The Polyakov Path Integral

We have found the spectrum of states of open + closed strings.

Open: $$\prod_{i=2}^{D-1}\prod_{n=1}^{\infty}\frac{(\alpha^i_{-n})^{N_{in}}}{\sqrt{n^{N_{in}}N_{in}!}}|0;k\rangle \equiv |N,k\rangle$$

$$N \equiv \sum_{i=2}^{D-1}\sum_{n=1}^{\infty} n N_{in}$$

with $M^2 = \frac{1}{\alpha'}(N-1)$

and low states

ground state $|0;k\rangle$ $\qquad M^2 = -\frac{1}{\alpha'}$ $\qquad$ tachyon

1st excited state $\alpha^i_{-1}|0;k\rangle$ $\qquad M^2 = 0$ $\qquad$ massless vector

Closed:

$$\left[\prod_{i=2}^{D-1}\prod_{n=1}^{\infty}\frac{\left(\alpha_{-n}^{i}\right)^{N_{in}}\left(\tilde{\alpha}_{-n}^{i}\right)^{\tilde{N}_{in}}}{\sqrt{n^{N_{in}}N_{in}!\,n^{\tilde{N}_{in}}\tilde{N}_{in}!}}\right]|0,0;k\rangle \equiv |N,\tilde{N};k\rangle$$

with $N=\tilde{N}$ (level-matching constraint)

and $M^2 = \frac{4}{\alpha'}[N-1]$

and low states

ground state $\quad |0;0;k\rangle \quad M^2 = -4/\alpha'$

1^{st} excited $\quad \alpha_{-1}^{i}\tilde{\alpha}_{-1}^{j}|0,0;k\rangle \quad M^2=0$ massless tensor

Let's study the massless tensor.

$\alpha_{-1}^{i}\tilde{\alpha}_{-1}^{j}|0;0;k\rangle$ is a basis state; a general massless state is

$\sum_{ij} e_{ij}\alpha_{-1}^{i}\tilde{\alpha}_{-1}^{j}|0,0;k\rangle$ for e_{ij} a tensor of coeffs.

Path integral formulation of string theory

We have studied the action

$$S_P = -\frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{\gamma}\, \gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu$$

in LCG, and now also in complex coordinates, with unit gauge for the metric:

$$S_P = \frac{1}{2\pi\alpha'}\int d^2 z\, \partial X^\mu \bar{\partial} X_\mu$$

This is actually not the most general 2D action that is diff×Weyl and Poincaré invariant.

$$\Delta S = \frac{\lambda}{4\pi}\int d^2\sigma \sqrt{\gamma}\, R$$

is allowed. Here λ is a dimensionless coefficient.

However, this term does not contribute to the eom, because 2D gravity is trivial: the Einstein eqns are identically obeyed (easy GR exercise).

That is, in 2D the E-H action depends not on the metric, but only on the WS topology.

In fact $\frac{1}{4\pi}\int d^2\sigma \sqrt{\gamma}\, R \equiv \chi$

is the Euler number of the WS

($\chi = 2$ sphere
$\chi = 0$ torus
etc.)

($\chi = 2 - 2g$ for oriented surfaces w/o boundary, with g handles.)

So we should consider a theory of strings governed by

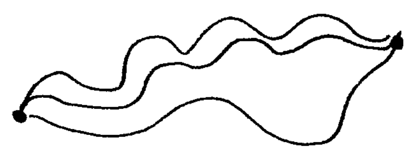
$$S = -\int d^2\sigma \sqrt{\gamma} \left\{ \frac{1}{4\pi\alpha'} \gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu + \frac{\lambda}{4\pi} R \right\}$$

$$= S_P + \lambda \chi$$

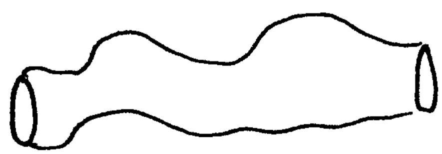
for some λ.

