The nonlinear σ-model for strings

We begin with (Euclidean signature)

$$S_p = \frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{g}\, g^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

To generalize replace $\eta_{\mu\nu}$ with $G_{\mu\nu}(X)$:

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{g}\, g^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X).$$

We've seen that if $G_{\mu\nu} = \eta_{\mu\nu} + \chi_{\mu\nu}(X)$

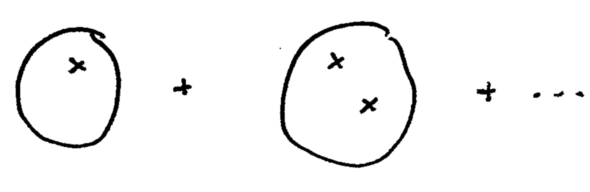
then

$$e^{-S} = e^{-S_p}\left(1 - \frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{g}\, g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \chi_{\mu\nu} + \cdots\right)$$

and since $\frac{g_c}{\alpha'}\int d^2\sigma \sqrt{g}\, g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu s_{\mu\nu} e^{ikX}$

$\Leftrightarrow V_{\text{graviton}}(s_{\mu\nu}, k)$

$$\langle \cdots \rangle_G = \langle \cdots \rangle_{G=\eta} + \sum_{j=1}^{\infty} \left\langle \cdots \prod_{i=1}^{j} V_i(s^i_{\mu\nu}, k_i) \right\rangle_{G=\eta}$$

$=$ 

More generally, may study strings in any background of the massless fields

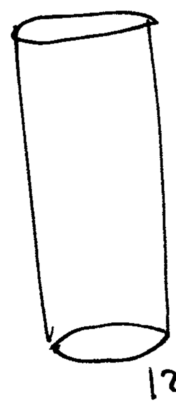
$$G_{\mu\nu}(X),\ B_{\mu\nu}(X),\ \Phi(X)$$

$$S_\sigma = \frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{g}\left\{\left[g^{\alpha\beta}G_{\mu\nu}(X) + i\epsilon^{\alpha\beta}B_{\mu\nu}(X)\right]\partial_\alpha X^\mu \partial_\beta X^\nu + \alpha' R^{(2)}\Phi(X)\right\}$$

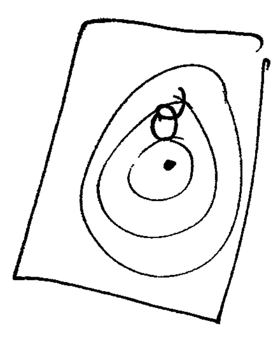
corresponding to coherent states of the massless fields, 'generated' by exponentiating the (massless) vertex operators

$$\left\{\begin{array}{l} g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu s_{\mu\nu}\, e^{ikX} \\ i\epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu a_{[\mu\nu]}\, e^{ikX} \\ \alpha'\phi R^{(2)} e^{ikX} \end{array}\right\}.$$

$\Leftrightarrow$ insertions of the states $\quad \alpha_{-1}^\mu \tilde{\alpha}_{-1}^\nu \left[s_{\mu\nu} + i\, a_{\mu\nu} + \alpha'\phi\,\delta_{\mu\nu}\right]|0,0;k\rangle$



$\Leftrightarrow$



Sigma Model Expansion

When $G_{\mu\nu}(X)$, $B_{\mu\nu}(X)$, $\Phi(X)$ are nontrivial, we can expand,

$$X^\mu = x_0^\mu + Y^\mu(\sigma)$$

$$S_\sigma = \frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{g}\Big\{ g^{\alpha\beta}\partial_\alpha Y^\mu \partial_\beta Y^\nu \Big[G_{\mu\nu}(x_0) + G_{\mu\nu,\omega}(x_0) Y^\omega + \frac{1}{2} G_{\mu\nu,\omega\rho} Y^\omega Y^\rho + \dots\Big]$$

$$+ i\epsilon^{\alpha\beta}\partial_\alpha Y^\mu \partial_\beta Y^\nu \Big[B_{\mu\nu}(x_0) + B_{\mu\nu,\omega}(x_0) Y^\omega + \frac{1}{2} B_{\mu\nu,\omega\rho} Y^\omega Y^\rho + \dots\Big]$$

$$+ \alpha' R^{(2)} \Big[\Phi(x_0) + \Phi_{,\omega}(x_0) Y^\omega + \frac{1}{2}\Phi_{,\omega\rho}(x_0) Y^\omega Y^\rho + \dots\Big]\Big\}$$

This is an interacting QFT in 2D with interaction strengths (couplings) set by $\left\{ \begin{matrix} G_{,\omega\rho\dots} \\ B_{,\omega\rho\dots} \\ \Phi_{,\omega\rho\dots} \end{matrix} \right.$

We can make dimensionless coords $\phi^\mu = \frac{Y^\mu}{\sqrt{\alpha'}}$

so that the derivative expansion

is an expansion in powers of $\left(\frac{\alpha'}{r^2}\right)$

with r the radius of curvature of the target space.

This is the "σ-model expansion".

When is string propagation in a background

$$\{ G_{\mu\nu}(X), B_{\mu\nu}(X), \Phi(X) \}$$

consistent?

Well, we must require that the Weyl anomaly vanishes,

$$\langle T^\alpha{}_\alpha \rangle = 0.$$

In particular, require rigid scale invariance,

$$\beta_\lambda = 0 \quad \forall \text{ couplings } \lambda.$$

Now, in our 2D QFT we may compute $\langle T^\tau{}_\tau \rangle$ to any desired order in $\left(\frac{\alpha'}{r_c^2}\right)$.

We've already argued that at $\mathcal{O}\left(\left[\frac{\alpha'}{r_c^2}\right]^0\right)$, i.e. even in ~~infinite~~ flat space,

$$T^\alpha{}_\alpha = -\frac{1}{12}(D-26)\, R^{(2)}$$

But these are corrections at $\mathcal{O}\left(\left(\frac{\alpha'}{r^2}\right)^{1,2\ldots}\right)$.

Best to think of these as <u>loop corrections</u> (in σ-model) to the β-fns

$$\beta_\lambda(M_0) = M \frac{\partial}{\partial M} \lambda(M)\Big|_{M_0}$$

What are the couplings? $G_{\mu\nu}(x)$
$B_{\mu\nu}(x)$
$\Phi(x)$

functions! (or, ∞ many coupling 'constants'.)

$\Rightarrow$ β-functionals

$$\begin{cases} \beta_G(x) = 0 \\ \beta_B(x) = 0 \\ \beta_\Phi(x) = 0 \end{cases}$$

must vanish.

After hard work one finds

$$T^{\alpha}{}_{\alpha} = -\frac{1}{2\alpha'}\beta^{G}_{\mu\nu}\, g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu$$
$$-\frac{i}{2\alpha'}\beta^{B}_{\mu\nu}\,\epsilon^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu$$
$$-\frac{1}{2}\beta^{\Phi} R^{(2)}$$

with

$$\beta^{G}_{\mu\nu} = \alpha' R_{\mu\nu} + 2\alpha' \nabla_\mu \nabla_\nu \Phi - \frac{\alpha'}{4} H_{\mu\lambda\omega} H_\nu{}^{\lambda\omega} + \mathcal{O}(\alpha'^2)$$

$$\beta^{B}_{\mu\nu} = -\frac{\alpha'}{2}\nabla^\omega H_{\omega\mu\nu} + \alpha' \nabla^\omega \Phi\, H_{\omega\mu\nu} + \mathcal{O}(\alpha'^2)$$

$$\beta^{\Phi} = \frac{D-26}{6} - \frac{\alpha'}{2}\nabla^2\Phi + \alpha' \nabla_\omega \Phi \nabla^\omega \Phi$$
$$- \frac{\alpha'}{24} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + \mathcal{O}(\alpha'^2).$$

These are <u>differential</u> eqns for $G_{\mu\nu}(X)$, $B_{\mu\nu}(X)$, $\Phi(X)$!

