\alpha}_n^- = \frac{2}{p_+}\tilde{L}_n^\perp$$

$$\sqrt{2\alpha'}\,\alpha_n^- = \frac{2}{p_+}L_n^\perp$$

but we showed $\alpha_0^\mu = \tilde{\alpha}_0^\mu$

$$\Rightarrow \quad L_0^\perp = \tilde{L}_0^\perp .$$

$$\Rightarrow \quad \frac{1}{2}\cancel{\alpha_0^i\alpha_0^i} + \underbrace{\sum_{p=1}^{\infty}\alpha_{-p}^i\alpha_p^i}_{\equiv N} = \frac{1}{2}\cancel{\tilde{\alpha}_0^i\tilde{\alpha}_0^i} + \underbrace{\sum_{p=1}^{\infty}\tilde{\alpha}_{-p}^i\tilde{\alpha}_p^i}_{\equiv \tilde{N}}$$

Easy to quantize now.
(recalling $X^\mu = X^\mu_L(\tau+\sigma) + X^\mu_R(\tau-\sigma)$, we have the mode expansion).

d.o.f.: α^i_n, $\tilde{\alpha}^i_n$, x^i, p^i, x^-, p^+

canonical commutators

$$[x^-, p^+] = -i$$
$$[x^i, p^j] = i\delta^{ij}$$
$$[\alpha^i_m, \alpha^j_n] = m\,\delta^{ij}\,\delta_{m,-n}$$
$$[\tilde{\alpha}^i_m, \tilde{\alpha}^j_n] = m\,\delta^{ij}\,\delta_{m,-n}$$

all others vanish.

ground state $|0;0;k\rangle$ annihil. by α^i_m, $\tilde{\alpha}^i_n$ for $m,n>0$.

general state

$$|N,\tilde{N};k\rangle = \left[\prod_{i=2}^{D-1}\prod_{n=1}^{\infty}\frac{\left(\alpha^i_{-n}\right)^{N_{in}}\left(\tilde{\alpha}^i_{-n}\right)^{\tilde{N}_{in}}}{\sqrt{n^{N_{in}}\,N_{in}!\;n^{\tilde{N}_{in}}\,\tilde{N}_{in}!}}\right]|0,0;k\rangle$$

Now the constraint: $N = \tilde{N}$

"level-matching"

· physical origin is σ-translation-invariance.

$$M^2 = 2p^+p^- - p^ip^i$$

$$= \frac{2}{\alpha'}\left[\sum_{n=1}^{\infty}\left(\alpha_{-n}^i \alpha_n^i + \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i\right) + A + \tilde{A}\right]$$

ought to be equal; we'll see this soon. in fact, we'll see $A = \tilde{A} = -1$.

$$= \frac{2}{\alpha'}\left[N + \tilde{N} + A + \tilde{A}\right]$$

$$= \frac{2}{\alpha'}\left[2N - 2\right] = \boxed{\frac{4}{\alpha'}\left[N-1\right]}$$

Closed String spectrum

$|0,0;k\rangle$ $\qquad M^2 = \frac{4}{\alpha'}(N-1)$

$$= -\frac{4}{\alpha'}$$

$\alpha^i_{-1}\tilde{\alpha}^j_{-1}|0,0;k\rangle$ $\qquad M^2 = 0$

general massless state

$$\sum_{ij} e^{ij}\alpha^i_{-1}\tilde{\alpha}^j_{-1}|0;0;k\rangle$$

general tensor $e^{ij} = \frac{1}{2}\left(e^{ij}+e^{ji}-\frac{2}{D-2}\delta^{ij}e^{kk}\right)$
(symm, traceless)

$$+\ \frac{1}{2}(e^{ij}-e^{ji})$$

$$+\ \frac{1}{D-2}\delta^{ij}e^{kk}$$

So we have

"Φ" a scalar
"$B_{\mu\nu}$" an antisymmetric tensor
"$g_{\mu\nu}$" a traceless symmetric tensor

} all massless

Comments on consistency + zero-point energy

We reordered

$$\frac{1}{2}\sum_{n=1}^{\infty} \alpha^i_{-n}\alpha^i_n + \alpha^i_n\alpha^i_{-n} \to \sum_{n=1}^{\infty} \alpha^i_{-n}\alpha^i_n + A$$

$A =$ sum of zero point energies for all modes, $\sum \frac{1}{2}\omega$

$$= \sum_{i=2}^{D-1}\sum_{n=1}^{\infty} \frac{1}{2}n = \frac{D-2}{2}\sum_{n=1}^{\infty} n$$

Well, $\zeta(s) \equiv \sum_{n=1}^{\infty} n^{-s}$

so this sum is $\zeta(-1)$.

(unique) analytic continuation of ζ gives $\zeta(-1) = -\frac{1}{12}$.

$$\Rightarrow \sum_{n=1}^{\infty} n \text{ ``$=$'' } -\frac{1}{12}$$

$$\Rightarrow A = \frac{D-2}{2}\cdot\left(-\frac{1}{12}\right) = \frac{2-D}{24}$$

but we saw $A=-1$ required for Lorentz inv.

$$\Rightarrow \boxed{D = 26}$$

We'll have a much more rigorous derivation using conformal field theory.

Other rigorous derivations exist (e.g. in Polchinski).

All give $D = 26$.

ex. demand existence of Lorentz generators in LCG.

usually have Lorentz algebra

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\eta^{\mu\rho}M^{\nu\sigma} - i\eta^{\nu\rho}M^{\mu\sigma} + i\eta^{\mu\sigma}M^{\rho\nu} - i\eta^{\nu\sigma}M^{\rho\mu}$$

where $M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$

demanding that this still works in LCG (with x, p operators made of string oscillators α)

gives $D = 26$.