Path integral prescription: sum over all paths, weighted by

$$e^{\frac{iS}{\hbar}}$$

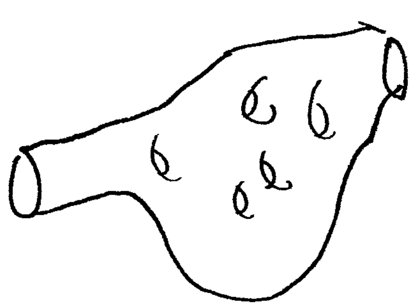


point particle



string

but should certainly allow more complicated intermediaries,



etc.

$\Rightarrow$ for fixed initial + final boundary data,

sum over all topologies (suitable ones, more soon)

and $\int$ over all embeddings $\{X, g\}$

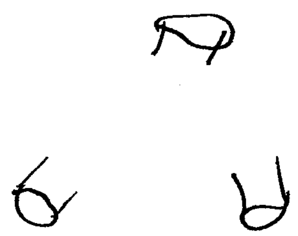
weighted by e^{-S}

and divide out any overcounting.

$$Z \equiv \sum_{\substack{\text{suitable} \\ \text{topologies}}} \int \frac{[DX][Dg]}{(\text{overcounting})} e^{-S}$$

overcounting = volume of gauge group = $V_{(\text{diff} \times \text{Weyl})}$

How to include data of $|i\rangle, |f\rangle$?



Well, $S_p = -\frac{1}{2\pi\alpha'}\int d^2z\, \partial X^\mu \bar\partial X_\mu$ unit gauge, $\mathbb{C}$ coords

enjoys conformal invariance.

$$ds^2 = g_{z\bar z}\, dz\, d\bar z$$

diff: $z \to z'(z)$

$$ds'^2 = \left|\frac{\partial z}{\partial z'}\right|^2 g_{z\bar z} \left|\frac{\partial z'}{\partial z}\right|^2 dz\, d\bar z = ds^2$$

now metric has transformed, no longer in unit gauge! Fix that!

use Weyl $\omega = \ln\left|\frac{\partial z'}{\partial z}\right|$

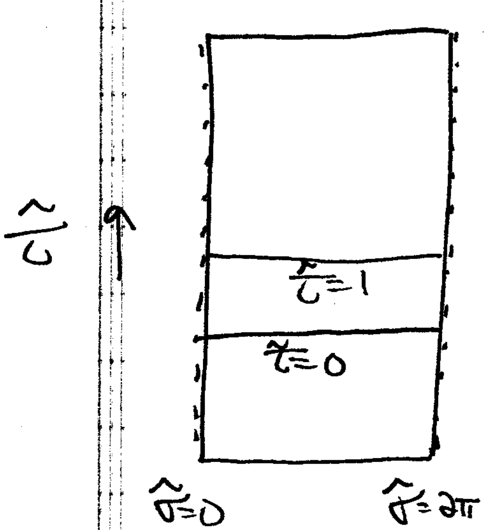
$$ds''^2 = e^{2\omega} ds'^2 = \left|\frac{\partial z'}{\partial z}\right|^2 g_{z\bar z}\, dz\, d\bar z''$$

$$= g_{z\bar z}\, dz'\, d\bar z' = \left|\frac{\partial z'}{\partial z}\right|^2 ds^2$$

metric invariant, line element changed!

But the action is diff $\times$ Weyl invariant $\Rightarrow$ action unchanged.

First, let's think about a simpler conformal map.



$0 \le \tilde\sigma \le 2\pi$

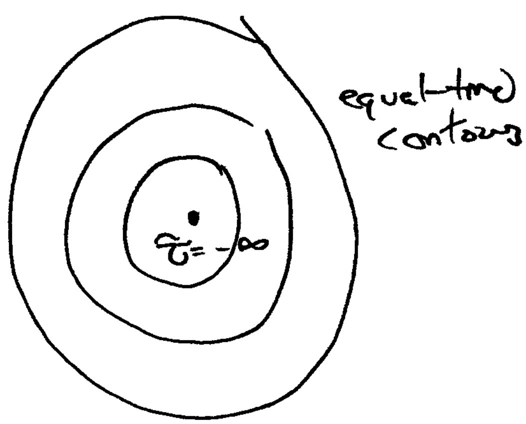
$-\infty < \tilde\tau < \infty$

define complex coord $w = \tilde\sigma + i\tilde\tau$

and $z = e^{-iw} = e^{-i\tilde\sigma + \tilde\tau}$

identify $\tilde\sigma = 0 \Leftrightarrow \tilde\sigma = 2\pi$

In z coords, origin $z = 0 \Leftrightarrow \tilde\tau = -\infty$.



time runs radially.

cf. cylinder.