The eom (for spacetime fields)

$$\beta^{G}_{\mu\nu} = \beta^{B}_{\mu\nu} = \beta_{\Phi} = 0 \qquad \text{(note: setting } D=26)$$

follow from extremizing

$$S = \frac{1}{2\kappa_0^2}\int d^{26}x\sqrt{G}\, e^{-2\Phi}\Big\{R - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + 4\partial_\mu\Phi\partial^\mu\Phi + \mathcal{O}(\alpha')\Big\}$$

where $\kappa_0 = $ const (not yet determined)

and

$$H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu}$$

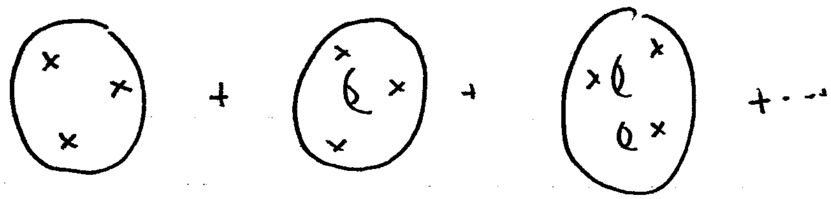
$(H = dB$ as forms$)$

Moreover, at higher order in α', one gets

$$\beta^{G}_{\mu\nu} = \alpha' R_{\mu\nu} + \underbrace{\frac{\alpha'^2}{2} R_{\mu\kappa\lambda\tau} R_{\nu}{}^{\kappa\lambda\tau}}_{\text{string correction to GR.}} + \mathcal{O}(\alpha'^3)$$

Different way to get low-energy EFT for massless modes:

compute S-matrix elements of massless states



$\times$: V_{massless} (graviton, $B_{\mu\nu}$, Φ).

$$\left(\int DX \, e^{-S_p - \lambda \chi} V_1 \cdots V_n \right)$$

and guess a QFT + gravity Lagrangian, in spacetime, that gives the same scattering amplitudes.

Expansion: string loop (genus) expansion.

Now since $S \to S_p + \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} \, \alpha' R^{(2)} \Phi(X)$

in a nontrivi. Φ bg,

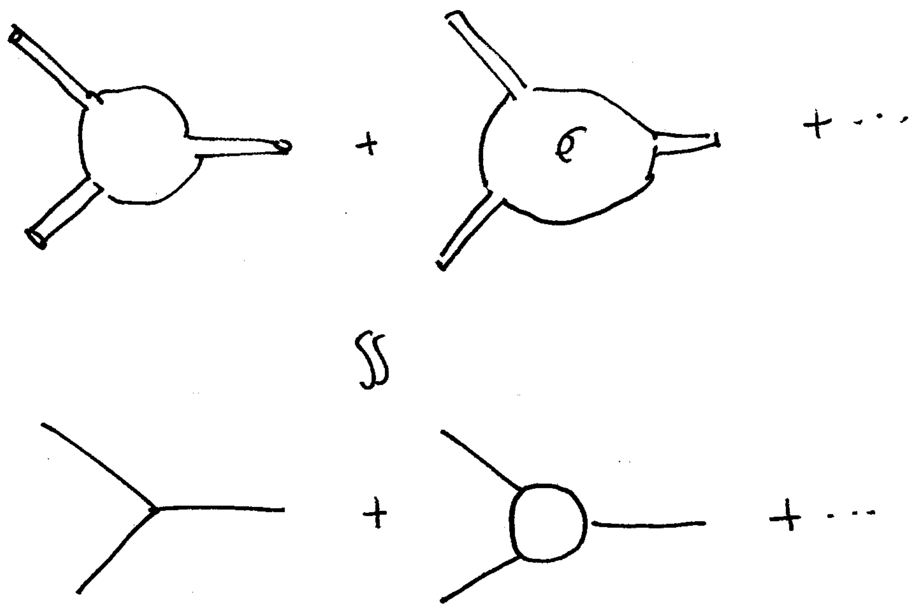
we have that $\lambda = \langle \Phi \rangle$ (if $\frac{\partial}{\partial X} \langle \Phi \rangle$ is small.)

We'll define $g_s = e^{-\Phi}$

so that $(\chi = 2 - 2g)$

$$\sum_{g=0}^{\infty} A_g e^{-\lambda \chi} = \sum_{g=0}^{\infty} A_g e^{-2\Phi + 2g\Phi} = \frac{1}{g_s^2} \sum_{g=0}^{\infty} g_s^{2g} A_g$$

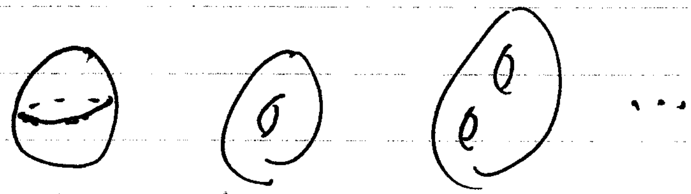
So the g_s expansion (string loop expansion) is the spacetime loop expansion



<u>NB</u> $\hbar \sim g_s$. since spacetime quantum effects require spacetime loops $\Leftrightarrow$ string loops.

The effective action that 'reproduces' the string S-Matrix is precisely the effective action that gives the β-fn eqns.

Often-confusing point: there are two expansions in string theory!

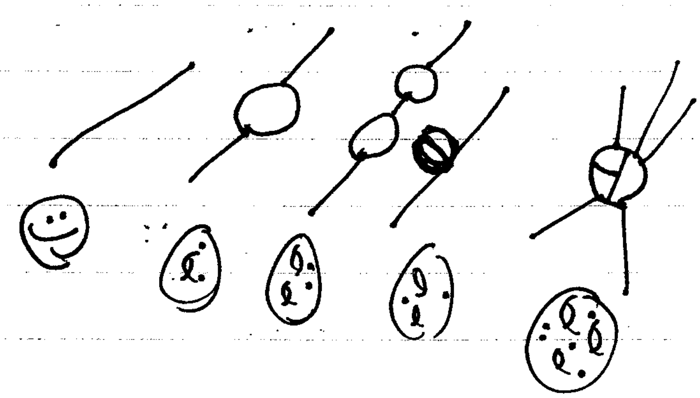


genus expansion

counted by g_c (via χ)

"string loop expansion"

corresponds to loop expansion of SPACETIME QFT (YM, GR quantized, ...)



 fixed-genus, 2D QFT

expanded in interactions coming from $G_{\mu\nu}$, ω

(expansion around flat space)

sigma model expansion ("α' expansion")
counted by $\frac{\alpha'}{r^2}$ (via $G_{\mu\nu,\omega\rho}\cdots$

loops in 2D QFT (not necessarily a CFT in ~~all~~ all cases, though usually we will find this is necessary)

So if I want to use string theory to study e.g. four-graviton scattering

I'll have to quote a result

$$
\begin{aligned}
A = \; & A_0^0 + A_1^0 g_s + A_2^0 g_s^2 + \cdots \\
& + \qquad\quad + \qquad\quad + \\
& A_0^1\left(\frac{\alpha'}{r^2}\right) + A_1^1\left(\frac{\alpha'}{r^2}\right)g_s + A_2^1 g_s^2\left(\frac{\alpha'}{r^2}\right) + \cdots \\
& + \qquad\quad + \qquad\quad + \\
& \vdots \qquad\quad \vdots \qquad\quad \vdots
\end{aligned}
$$

sphere   torus   two-hole-torus

brute force never gets farther than this.
Clever use of symmetries + nonrenormalization thms. does get results ~~to~~ to all orders in g_s and/or to all orders in α'.

Remark:

The true quantum gravity effects are g_s effects
(since quantum gravity in spacetime requires $\hbar \neq 0$)

But string theory provides other high-scale effects,
from 'stringy' α' effects.

e.g. GR $\mathcal{L} = \sqrt{g} R$

string tree level (g_s^{-2}, S^2) $\mathcal{L} = \sqrt{g}\left(R + \alpha'^3 \zeta(3)\left(R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} + \cdots\right)\right)$

in IIB.

coeff gets g_s corrections from

$$\langle h \cdots h \rangle_{S^2} + \langle h \cdots h \rangle_{T^2} + \cdots$$

The vertex operator for $B_{\mu\nu}$ is

$$\frac{g_c}{\alpha'}\int d^2\sigma \sqrt{g}\, \epsilon^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu a_{\mu\nu}\, e^{ik\cdot X}$$

for $k=0$, we have

$$\frac{g_c}{\alpha'}\int d^2\sigma \sqrt{g}\, \partial_\alpha\left(\epsilon^{\alpha\beta}\partial_\beta X^\mu X^\mu\right) a_{\mu\nu}$$

which is a total derivative on the WS.

$\Rightarrow$ if $\partial(WS)=0$, this coupling vanishes.

$\Rightarrow$ for WS w/o ∂,

the zero-momentum $B_{\mu\nu}$ coupling vanishes to all orders in α'
and in g_s (since genus did not matter).

$\Rightarrow$ the corresponding spacetime action must have no non-derivative couplings of $B_{\mu\nu}$ to any order in g_s, α'!

How can this fail?

1) if WS has ∂, total derivative can contribute

2) $$\int_\Sigma d^2\sigma \, \cancel{\sqrt{h}}^{\text{unit gauge}} \epsilon^{\alpha\beta} B_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu$$

$$= \int_{\Sigma_2} B_2$$

in σ-model pert. thy, Σ_2 is contractible.

But can consider topologically nontrivial Σ_2,

maps WS $\to$ target

where Σ_2 wraps nontrivial 2-cycle in target.

This is an instanton in the σ-model

"worldsheet instanton"

with $$S_{inst} = \frac{1}{2\pi\alpha'}\left(\int \sqrt{g} + i\int B\right)$$

contributions $\sim e^{-S_{inst}}$

So, the spacetime action can depend on

$$H = dB \quad \text{at} \quad \mathcal{O}((\alpha')^k (g_s)^j)$$

but on B itself only as $e^{-\frac{in}{2\pi\alpha'}\int B + \frac{n}{2\pi\alpha'}\int\sqrt{g}}$

$$= \exp\left(-n\left[\int\frac{\sqrt{g}}{2\pi\alpha'} + \frac{i}{2\pi\alpha'}\int B\right]\right)$$

$$= e^{-\frac{n}{2\pi\alpha'}\left[\mathrm{Vol}\,\Sigma_2 + i\int_{\Sigma_2} B\right]}$$

$$\text{“} \quad e^{-n[t+ib]}$$

Schematically,

$$\mathcal{L}_{pert} \supset c_2 H_{\mu\nu\rho}H^{\mu\nu\rho} + c_4\alpha' H_{\mu\nu\rho}H^{\mu\nu\rho}H_{\sigma\tau\lambda}H^{\sigma\tau\lambda} + \cdots$$

$$\text{with} \quad c_2 = \sum_{j=0}^{\infty} c_2^{(j)} g_s^j \quad \text{etc.}$$

plus,

$$\mathcal{L}_{NP} \supset \sum_{\Sigma_2}\sum_j d_j\, e^{-j\left[\frac{\mathrm{Vol}\,\Sigma_2}{2\pi\alpha'} + i\int_{\Sigma_2} B\right]}.$$

Superstrings

As you have realized, our bosonic string theory

$$S = \frac{1}{2\pi\alpha'}\int d^2z\, \partial X^\mu \bar{\partial} X_\mu = -\frac{1}{4\pi\alpha'}\int d^2\sigma\, \partial_\alpha X^\mu \partial^\alpha X_\mu$$

is not suitable to describe reality. Most painful is its lack of fermions.

We can remedy this. Consider

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma \left(\partial_\alpha X^\mu \partial^\alpha X_\mu + \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu \right)$$

where for any $\mu \in \{0, .. D-1\}$, ψ_μ is a fermion on the worldsheet,

and ρ^α $\alpha = 0,1$ are 2D Dirac matrices.

(and $\bar{\psi} \equiv \psi^\dagger i \rho^0$)

Namely, ρ^α obey the Dirac algebra

$$\{\rho^\alpha, \rho^\beta\} = 2\eta^{\alpha\beta} \qquad \alpha,\beta \in \{0,1\}$$

Choose

$$\rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad \text{purely real.}$$

$\Rightarrow$ by def, a Majorana repn.

So ψ^μ has 2 components $\psi^\mu \equiv \begin{pmatrix} \psi^\mu_- \\ \psi^\mu_+ \end{pmatrix}$

Note that $\Gamma \equiv \rho^0\rho^1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

has the properties $\Gamma^2 = \mathbb{1}$ (just like γ_5 in 4d)

$$\{\Gamma, \rho^\alpha\} = 0$$

$\Rightarrow$ Γ-eigenstates $\begin{matrix} + \\ - \end{matrix}$ do not mix.

Hence notation $\begin{pmatrix} \psi^\mu_- \\ \psi^\mu_+ \end{pmatrix}$.

So, the <u>one real</u> state $\begin{pmatrix} \psi_-^\mu \\ 0 \end{pmatrix}$ for fixed μ

furnishes a representation of the Dirac algebra, with eigenvalue $-$ under Γ

and $\begin{pmatrix} 0 \\ \psi_+^\mu \end{pmatrix}$ for fixed μ

is a representation of the Dirac algebra with eigenvalue $+$ under Γ.

We call the Γ-eigenvalue "chirality"

and name these fixed-chirality fields "Weyl spinors".

2-$\mathbb{R}$-dim Dirac spinor (reducible rep) decomposes into 2 1-$\mathbb{R}$-dim Weyl spinors (irreps).

Now since ρ^α are <u>real</u>, everything can be called "Majorana"!

Upshot: ψ^μ_+ is a D-vector of positive-chirality Majorana-Weyl spinors (irreps of 2D Dirac algebra).

Dirac eqn:

$$\rho^\alpha \partial_\alpha \psi = 0$$

$$\begin{pmatrix} 0 & -\partial_0 + \partial_1 \\ \partial_0 + \partial_1 & 0 \end{pmatrix} \begin{pmatrix} \psi^\mu_- \\ \psi^\mu_+ \end{pmatrix} = 0$$

$$\tfrac{1}{2}(\partial_0 \pm \partial_1) \equiv \partial_\pm$$

$$\Rightarrow \begin{cases} \partial_- \psi_+ = 0 \\ \partial_+ \psi_- = 0 \end{cases}$$

indeed, can write

$$S = \frac{i}{2\pi} \int d^2\sigma \left(\psi_- \partial_+ \psi_- + \psi_+ \partial_- \psi_+ \right)$$

and see eom this way.

This action is SUSY:

take $\varepsilon =$ Majorana spinor, $\varepsilon = \begin{pmatrix} \varepsilon_- \\ \varepsilon_+ \end{pmatrix}$

then
$$\begin{Bmatrix} \delta X^\mu = \bar{\varepsilon}\psi^\mu \\ \delta\psi^\mu = \rho^\alpha \partial_\alpha X^\mu \varepsilon \end{Bmatrix}$$
is an invariance of the action (up to a total derivative)

In fact, this action enjoys superconformal symmetry as we'll see soon.

Before solving the eom, let's think about these fermions from a CFT p.o.v.

Think back to bc CFT

$$S = \frac{1}{2\pi}\int d^2z\, b\bar{\partial}c$$

		h	$\tilde{h}$
weights	b	λ	0
	c	$1-\lambda$	0

$c = -3(2\lambda-1)^2+1$

for $\lambda = \frac{1}{2}$, $c=1$.

$h(b) = h(c) = \frac{1}{2}$.

$b \equiv \psi$

$c \equiv \bar{\psi}$

if we define

$$\psi = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2)$$

$$\bar{\psi} = \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2)$$

we have

$$S = \frac{1}{4\pi}\int d^2z\left(\psi_1\bar{\partial}\psi_1 + \psi_2\bar{\partial}\psi_2\right)$$

two $c=\frac{1}{2}$ fermionic theories.

So may introduce just ψ_1, say:

$$S = \frac{1}{4\pi}\int d^2z \left(\psi_1^\mu \bar\partial \psi_{1\mu} \right) \qquad \mu = 0, \ldots D-1$$

should add c.c.

$$S = \frac{1}{4\pi}\int d^2z \left(\psi_1^\mu \bar\partial \psi_{1\mu} + \tilde\psi_1^\mu \partial \tilde\psi_{1\mu} \right)$$

$$\equiv \frac{1}{4\pi}\int d^2z \left(\underbrace{\psi^\mu \bar\partial \psi_\mu}_{c = \frac{D}{2}} + \underbrace{\tilde\psi^\mu \partial \tilde\psi_\mu}_{\tilde c = \frac{D}{2}} \right)$$

The OPEs are

$$\psi^\mu(z)\,\psi^\nu(0) \sim \frac{\eta^{\mu\nu}}{z}$$

$$\tilde\psi^\mu(\bar z)\,\tilde\psi^\nu(0) \sim \frac{\eta^{\mu\nu}}{\bar z}$$

$$\psi^\mu(z)\,\tilde\psi^\nu(0) \sim \text{reg.}$$

(and $X^\mu(z) X^\nu(0) \sim -\eta^{\mu\nu} \ln|z|^2$)

One can derive the stress-energy tensor

$$T_B(z) = -\frac{1}{\alpha'}\partial X^\mu \partial X_\mu - \frac{1}{2}\psi^\mu \partial \psi_\mu$$

$$\left[\tilde{T}_B(\bar{z}) = -\frac{1}{\alpha'}\bar{\partial} X^\mu \bar{\partial} X_\mu - \frac{1}{2}\tilde{\psi}^\mu \bar{\partial} \tilde{\psi}_\mu\right]$$

NB. $::$ implicit forevermore.