Now write our mode expansion in terms of z

$$X^\mu_R = \frac{1}{2} x_0^{R\mu} + \sqrt{\frac{\alpha'}{2}}\,\tilde{\alpha}_0^\mu \cdot (\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\tilde{\alpha}_n^\mu}{n} e^{-in(\tau+\sigma)}$$

$$X^\mu_L = \frac{1}{2} x_0^{L\mu} + \sqrt{\frac{\alpha'}{2}}\,\alpha_0^\mu\,(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n} e^{-in(\tau-\sigma)}$$

finally, the analytic continuation is as follows:

$$ds^2 = -d\tau^2 + d\sigma^2 \qquad \text{original}$$

$$\tau \to -i\tilde{\tau}$$

$$ds^2 = d\tilde{\tau}^2 + d\tilde{\sigma}^2$$

since $\tilde{\tau} = i\tau$

$$ds^2 = dz\,d\bar{z} \qquad z = e^{-i\tilde{\sigma}+\tilde{\tau}} = e^{-i(\sigma+\tau)}$$

so $\ln z = -i(\sigma+\tau)$

and $\sigma+\tau = i\ln z$

$-\sigma+\tau = i\ln\bar{z}$ or $\bar{z} = e^{+i(\sigma+\tau)}$

Thus,

$$X^\mu_R = \frac{1}{2} x_0^{R\mu} + \sqrt{\frac{\alpha'}{2}}\,\tilde{\alpha}_0^\mu(-i\ln\bar{z}) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\tilde{\alpha}_n^\mu}{n}\,\bar{z}^{-n}$$

$$X^\mu_L = \frac{1}{2} x_0^{L\mu} + \sqrt{\frac{\alpha'}{2}}\,\alpha_0^\mu(-i\ln z) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}\,z^{-n}$$

Next, take $X^\mu(z,\bar z) = X^\mu_R(\bar z) + X^\mu_L(z)$

$$\partial_z X^\mu(z) = -i\sqrt{\tfrac{\alpha'}{2}}\sum_n \alpha^\mu_n z^{-n-1}$$

$$\partial_{\bar z} X^\mu(\bar z) = -i\sqrt{\tfrac{\alpha'}{2}}\sum_n \tilde\alpha^\mu_n \bar z^{-n-1}$$

or,
$$\oint \frac{dz}{2\pi i} z^{-m}\partial_z X^\mu(z)$$

$$= -i\sqrt{\tfrac{\alpha'}{2}}\,\alpha^\mu_{-m}$$

so
$$\alpha^\mu_{-n} = \sqrt{\tfrac{2}{\alpha'}}\oint\frac{dz}{2\pi} z^{-n}\partial_z X^\mu(z)$$

Now $\oint \underset{\substack{\text{in}\\ \text{radial}\\ \text{quantization}}}{\Leftrightarrow} \int_{\substack{\text{spacelike}\\ \text{surface}}}$ which is how we usually define conserved charges.

(hol $\Rightarrow$ contour-indep
$\Rightarrow$ time-trans-indep
$\Rightarrow$ conserved.)

finally, spell out

$$\alpha^\mu_{-n} = i\sqrt{\tfrac{2}{\alpha'}}\oint\frac{dz}{2\pi i} z^{-n}\,\partial_z^{n-1}\partial_z X^\mu(z)\,\frac{1}{n-1!}$$

$$= i\sqrt{\tfrac{2}{\alpha'}}\,\partial_z^n X^\mu(0) \equiv \frac{1}{n-1!}\, i\sqrt{\tfrac{2}{\alpha'}}\,\partial^m X^\mu(0)$$