But we can find more conserved (i.e, holomorphic) currents:

[... remember radial quantization.]

define $T_F(z) = i\sqrt{\frac{2}{\alpha'}}\,\psi^\mu(z)\partial X_\mu(z)$

$$\tilde{T}_F(\bar{z}) = i\sqrt{\frac{2}{\alpha'}}\,\tilde{\psi}^\mu(\bar{z})\bar{\partial} X_\mu(\bar{z})$$

Now define $j^\eta(z) \equiv \eta(z)\, T_F(z)$

$$\left(\quad \tilde{j}^\eta(\bar{z}) \equiv \eta^*(\bar{z})\, \tilde{T}_F(\bar{z}) \quad\right) \qquad [\eta^*(z)]$$

and $\bar{\partial} j^\eta = 0 \Rightarrow$ still conserved.

Find resulting symmetry as usual.

$$\frac{1}{i\epsilon}\delta_\eta X^\mu(0) = [Q_\eta, X^\mu(0)] = \operatorname*{Res}_{z\to 0} j^\eta(z) X^\mu(0)$$

$$T_F(z) X^\mu(0) = i\sqrt{\frac{2}{\alpha'}}\,\eta(z)\psi^\nu(z)\underbrace{\partial X_\nu(z) X^\mu(0)}_{-\frac{\alpha'}{2}\delta_\nu{}^\mu \frac{1}{z}}$$

$$= -i\sqrt{\frac{2}{\alpha'}}\,\eta(z)\psi^\mu(z)$$

$$\Rightarrow \delta X^\mu \sim \epsilon\sqrt{\frac{\alpha'}{2}}\left[\eta(z)\psi^\mu(z) + \eta^*(\bar z)\tilde\psi^\mu(\bar z)\right]$$

Similarly

$$\delta\psi^\mu \sim -\epsilon\sqrt{\frac{2}{\alpha'}}\left[\eta(z)\partial X^\mu(z)\right]$$

$$\delta\tilde\psi^\mu \sim -\epsilon\sqrt{\frac{2}{\alpha'}}\left[\eta^*(\bar z)\bar\partial X^\mu(\bar z)\right]$$

Now X commutes + ψ does not

$\Rightarrow$ necessarily η is an anticommuting parameter.

Schematically, $\delta X^\mu(z,\bar{z}) \sim + \epsilon\,\eta(z)\,\psi^\mu(z)$

$$\delta\psi^\mu(z) \sim -\epsilon\,\eta(z)\,\partial X^\mu(z)$$

This superconformal transformation, together with the familiar conformal transformation, generates the superconformal algebra.

Easy exercise:

$$\begin{cases} T_B(z)\,T_B(0) \sim \dfrac{3D}{4z^4} + \dfrac{2}{z^2}T_B(0) + \dfrac{1}{z}\partial T_B(0) \\ T_B(z)\,T_F(0) \sim \dfrac{3}{2z^2}T_F(0) + \dfrac{1}{z}\partial T_F(0) \\ T_F(z)\,T_F(0) \sim \dfrac{D}{z^3} + \dfrac{2}{z}T_B(0). \end{cases}$$

Similarly for $\tilde{T}$ sector.

Observe: • $T_F(z)$ is a tensor of weight $\left(\frac{3}{2}, 0\right)$.
$\tilde{T}_F(\bar{z})$ " " $\left(0, \frac{3}{2}\right)$.

• $c = \frac{3}{2}D$. $\left(T(z)T(0) \sim \frac{c}{2}\frac{1}{z^4} \right.$
$\tilde{c} = \frac{3}{2}D$. by <u>def</u> of c $\Big)$

Namely, scalars give 1 each $\times D$ X^μ $= +D$
fermions $\frac{1}{2}$ each $\times D$ ψ^μ $= +\frac{D}{2}$
$c = \frac{3}{2}D$.

<u>Def.</u> with $\psi, \tilde{\psi}$ present, hence $T_F(z)$ and $\tilde{T}_F(\bar{z})$, we call this the

$$(N, \tilde{N}) = (1,1) \text{ SCFT.}$$

"(1,1) worldsheet SCFT"

[hol + antihol sectors do <u>not</u> talk to each other, as usual, so makes sense to have $N \neq \tilde{N}$. This is the case in the <u>heterotic</u> string.]

When we gauge-fixed the bosonic string, we saw:

$$\int [DX][Dg] \exp\left(-\frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{g}\, g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu - \lambda\chi\right)$$

$$\rightarrow \int [DX][Db][Dc] \exp\left(-\frac{1}{4\pi\alpha'}\int d^2\sigma\, \eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu - \lambda\chi - \frac{1}{2\pi}\int d^2\sigma\, b_{\alpha\beta}\nabla^\alpha c^\beta\right)$$

namely, in complex coords,

$$S \rightarrow \frac{1}{2\pi\alpha'}\int d^2z\, \partial X \bar{\partial} X + \frac{1}{2\pi}\int d^2z\, b\bar{\partial}c + \tilde{b}\partial\tilde{c}$$

with the (b,c) CFT the ghosts of the fixed symmetry.

Now the (now-fixed) diff x Weyl symmetry is what protects us from unphysical wrong-sign oscillators,

$$[\alpha^\mu_m, \alpha^\nu_n] = m\,\delta_{m+n}\,\eta^{\mu\nu}$$

With ghosts present, we may (correctly) think that the ghosts run around to cancel the effects of the timelike oscillators.

We have a new problem: and must seek a solution.

$$\{\psi^\mu_m, \psi^\nu_n\} = \eta^{\mu\nu}\delta_{m+n}$$

The guiding structure of the course so far has been conformal symmetry (eg., Virasoro algebra)

This symmetry algebra is generated by holomorphic currents

$$j(z) = v(z)\, T(z)$$

namely, by $T(z)$.

In brief: $T(z)$ generates the conformal symmetry that is the core structure of bosonic string theory.

We have seen that

$$S = \frac{1}{2\pi\alpha'}\int d^2z \left(\partial X^\mu \bar\partial X_\mu\right) + \frac{1}{4\pi}\int d^2z \left(\psi^\mu \bar\partial \psi_\mu + \tilde\psi^\mu \partial \tilde\psi_\mu\right)$$

enjoys superconformal symmetry, generated by

$$T_B,\ T_F.$$

Can we use SC symmetry to eliminate unwanted ψ^0 oscillators?

Polyakov action has diff + Weyl symmetry
⇒ conformal invariance
⇒ can choose LCG and have manifestly positive spectrum,
or can do Polyakov path-integral. + let ghosts ensure positivity.

Local (diff + Weyl) symmetry ⇒ constraint algebra (Virasoro algebra)
[$g_{\alpha\beta}$ eom ⇔ $T_{\alpha\beta}=0$ ⇔ conformal invariance]

In the superstring, the analogous procedure would be to formulate a theory with a local worldsheet supersymmetry (gauge invariance ⇔ SUSY)
and derive the resulting constraint algebra.
[Ψ^μ eom ⇔ $T_F=0$ ⇒ superconformal algebra as constraint algebra on states]

~~T~~his is rather advanced, and we will not get into it.

Instead, following Polchinski (+BBS), we will impose the superconformal algebra as a constraint algebra.

When one formulates a superconformally-inv't theory via Polyakov path integral,

just as

$$(2,0) \overset{\text{commuting}}{\text{constraint}} \; T_B(z) \Rightarrow (2,0) \text{ ghost } b \text{ (anticommuting)}$$

we must also have

$$\left(\tfrac{3}{2},0\right) \overset{\text{anticommuting}}{\text{constraint}} \; T_F(z) \Rightarrow \left(\tfrac{3}{2},0\right) \text{ commuting ghost } \beta.$$

$\exists$ free SCFT of <u>ghosts</u>:

$$S_{BC} = \frac{1}{2\pi}\int d^2z \left(b\bar\partial c + \beta\bar\partial\gamma \right)$$

($\exists$ right-moving analogue, $\tilde b, \tilde c, \tilde\beta, \tilde\gamma$)

with

$$\beta(z_1)\gamma(z_2) \sim -\frac{1}{(z_1-z_2)}$$

$$\gamma(z_1)\beta(z_2) \sim \frac{1}{(z_1-z_2)}$$

$$\left.\begin{array}{l} b(z_1)c(z_2) \sim \dfrac{1}{z_1-z_2} \\ c(z_1)b(z_2) \sim \dfrac{1}{z_1-z_2} \end{array}\right\} \text{as usual}$$

others nonsingular

and with $h_b = \lambda$

$$h_c = 1-\lambda$$

$$h_\beta = \lambda - \tfrac{1}{2}$$

$$h_\gamma = \tfrac{3}{2} - \lambda$$

and $C = C_{bc} + C_{\beta\gamma}$

$$= \left[-3(2\lambda-1)^2+1\right] + \left[3(2\lambda-2)^2-1\right]$$

(easy to check...)

Now for us, $\lambda = 2$ $\left[(h_b, h_c) = (2,-1)\right]$

$$\Rightarrow C_{bc} = \left[-3(3)^2+1\right]$$

$$= -26$$

and $C_{\beta\gamma} = \left[3(2)^2-1\right] = 11$

$$\Rightarrow C_{ghost} = -26+11 = -15.$$

but we saw that $C_X = D$

$$C_\psi = \tfrac{D}{2}$$

$$\Rightarrow C_X + C_\psi + C_{bc} + C_{\beta\gamma} = 0$$

$$\Leftrightarrow \tfrac{3}{2}D - 15 = 0 \Leftrightarrow \boxed{D=10}.$$

Summary so far:

$$S_m = \frac{1}{2\pi\alpha'}\int d^2z\, \partial X^\mu \bar{\partial} X_\mu + \frac{1}{4\pi}\int d^2z \left(\psi^\mu \bar{\partial} \psi_\mu + \tilde{\psi}^\mu \partial \tilde{\psi}_\mu \right)$$

enjoys $(N,\tilde{N}) = (1,1)$ WS SC symmetry

and, $S_m + \frac{1}{2\pi}\int d^2z \left(b\bar{\partial}c + \beta\bar{\partial}\gamma \right) \qquad (\lambda = 2)$

corresponds ... I asserted ... to the gauge-fixed form of an action with $\underbrace{\text{local}}_{\text{(i.e. gauged)}}$ WS SUSY.

The new ghosts β, γ give $c = +11$

The new matter $(\psi, \tilde{\psi})$ gives $c = \frac{D}{2}$

The old ghosts gave $c = -26$

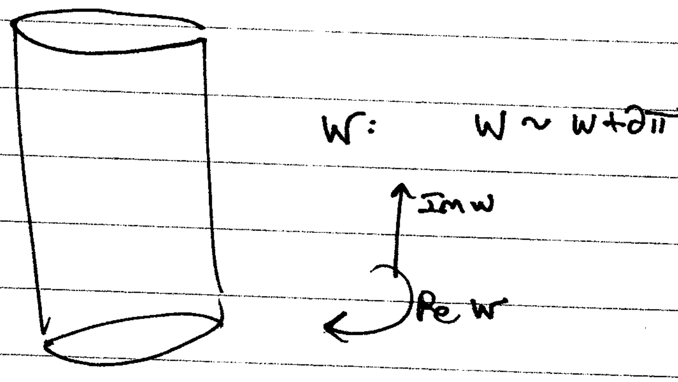
The old matter (X) gives $c = D$

$$\Rightarrow \boxed{D = 10} \Leftrightarrow c_{tot} = 0$$

Enough SCFT.

Boundary conditions

$$S_\psi = \frac{1}{4\pi}\int d^2w\left(\psi\bar\partial\psi + \tilde\psi\partial\tilde\psi\right) \qquad \left(\text{for some fixed } \mu\right)$$



for reference only.

varying action, get

$$\delta S_\psi = \frac{1}{4\pi}\int d^2w\left(\delta\psi\bar\partial\psi + \psi\bar\partial\delta\psi + \delta\tilde\psi\partial\tilde\psi + \tilde\psi\partial\delta\tilde\psi\right)$$

$$= \frac{1}{4\pi}\int d^2w\left[\delta\psi(\bar\partial\psi) + \delta\tilde\psi(\partial\tilde\psi)\right] + \frac{1}{4\pi}\int d\,\mathrm{Im}\,w\left(\psi\delta\psi + \tilde\psi\delta\tilde\psi\right)\Big|_{\mathrm{Re}\,w=0}^{\mathrm{Re}\,w=2\pi}$$

EOM $\quad \bar\partial\psi = 0 = \partial\tilde\psi$

b.c.: $\quad \psi\delta\psi + \tilde\psi\delta\tilde\psi\Big|_0^{2\pi} = 0$

now $\quad \psi(w) = \pm\psi(w+2\pi) \Rightarrow \psi\delta\psi$ term does not contribute

$\tilde\psi(w) = \pm\tilde\psi(w+2\pi) \Rightarrow \tilde\psi\partial\tilde\psi$ likewise.

Clearer: invariance of S_ψ under $w \to w + 2\pi$

$$\Rightarrow \quad \begin{aligned} \psi(w+2\pi) &= \pm\, \psi(w) \\ \tilde{\psi}(w+2\pi) &= \pm\, \tilde{\psi}(w) \end{aligned}$$ choices of $\pm$ distinct

We'll name these:

$$\psi^\mu(w+2\pi) = +\,\psi^\mu(w) \qquad \text{Ramond (R)}$$

$$\psi^\mu(w+2\pi) = -\,\psi^\mu(w) \qquad \text{Neveu-Schwarz (NS)}$$

sign same $\forall \mu$.

Choice for $\tilde{\psi}$ is independent.

so $\exists$ four sectors: RR, R NS, NS R, NS NS
$(++), (+-), (-+), (--)$.

Notation:
$$\left.\begin{aligned} \psi^\mu(w+2\pi) &= e^{2\pi i\nu}\, \psi^\mu(w) \\ \tilde{\psi}^\mu(\bar{w}+2\pi) &= e^{-2\pi i\tilde{\nu}}\, \tilde{\psi}^\mu(\bar{w}) \end{aligned}\right\} \quad \begin{aligned} \nu, \tilde{\nu} &= 0 \ (R) \\ \nu, \tilde{\nu} &= \tfrac{1}{2} \ (NS) \end{aligned}$$

Let's transform to z coords.

ψ has weight $h=\frac{1}{2}$ (check: $T_B = -\frac{1}{\alpha'}\partial X\partial X - \frac{1}{2}\psi\partial\psi$

$$\psi(z)\psi(w) \sim \frac{1}{z-w}$$

$$T_B(z)\psi(w) \sim -\frac{1}{2}\psi(z)\,\partial_z\left(\frac{1}{z-w}\right)$$

$$\sim \frac{1}{2}\frac{1}{(z-w)^2}\psi(z) \quad)$$

$$\Rightarrow \quad \psi'(w',\bar{w}') = (\partial_w w')^{-1/2}\,\psi(w,\bar{w})$$

$$w' = z = e^{-iw}$$

$$\Rightarrow \psi(z) = \left(-ie^{-iw}\right)^{-1/2}\psi(w)$$

$$= i^{1/2} z^{-1/2}\psi(w)$$

So an expansion in $\underline{w}$ ~~fourier~~ modes

$$\psi(w) = i^{-1/2}\sum_{r\in\mathbb{Z}+\nu}\psi_r e^{irw}$$

maps to

$$\psi(z) = \sum_{r\in\mathbb{Z}+\nu}\frac{\psi_r}{z^{r+1/2}}$$

NOTE: $\nu=0 \Leftrightarrow r\in\mathbb{Z} \Leftrightarrow$ R $\Leftrightarrow r+\frac{1}{2}\in\mathbb{Z}+\frac{1}{2}$

$\nu=\frac{1}{2} \Leftrightarrow r\in\mathbb{Z}+\frac{1}{2} \Leftrightarrow$ NS $\Leftrightarrow r+\frac{1}{2}\in\mathbb{Z}.$

In sum:

$$\psi^\mu(z) = \sum_{r \in \mathbb{Z}+\nu} \frac{\psi^\mu_r}{z^{r+\frac{1}{2}}} \qquad \tilde{\psi}^\mu(\bar{z}) = \sum_{r \in \mathbb{Z}+\tilde{\nu}} \frac{\tilde{\psi}^\mu_r}{\bar{z}^{r+1/2}}$$

$$\begin{array}{lll} (\nu, \tilde{\nu}) = & (0,0) & \text{RR} \\ & (0, \tfrac{1}{2}) & \text{RNS} \\ & (\tfrac{1}{2}, 0) & \text{NS R} \\ & (\tfrac{1}{2}, \tfrac{1}{2}) & \text{NS NS} \end{array}$$

and
$$\partial X^\mu(z) = -i\sqrt{\frac{\alpha'}{2}} \sum_{m=-\infty}^{\infty} \frac{\alpha^\mu_m}{z^{m+1}}$$
$$\bar{\partial} X^\mu(z) = -i\sqrt{\frac{\alpha'}{2}} \sum_{m=-\infty}^{\infty} \frac{\tilde{\alpha}^\mu_m}{\bar{z}^{m+1}}$$

Algebra:
$$\{\psi^\mu_r, \psi^\nu_s\} = \{\tilde{\psi}^\mu_r, \tilde{\psi}^\nu_s\} = \eta^{\mu\nu}\delta_{r+s}$$
$$[\alpha^\mu_m, \alpha^\nu_n] = [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m\,\eta^{\mu\nu}\delta_{m+n}$$

Now the spectrum,

NS sector. May take $|0\rangle_{NS}$ to obey

$$\psi^{\mu}_{r}|0\rangle_{NS} = 0 \quad r > 0 \quad (r = \tfrac{1}{2}, \tfrac{3}{2}, \ldots)$$

Can then build states as

$\psi^{\mu}_{-\frac{1}{2}}|0\rangle_{NS}$ etc. (but only 1 excitation of each mode allowed).