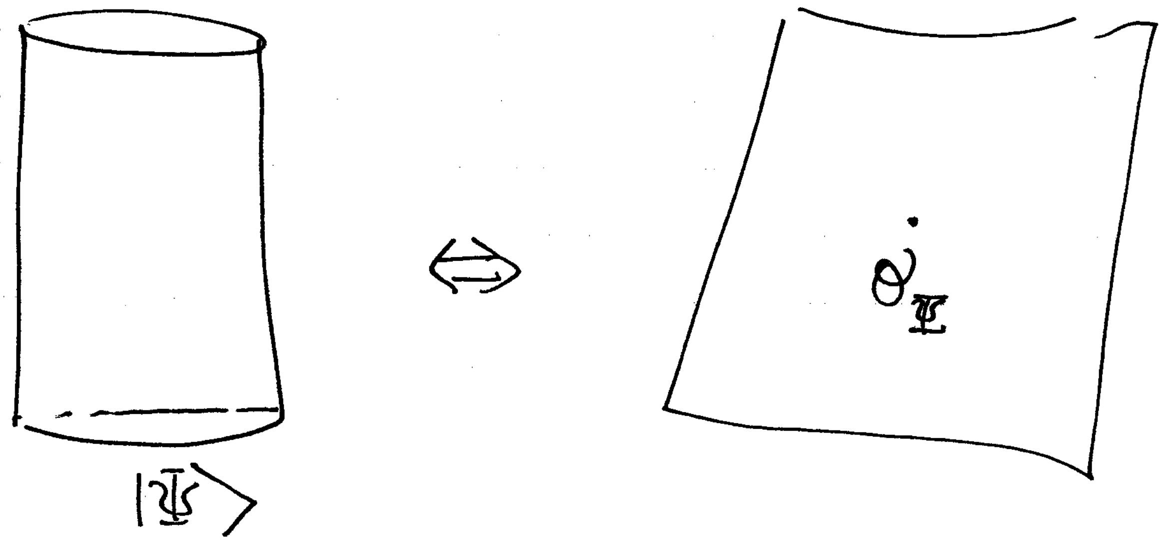
Finally, we can compute α^{μ}_{-n}.

$$\alpha^{\mu}_{-n} = i\sqrt{\frac{2}{\alpha'}}\oint \frac{dz}{2\pi i z}\, \underbrace{z^{-(n-1)}\,\partial_z X^{\mu}(z)}_{\text{expand around } z=0\text{, get}}$$

$$z^{-(n-1)}\left[\frac{1}{n-1!}\, z^{n-1}\,\partial_z^{n} X^{\mu}(0)\right]$$

$$= \sqrt{\frac{2}{\alpha'}}\,\frac{i}{n-1!}\,\partial^{n} X^{\mu}(0)$$

We have an isomorphism between <u>initial states</u> and <u>local operators</u>:



with $\alpha^\mu_{-m}\tilde{\alpha}^\nu_{-m}|0,0\rangle = |\Psi\rangle$

$$\Leftrightarrow$$

$$\mathcal{O}_\Psi = \left(\sqrt{\frac{2}{\alpha'}}\frac{i}{m-1!}\partial^m X^\mu(0)\right)\left(\sqrt{\frac{2}{\alpha'}}\frac{i}{m-1!}\bar{\partial}^m X^\nu(0)\right)$$

More on the state-operator correspondence

Consider the path integral in QM,

$$\langle q_f, T | q_i, 0\rangle = \int [dq]\, e^{i/\hbar \int_0^T dt\, L(q,\dot{q})}$$

'cut open' at t, $0 < t < T$

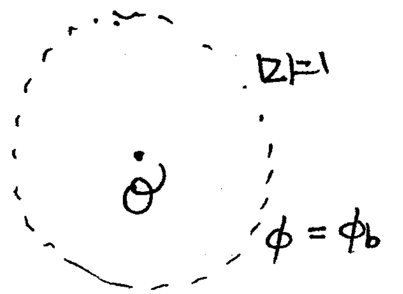
$$\langle q_f, T | q_i, 0\rangle = \underbrace{\int dq(t)}_{\text{regular } \int} \int_{q,t}^{q_f,T} [dq]\, e^{i/\hbar \int_t^T L} \int_{q_i,0}^{q,t} [dq]\, e^{i/\hbar \int_0^t L}$$

$$= \int dq(t)\, \langle q_f, T | q, t\rangle \langle q, t | q_i, 0\rangle$$

now could replace $\int dq\, |q,t\rangle\langle q,t|$ with any complete set of states.