R sector. Now $\{\psi^{\mu}_{0}, \psi^{\nu}_{0}\} = \eta^{\mu\nu}$.

$\rightarrow$ defining $\Gamma^{\mu} = \sqrt{2}\,\psi^{\mu}_{0}$

we have $\{\Gamma^{\mu}, \Gamma^{\nu}\} = 2\eta^{\mu\nu}$ Dirac algebra in 10D!

Before working out rep'ns, note that

$\psi^{\mu}_{r}|0\rangle_{R} = 0$ for $r > 0$ is consistently imposable

but $\psi^{\mu}_{0}|0\rangle_{R} = 0$ is not:

$$\psi^{\nu}_{0}(\psi^{\mu}_{0}|0\rangle_{R}) + \psi^{\mu}_{0}(\psi^{\nu}_{0}|0\rangle_{R}) = \eta^{\mu\nu}|0\rangle_{R} \neq 0 \text{ for } \mu = \nu$$

So the Ramond ground state is degenerate and furnishes a repn of the 10D Dirac algebra.

Dirac algebra in 10D

We have $\Gamma^0, \dots \Gamma^9$ $\qquad (\Gamma^\mu \equiv \sqrt{2}\,\psi_0^\mu)$

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$$

Define
$$\Gamma^{0\pm} = \frac{1}{2}(\pm\Gamma^0 + \Gamma^1)$$
$$\Gamma^{\alpha\pm} = \frac{1}{2}(\Gamma^{2\alpha} \pm i\Gamma^{2\alpha+1}) \qquad \alpha = 1, \dots 4$$

Then $\{\Gamma^{a+}, \Gamma^{b-}\} = \delta^{ab}$ $\qquad a = 0, \dots 4$ (incl. 0 and α)

$$\{\Gamma^{a+}, \Gamma^{b+}\} = \{\Gamma^{a-}, \Gamma^{b-}\} = 0.$$

$$\Rightarrow (\Gamma^{a+})^2 = 0 = (\Gamma^{a-})^2.$$

So, given some object $|\xi\rangle$ on which the Γ act, Γ^{a-} can be taken to be lowering (annihilation) ops, and $\exists\ |\zeta\rangle$ s.t. $\Gamma^{a-}|\zeta\rangle = 0 \quad \forall a$.

(at worst, take $|\zeta\rangle = \Gamma^{1-}\Gamma^{2-}\Gamma^{3-}\Gamma^{4-}|\xi\rangle$).

Now use Γ^{a+} as raising ops (at most once each) acting on $|\xi\rangle$.

$\Rightarrow$ with 5 choices, we have $2^5 = 32$ states.

This is a Dirac spinor, which is a (reducible) repn. of the Dirac algebra.

[Can write $|\Psi\rangle = \prod_{a=0}^{4} (\Gamma^{a})^{S_a + \frac{1}{2}} |\xi\rangle \quad S_a = \pm\frac{1}{2}$.]

To reduce to irreps, define $\Gamma = \Gamma^0 \cdots \Gamma^9$

obeying $\Gamma^2 = 1, \quad \{\Gamma, \Gamma^\mu\} = 0$

(and $[\Gamma, \Sigma^{\mu\nu}] = 0$ where $\Sigma^{\mu\nu} = \frac{i}{4}[\Gamma^\mu, \Gamma^\nu]$ are generators of $SO(9,1)$.]

and

$|\Psi\rangle$ is a simultaneous eigenstate of

$$\Gamma^{a+}\Gamma^{a-} - \frac{1}{2}$$

with eigenvalues S_a.

So Γ has eigenvalues ± 1.

$\Rightarrow$ split $+,-$ eigenspaces

"spinors of positive chirality"
(negative)

Weyl representations. (16 states each).

$\underline{16}, \underline{16}'$.

Just as in 2D (generally, $D = 2,3,4 \bmod 8$)

we can find a basis of purely real Γ-matrices

$\Rightarrow$ can impose that spinors are real as well

cuts # dof in half.

$$|S\rangle^* \xleftrightarrow[\text{transformation}]{\text{some similarity}} |S\rangle$$

Now Majorana + Weyl conditions not generally compatible: if Γ is not real,

$\frac{1}{2}(1 \pm \Gamma)|S\rangle$ are not (in general) real.

But if Γ is real also, M + W conditions can be imposed together.

Works in $D = 2 \bmod 8$ $(D=2, D=10)$.

So: $32_{\mathbb{C}}$ Dirac

$$\begin{tikzcd}[column sep=small] & 32_{\mathbb{C}} \arrow[dl] \arrow[dr] & & \text{Dirac} \\ 16_{\mathbb{C}} \arrow[d] & & 16'_{\mathbb{C}} \arrow[d] & \text{Weyl} \\ 16_{\mathbb{R}} & & 16'_{\mathbb{R}} & \text{Majorana-Weyl} \end{tikzcd}$$

and finally, Dirac eqn $\Gamma^{\mu}\partial_{\mu}|\Psi\rangle = 0$

$\Rightarrow$ can solve for half of the components

$\Rightarrow$ 8 real dof in an on-shell MW spinor in 10D.

[cf 2D: 1 real (Hermitian...) component].

Light-cone gauge

Take $X^+(\sigma,\tau) = x^+ + p^+\tau$ as in bosonic case

this uses $T_B(z)$ conformal symm.

Use $T_F(z)$ fermionic symm. to set

$$\psi^+(\sigma,\tau) = 0 \qquad (\psi^\mu(z))$$

~~in NS sector~~
in R sector, may want to keep zero mode (Dirac matrix)

(since $\delta\psi^+ \sim \eta(z)\,\partial_z X^+$

Constraints:

WS metric EOM $\Rightarrow T^{(B)}_{\alpha\beta} = 0$.

fermionic constraint: $T_F = 0$.

$$\left(\text{recall } T_F = i\sqrt{\tfrac{2}{\alpha'}}\,\psi^\mu(z)\,\partial X_\mu(z)\right)$$

As in bosonic string, can solve for

$\partial_\pm X^-$ using $T_{\alpha\beta}$ constraint + LCG.

Now also, solve for

$\psi^-_\pm$ using T_F constraint + LCG.

Details can be worked out without subtlety.

Result: $\begin{cases} \alpha^{\mu}_{-n} \xrightarrow[\text{to}]{\text{restricts}} \alpha^{i}_{-n} \\ \psi^{\mu}_{-r} \longrightarrow \psi^{i}_{-r} \end{cases}$

and the mass spectrum is

$$m^2 = \frac{2}{\alpha'}\Big\{ \sum_{n>0} \alpha^i_{-n}\alpha^i_n + \sum_{r>0} r\psi^i_{-r}\psi^i_r - a$$

$$+ \sum_{n>0} \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n + \sum_{r>0} r\tilde{\psi}^i_{-r}\tilde{\psi}^i_r - \tilde{a} \Big\}$$

$$\equiv m_L^2 + m_R^2$$

[level matching gives $m_L^2 = m_R^2$]

Open superstring spectrum

Neumann bcs: $\left.\frac{\partial X^\mu}{\partial \sigma}\right|_{\sigma=0,\pi} = 0.$

$$S_F = \frac{i}{2\pi}\int d^2\sigma \left(\psi_- \partial_+ \psi_- + \psi_+ \partial_- \psi_+ \right)$$

vary + get boundary term

$$\frac{i}{2\pi}\int d^2\sigma \left[\partial_+(\psi_- \delta\psi_-) + \partial_-(\psi_+ \delta\psi_+)\right]$$

$$\rightarrow \frac{i}{2\pi}\left[\psi_- \delta\psi_- - \psi_+ \delta\psi_+ \right]_0^\pi = 0.$$

Can solve by taking $\left(\psi_- \delta\psi_- = \psi_+ \delta\psi_+ \text{ at } 0 \text{ } \underline{\text{and}} \text{ at } \pi. \right)$

$$\psi_+(\tau, 0) = \pm \psi_-(\tau, 0)$$

$\pm$ choices independent.

$$\psi_+(\tau, \pi) = \pm \psi_-(\tau, \pi)$$

Now whether $\psi_+(\tau, 0) = +\psi_-(\tau, 0)$ or $-\psi_-(\tau, 0)$ is convention.