$$\Rightarrow \int_{q_i,0}^{q,t} [dq]\, e^{i/\hbar \int_0^t L} \Leftrightarrow \text{a } \underline{\text{state}}.$$

Alternative view:



$$\int [d\phi_i]_{\phi_b} \, e^{-S[\phi_i]} \, \mathcal{O}(0) \equiv \Psi_{\mathcal{O}}[\phi_b]$$

interior field configs

with $\phi = \phi_b$ on $|z|=1$ (∂ of disk)

here $\Psi_{\mathcal{O}}[\phi_b]$ maps field configs $\phi_b \to \mathbb{C}$

i.e. $\Psi_{\mathcal{O}}$ is a functional of fields $\Leftrightarrow$ a <u>state</u>

but the state depends on $\mathcal{O}$

$\Rightarrow$ $\Psi_{\mathcal{O}}$ gives a map from operators ($\mathcal{O}$ inserted) to states $\Psi_{\mathcal{O}}$

Also for k^μ: $\quad p^\mu|0,0;k\rangle = k^\mu|0,0;k\rangle$

$$p^\mu = \sqrt{\frac{2}{\alpha'}}\,\alpha_0^\mu = \sqrt{\frac{2}{\alpha'}}\,\tilde{\alpha}_0^\mu$$

we take

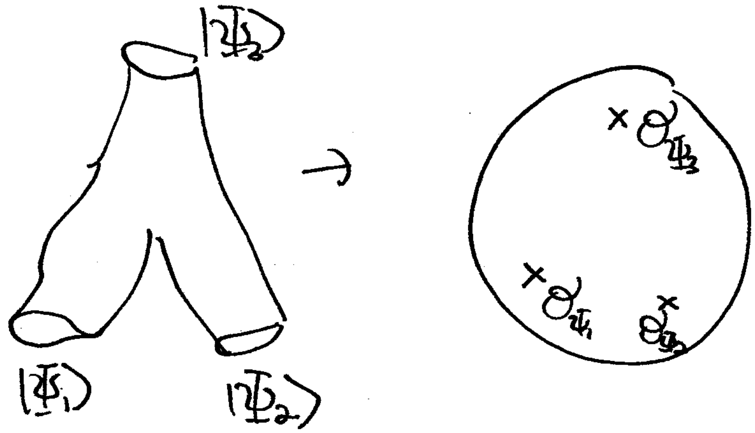
$$|0,0;k\rangle \Leftrightarrow e^{ik_\mu X^\mu(0)}$$

so that

$$\alpha_{-m}^\mu|0;k\rangle \Leftrightarrow \sqrt{\frac{2}{\alpha'}}\frac{i}{m-1!}\,\partial^m X^\mu(0)\,e^{ikX(0)}$$

$$\alpha_{-m}^\mu\,\tilde{\alpha}_{-m}^\nu|0,0;k\rangle \Leftrightarrow \left(\sqrt{\frac{2}{\alpha'}}\frac{i}{m-1!}\right)^2 \partial^m X^\mu(0)\,\bar{\partial}^m X^\nu(0)\,e^{ikX(0)}$$

So we've learned how to represent an initial state $|\Psi\rangle$ by inserting a local operator $\mathcal{O}_{|\Psi\rangle}$.



But we chose special coordinates, inserting the $\mathcal{O}$ at $z = \bar{z} = 0$.

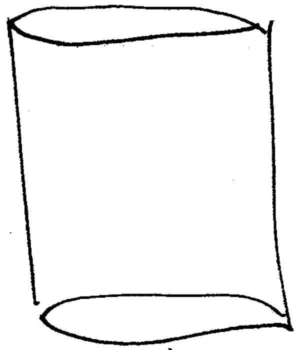
Let us now obtain a diff-inv't result by integrating over possible locations for the insertion.

$$V_0 \equiv \underset{\substack{\text{(constant} \\ \text{normaliz.)}}}{g_c} \int d^2z \; e^{ik_\mu X^\mu(z,\bar{z})}$$

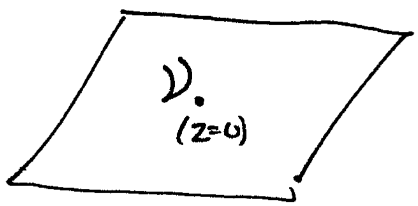
'vertex operator for $|0;k\rangle$' in O.S.
$|0,0;k\rangle$ in C.S.

$$V_{-1,-1} = g_c \int d^2z \left(\frac{2}{\alpha'}\right) \partial X^\mu(z,\bar{z})\, \bar{\partial} X^\nu(z,\bar{z})\, e^{ikX(z,\bar{z})}$$

$$= \frac{2}{\alpha'} g_c \int d^2z\, \partial X^\mu \bar{\partial} X^\nu e^{ikX}$$



For initial + final data, we know how to insert a particular on-shell state at $t = -\infty$:

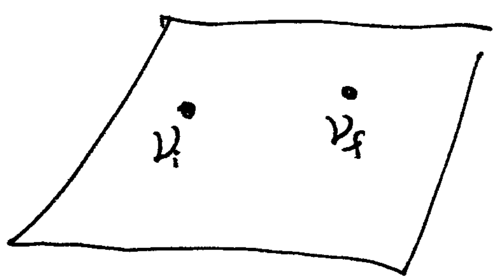


$$V = :\partial X^\mu_{(0)} \bar{\partial} X^\nu_{(0)} e^{ikX(0)}: \frac{2}{\alpha'}, \text{ e.g.}$$

and can make this diff-invariant by integrating:

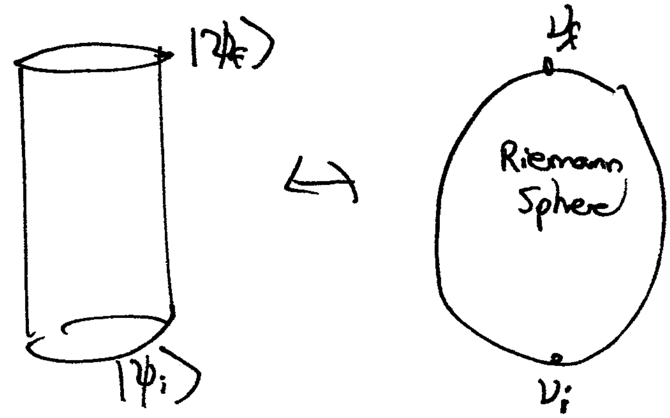
$$V_{\text{integrated}} = \frac{2g_c}{\alpha'} \int d^2z \, :\partial X^\mu \bar{\partial} X^\nu e^{ikX}:$$

Inserting a final state is similar



(positions illustrative only)

but really we want



We will soon see that the $e^{-\lambda\chi}$ factor suppresses complicated topologies. The dominant contributions in string perturbation theory come from

sphere	$g=0, b=0$	$\chi=2$	$b=$ # boundaries
disk	$g=0, b=1$	$\chi=1$	
annulus	$g=0, b=2$	$\chi=0$	
torus	$g=1, b=0$	$\chi=0$	

$$(\chi = 2-2g-b)$$

and their unoriented friends (more later).

So to start off, we'd really need to know how to compute

$$\langle V_{j_1} \cdots V_{j_n} \rangle_{\text{sphere}}$$

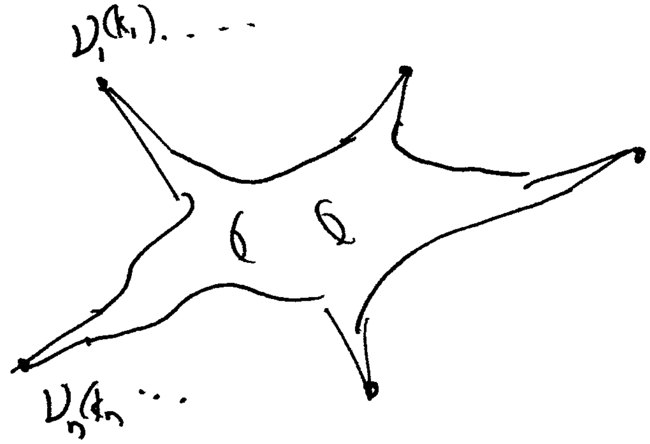
and in particular, compute $V_{\text{diff}\times\text{Weyl}}$.