Set $\psi_+^\mu(\tau, \pi) = \psi_-^\mu(\tau, \pi)$

and the sign at $\sigma = 0$ then matters.

Now define

$$\psi^\mu(\tau,\sigma) = \begin{cases} \psi_+^\mu(\tau,\sigma) & 0 \leq \sigma \leq \pi \\ \psi_-^\mu(\tau, \underset{2\pi-\sigma}{\cdot}) & \pi \leq \sigma \leq 2\pi \end{cases}$$

valid for $0 \leq \sigma \leq 2\pi$.

Now $\psi_+^\mu(\tau,0) = \pm \psi_-^\mu(\tau,0) \Leftrightarrow \psi^\mu(\tau,0) = \pm \psi^\mu(\tau,2\pi)$
$\Leftrightarrow$ $\binom{R}{NS}$ sectors.

Mass spectrum:

$$m^2 = \frac{1}{\alpha'}\left(\sum_{n>0} \alpha^i_{-n}\alpha^i_n + \sum_{r>0} r\,\psi^i_{-r}\psi^i_r - a \right)$$

$$\left(\begin{matrix} r \in \mathbb{Z}+\frac{1}{2} & NS \\ \mathbb{Z} & R \end{matrix} \right)$$

NS sector

Ground state: $m^2 = -\frac{1}{\alpha'}a$

1st excited: $\psi^i_{-\frac{1}{2}}|0\rangle \qquad m^2 = \frac{1}{\alpha'}\left(\frac{1}{2} - a\right)$

Now this is a vector of $SO(8)$ $\rightarrow$ by usual Lorentz argument, $m^2 = 0 \Rightarrow a = \frac{1}{2} \Rightarrow$ ground state is a tachyon.

R sector

$|0\rangle_R$ obeys $\alpha^i_n |0\rangle_R = 0$

$$\psi^i_r |0\rangle_R = 0 \quad r>0$$

but $\psi^i_0 |0\rangle_R \neq 0$.

degeneracy: ψ^i_0 is a 32×32 Dirac matrix (with a transverse vector index)

$$\underbrace{(\psi^i_0)_\alpha{}^\beta}_{\substack{\text{real}\\ \Gamma\text{-matrices}}} \underbrace{(|0\rangle_R)_\beta}_{\text{32-component spinor}}$$

spinor may be taken Majorana $\Rightarrow 32_R$ dof.

but $\exists$ two choices:

$$\begin{cases} |0\rangle_R & \text{Majorana-Weyl with chirality } + \\ |0\rangle_R & \text{'' \quad '' \quad '' } - \end{cases}$$

each case is 16_R dof ($\to 8$ on-shell using Dirac eqn)

So in sum:

$|0\rangle_{NS}$ is a tachyon w/o degeneracy.

$\exists$ 2 R-sector ground states

$|0\rangle_R^+ \qquad |0\rangle_R^-$

each a 16_R component MW spinor of the 10D Dirac algebra.

OK. $\Gamma \equiv \Gamma^0 \cdots \Gamma^9$ anticommutes with ψ_0^μ $\to$ splits ground state into $+,-$ eigenspaces.

Would like to do the same for the whole spectrum, in both sectors.

$(-1)^F$

Define $(-1)^F \equiv (-1)^{\sum_{r=\frac{1}{2}}^{\infty} \psi^i_{-r}\psi^i_r + 1}$ in NS sector

$\equiv \Gamma (-1)^{\sum_{r=0}^{\infty} \psi^i_{-r}\psi^i_r}$ in R sector.

So F = WS fermion number mod 2.

i.e., if start with $|\lambda\rangle$ obeying $(-1)^F|\lambda\rangle = +|\lambda\rangle$

then $\psi^i_{-r}|\lambda\rangle$ obeys $(-1)^F \psi^i_{-r}|\lambda\rangle = -\psi^i_{-r}|\lambda\rangle$

$\Rightarrow \{(-1)^F, \psi^i_{-r}\} = 0.$

What?? Why?? (And, what is $(-1)^F$ eigenvalue of $|0\rangle_{NS}$, $|0\rangle_R$?)

$-(-1)^F \equiv$ "G-parity"

Study of the ghosts reveals that

$$\begin{cases} (-1)^F |0\rangle_{NS} = -|0\rangle_{NS} \\ \Gamma(-1)^F |0\rangle_R \text{ depends: } \begin{array}{l} + \text{ for } \underline{16} \\ - \text{ for } \underline{16}'. \end{array} \end{cases}$$

Idea (Gliozzi, Scherk, Olive): project onto $(-1)^F = +1$ (i.e. + eigenspace) in NS sector, and something similar in R sector. lowest states

Possible left-moving sectors

NS+	$\psi^i_{-1/2}|0\rangle_{NS}$	
NS−	$|0\rangle_{NS}$	
R+	$|0\rangle^+_R$	($\underline{16}$)
R−	$|0\rangle^-_R$	($\underline{16}'$)

similarly for right-movers

GSO: kills NS−.

can choose R+ or R−.

One more time: omit by def of $(-1)^F$ in R sector

hol.	$(-1)^F$ / or $\Gamma(-1)^F$	antihol	$(-1)^{\tilde{F}}$ / or $\Gamma(-1)^{\tilde{F}}$
$\vert 0\rangle_{NS}$	-1	$\vert 0\rangle_{NS}$	-1
$\psi^i_{-1/2}\vert 0\rangle_{NS}$	$+1$	$\tilde{\psi}^i_{-1/2}\vert 0\rangle_{NS}$	$+1$
$\vert 0\rangle^+_R$	$+1$	$\vert 0\rangle^+_R$	$+1$
$\vert 0\rangle^-_R$	-1	$\vert 0\rangle^-_R$	-1

Two choices lead to consistent string theories with supersymmetry.

[aside: modular invariance $\Leftrightarrow$ invariance under large diffeomorphisms of WS]

(i) $(-1)^F = +1$

$(-1)^{\tilde{F}} = +1$

(ii) $(-1)^F = +1$

$(-1)^{\tilde{F}} = +1$ NS

$\phantom{(-1)^{\tilde{F}} =} -1$ R

Choice (i):

$|0\rangle_R^+ \otimes |0\rangle_R^+$

$\psi^i_{-1/2}|0\rangle_{NS} \otimes \tilde{\psi}^j_{-1/2}|0\rangle_{NS}$

$\psi^i_{-1/2}|0\rangle_{NS} \otimes |0\rangle_R^+$

$|0\rangle_R^+ \otimes \tilde{\psi}^i_{-1/2}|0\rangle_{NS}$

Choice (ii):

$|0\rangle_R^+ \otimes |0\rangle_R^-$

$\psi^i_{-1/2}|0\rangle_{NS} \otimes \tilde{\psi}^j_{-1/2}|0\rangle_{NS}$

$|0\rangle_R^+ \otimes \tilde{\psi}^i_{-1/2}|0\rangle_{NS}$

$\psi^i_{-1/2}|0\rangle_{NS} \otimes |0\rangle_R^-$

In case (i), the L, R Ramond ground states have the same chirality $\Rightarrow$ chiral theory (2×16, not $16'$)

In case (ii), the L, R Ramond ground states have opposite chirality $\Rightarrow$ nonchiral theory (16 and $16'$).

Work out SO(8) rep'ns: (Appendix B)

$SO(9,1) \to SO(8)$

$16 \to 8$ $\quad (|0\rangle_R^+)$

$16' \to 8'$ $\quad (|0\rangle_R^-)$

$\psi^i_{-1/2}|0\rangle_{NS}$: 8_v

$\tilde{\psi}^i_{-1/2}|0\rangle_{NS}$: 8_v

Compute:
$$\begin{cases} 8\otimes 8 & RR \quad (IIB) \\ 8\otimes 8' & RR \quad (IIA) \\ 8_v\otimes 8 & NS\,R \quad (IIB) \\ 8_v\otimes 8' & NS\,R \quad (IIA) \\ 8_v\otimes 8_v & NS\,NS \end{cases}$$
[and R NS IIA, IIB]

$8_v\otimes 8_v$: $c^{ij} \to s^{ij} + a^{ij}$

$$\to \left(s^{ij} - \tfrac{1}{8}\delta^{ij} s^i{}_i\right) + a^{ij} + \tfrac{1}{8}\delta^{ij} s^i{}_i$$

$$64 \to \frac{8(8+1)}{2} - 1 + \frac{8(8-1)}{2} + 1$$

$$8_v\otimes 8_v = 35 \oplus 28 \oplus 1$$

graviton g_{ij} — antisymmetric tensor B_{ij} — dilaton Φ

just like bosonic string (but D=10 not D=26)

$8_v \otimes 8$

$\underbrace{\underbrace{\psi^i_{-1/2} |0\rangle_{NS}}_{\underbrace{V^i}_{\text{vector of } SO(8)}} \otimes \underbrace{|0\rangle^+_R}_{|S\rangle_s}}$

$|S\rangle_s$ spinor of $SO(8)$

s: spinor index

write $\Gamma^i_{ss'} |S\rangle_s \cdot V_i$

this is a spinor (s' index free) $\Rightarrow 8'$.