We conclude that

the leading-order propagation amplitude between given initial + final states is the correlation function, on the sphere, of V_i, V_f.

$$\Leftrightarrow \int \frac{[DX\,Dg]}{V_{\text{diff}\times\text{Weyl}}} e^{-S_p - \lambda\chi_{\text{sphere}}} \int d^2\sigma_i \sqrt{g}\, V_i \int d^2\sigma_f \sqrt{g}\, V_f$$

More generally,



is computed by

$$\sum_{\substack{\text{compact}\\ \text{suitable}\\ \text{topologies}}} \int \frac{[DX\,Dg]}{V_{\text{diff}\times\text{Weyl}}} e^{-S_p - \lambda\chi} \prod_{i=1}^{n} \int d^2\sigma_i \sqrt{g}\, V_{j_i}(k_i;\sigma_i)$$

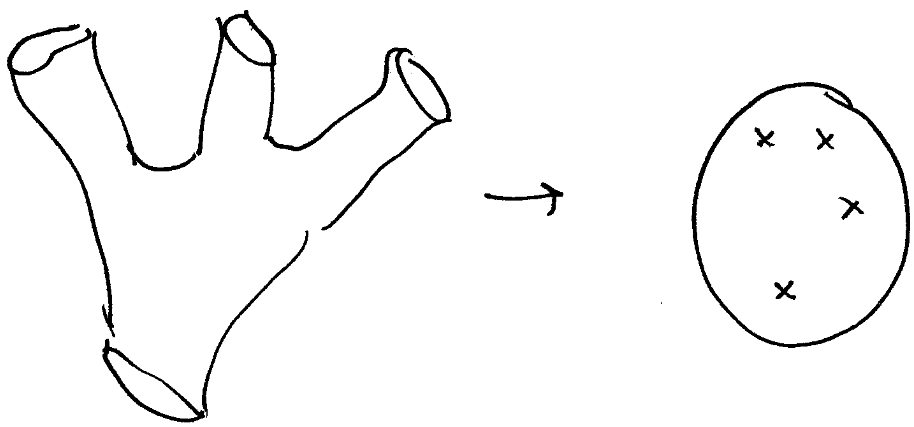
$$\equiv S_{j_1 \dots j_n}(k_1, \dots k_n).$$

This is the Polyakov path integral prescription.

or,

$$S_{j_1\cdots j_n}(k_1,\ldots k_n) = \sum_{\substack{\text{compact}\\ \text{suitable}\\ \text{topologies}}} \int \frac{[DX\, Dg]}{V_{\text{diff}\times\text{Weyl}}} e^{-\frac{1}{2\pi\alpha'}\int d^2z\, \partial X \bar\partial X - \lambda\chi} \prod_{i=1}^{n} d^2z_i\, V_{j_i}(k_i, z_i)$$

Now, conformal invariance tells us that



So $$S_{j_1\cdots j_n}(k_1 \ldots k_n) = \sum_{\substack{\text{suitable}\\ \text{cpt}\\ \text{topologies}\\ \Sigma_\alpha}} e^{-\lambda\chi} \left\langle V_{j_1}(k_1,z_1)\cdots V_{j_n}(k_n,z_n)\right\rangle_{\Sigma_\alpha}$$

$$= e^{-\lambda\cdot 2}\left\langle \prod V\right\rangle_{S^2} + e^{-\lambda\cdot 0}\left\langle \prod V\right\rangle_{T^2} + \cdots$$

btw, $Z = \sum \int \frac{(Dx\, Dg)}{V_{\text{Diff}\times W}} e^{-S_p - \lambda\chi} = \langle 0 \rangle.$

We can work out $\int \frac{[Dx\, Dg]}{V_{\text{diff}\times\text{Weyl}}}$

using the Faddeev-Popov method.

(for computing $[Dx\, Dg] = D\left\{\begin{matrix}\text{gauge-}\\\text{inequiv}\end{matrix}\right\} \cdot \underbrace{D\left\{\begin{matrix}\text{gauge}\\\text{equivalent}\end{matrix}\right\}}_{=V_{\text{diff}\times\text{Weyl}}}$

We won't do so here.

Key issue: Weyl anomaly

Classical theory is diff, Weyl, Poincaré invariant

Are these symmetries preserved in the quantum theory?

Suppose we regulate a UV divergence with a Pauli-Villars field Y^μ,

$$\Delta S = \mu^2 \int d^2\sigma \sqrt{\gamma}\, Y_\mu Y^\mu$$

Poincaré ✓

diff ✓

Weyl violated!

Does there exist a Weyl-invariant regulator?

We'd ~~better~~ find one, otherwise $Z[g]$ will depend on choice of metric and unitarity or covariance may fail to ~~per~~ exist.

We'll now sketch this issue; cf. Polchinski chapter 3 for a correct treatment.

$$\langle \cdots \rangle = \int \frac{[Dx\, Dg]}{V_{\text{diff}\times\text{Weyl}}} e^{-S[X,g]}$$

$$\langle \cdots \rangle_{\hat{g}} \equiv \int [Dx]\, e^{-S[X,\hat{g}]}$$

actually, will have $Db\, Dc$ and $S \to S[b,c$ in correct ~~treatment~~ with ghosts.

we want

$$\langle \cdots \rangle_{\hat{g}} \overset{!}{=} \langle \cdots \rangle_{\tilde{g}^{\mathcal{N}}} \quad \text{for } \tilde{g}^{\mathcal{N}} = \text{any Weyl transform of } \hat{g}.$$

i.e. $\tilde{g}^{\mathcal{N}}_{\alpha\beta} = e^{2\omega} \hat{g}_{\alpha\beta}$.

Well, $\delta \langle \cdots \rangle_{\hat{g}}$

$$= \int [DX] \left(- \frac{\delta S}{\delta g_{\alpha\beta}} \delta g_{\alpha\beta} \right) (\cdots) \quad + \delta(\cdots)$$

we'll neglect: this must itself be Weyl-inv't it turns out!

but $\frac{\delta S}{\delta g_{\alpha\beta}(\sigma)} = \frac{\sqrt{g}}{4\pi} T^{\alpha\beta}(\sigma)$

So, $\delta \langle \cdots \rangle_{\hat{g}}$

$$= \int d^2\sigma \sqrt{g} \frac{\delta}{\delta g_{\alpha\beta}(\sigma)} \langle \cdots \rangle_{\hat{g}}$$

$$= -\frac{1}{4\pi} \int d^2\sigma \; \delta g_{\alpha\beta}(\sigma) \langle T^{\alpha\beta}(\sigma) \cdots \rangle_{\hat{g}}$$

We'll take $\delta g_{\alpha\beta} = 0$ at locations of $\cdots$

so, we require $\delta g_{\alpha\beta} \langle T^{\alpha\beta}(\sigma) \rangle_{\hat{g}} = 0$

For a Weyl transformation,

$$\tilde{g}_{\alpha\beta} = e^{2\omega}\hat{g}_{\alpha\beta}$$

$$\delta g_{\alpha\beta} = \underbrace{(e^{2\omega}-1)}_{\tilde{\omega}}\hat{g}_{\alpha\beta}$$

so $\quad \langle \hat{g}_{\alpha\beta} T^{\alpha\beta} \cdots \rangle_{\tilde{g}} = 0$

or

$$\langle T^{\alpha}{}_{\alpha} \cdots \rangle_{\tilde{g}} = 0$$

So $T^{\alpha}{}_{\alpha} = 0$ must hold as an operator eqn.

We'll now check this.

$T^\alpha{}_\alpha = 0$ on flat worldsheet (unit gauge)

So, $T^\alpha{}_\alpha = a_1 R$ on dimensional grounds.

One can show that $a_1 = \frac{26-D}{12}$

$\Rightarrow$ Weyl invariance $\Leftrightarrow$ $D = 26$.

Now let's consider more general backgrounds for string propagation.