$\Rightarrow 8_v \otimes 8 \rightarrow 8' \oplus 56$

↳ vector-spinor irrep Ψ^i_s

$8_v \otimes 8' \rightarrow 8 \oplus 56'$

↳ $\Psi^i_{s'}$

$8 \otimes 8 \rightarrow [0] + [2] + [4]_+$

$1 \oplus 28 \oplus 35_+$

scalar — antisymm 2-form — 4-form w/ selfdual field strength

$C_0 \quad C_2 \quad C_4 \quad (dC_4 = *dC_4)$

$8 \otimes 8' \rightarrow [1] + [3]$

$C_1 + C_3$

$8_v + 56$ (tensor, not vector-spinor) $\left(\frac{8 \cdot 7 \cdot 6}{3!}\right)$

In sum:

IIA: $(1 \oplus 28 \oplus 35) \oplus (8 + 56') \oplus (8' + 56) \oplus (8_v + 56_+)$

NS NS — R NS — NS R — RR

$\Phi \quad B_{ij} \quad g_{ij} \qquad \lambda_s \quad \Psi^i_s \qquad \hat{\lambda}_s \quad \hat{\Psi}^i_s \qquad C_i \quad C_{ijk}$

$C_1 \quad C_3$

IIB $(1 \oplus 28 \oplus 35) \oplus (8' \oplus 56) \oplus (8' \oplus 56) \oplus (1 + 28 + 35_+)$

NS NS

$\Phi \quad B_{ij} \quad g_{ij} \qquad \lambda_{s'} \quad \Psi^i_s \qquad \hat{\lambda}_{s'} \quad \hat{\Psi}^i_s \qquad C_0 \quad C_2 \quad C_4$

IIA has opposite-chirality Ψ_s.
IIB " same " "

Universal NS NS sector:

$$g_{ij},\ B_{ij},\ \Phi.$$

NS-R R-NS

fermions: 2 gravitinos Ψ with $\begin{pmatrix}\text{same}\\ \text{opposite}\end{pmatrix}\begin{pmatrix}\text{IIB}\\ \text{IIA}\end{pmatrix}$ chirality.
2 dilatinos λ

NB Rarita-Schwinger eqn for $m=0$
$\epsilon^{\mu\nu\rho\sigma}\gamma^5\gamma_\nu\partial_\rho\psi_\sigma = 0$
(or $= -m\psi^\mu$)

RR:

antisymmetric p-forms

C_0 C_2 C_4 IIB

C_1 C_3 IIA

These spectra have the same # of bose + fermi states:
64 in each sector.
128 bosons, 128 fermions.

What to do with all this?

① Presence of massless gravitino ⇒ (local spacetime) supersymmetry (SUGRA).
So presumably the spectrum we've found is some SUGRA theory.

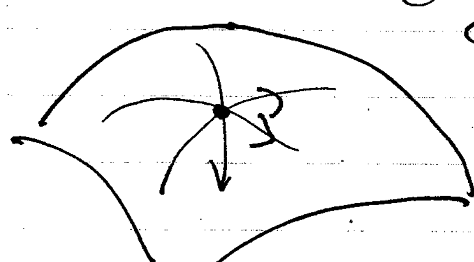
Step (i): review differential forms, to formulate RR sector action

(ii): write full action, examine e.o.m.

(iii): search for supersymmetric solutions (leads to Calabi-Yau geometry).

Differential forms + gauge fields

Consider a manifold $\{M, g, X^\mu\}$.



considering curves through a pt "p" allows definition of tangent space

T_pM.

Basis: $\left\{\frac{\partial}{\partial X^\mu}\right\}$

So any $v \in T_pM = v^\mu \frac{\partial}{\partial X^\mu}$.

Dual of this vector space:

recall given V, V^* defined by:

V^*: space of linear maps $f: V \to \mathbb{R}$

$v \in V$, $v = v^\mu \underset{\text{basis}}{\underbrace{e_\mu}}$

$f(v) = v^\mu f(e_\mu)$

then if f, g are such fns,

$(\alpha f + \beta g) v = \alpha v^\mu f(e_\mu) + \beta v^\mu g(e_\mu) = \alpha f(v) + \beta g(v)$

$\Rightarrow$ vector space V^*.

dual basis: $e^{*\mu}$ obeying $e^{*\mu}(e_\nu) = \delta^\mu{}_\nu$

$f = f_\mu e^{*\mu}$ $\quad f(v) = f_\mu e^{*\mu} v^\nu e_\nu = f_\mu v^\mu$

$= \langle f, v \rangle$.

The specific case $V = T_pM$ tangent space
$V^* = T_pM^*$ cotangent space

basis of T_pM^* (T_p^*M):

basis set of maps $\frac{\partial}{\partial x^\mu} \to \mathbb{R}$.

Notation: $\{dx^\mu\}$

$$\left(\frac{\partial}{\partial x^\mu} dx^\nu = \delta_\mu{}^\nu \right)$$

element of T_p^*M: $\omega_\mu dx^\mu$ one-form
↳ "cotangent vector"

Elements of $\underbrace{T_p^*M \otimes \cdots \otimes T_p^*M}_{k}$: k-index tensors with covariant (lower) indices.

<u>Define</u>: $dx^{\mu_1} \wedge \cdots dx^{\mu_r} \equiv \sum_{\text{Perms } P} \operatorname{sgn}(P)\, dx^{\mu_{P(1)}} \otimes \cdots dx^{\mu_{P(r)}}$

(wedge product)

creates totally antisymmetric objects.

Then $\frac{1}{r!}\underbrace{\omega_{\mu_1 \ldots \mu_r}}_{\text{totally antisymm}} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_r} \equiv \omega$ is an element of $\Omega_p^r(M)$

"vector space of r-forms at $p \in M$"

"r-form".

Exterior derivative: $d_r : \Omega^r(M) \to \Omega^{r+1}(M)$

$$d_r \omega = \frac{1}{r!}\left(\frac{\partial}{\partial x^\nu}\omega_{\mu_1 \ldots \mu_r}\right) dx^\nu \wedge dx^{\mu_1} \wedge \ldots dx^{\mu_r}$$

usually omit the r.

eg. $\omega = \frac{1}{2!} f(x,y,z)\, dx \wedge dy$

$$d\omega = \frac{1}{2!}\left[\frac{\partial f}{\partial z} dz \wedge dx \wedge dy + \cancel{\frac{\partial f}{\partial y} dy \wedge dx \wedge dy} + \cancel{\frac{\partial f}{\partial x} dx \wedge dx \wedge dy}\right]$$

note $d^2 \equiv 0$

since $\frac{\partial^2 \omega}{\partial x^\alpha \partial x^\beta} dx^\alpha \wedge dx^\beta \equiv 0$.

(for any ω)

Differential forms are topological because $\underline{d}$ does not depend on metric data (connection terms cancel.)

We need to define an inner product for p-forms.

Define isomorphism $\Omega^r(M) \to \Omega^{m-r}(M)$, the "Hodge $*$":

$$*: \Omega^r(M) \to \Omega^{m-r}(M)$$

uses metric!

$$*(dx^{\mu_1} \wedge \dots dx^{\mu_r}) = \frac{\sqrt{g}}{(m-r)!} \varepsilon^{\mu_1 \dots \mu_r}{}_{\nu_{r+1} \dots \nu_m} \cdot$$
$$\cdot dx^{\nu_{r+1}} \wedge \dots dx^{\nu_m}$$

where $\varepsilon_{\mu_1 \dots \mu_m} = \begin{cases} +1 & \text{even perm} \\ -1 & \text{odd perm} \end{cases}$

and $\varepsilon^{\mu_1 \dots \mu_r}{}_{\nu_{r+1} \dots \nu_m} \equiv g^{\mu_1 \nu_1} \dots g^{\mu_r \nu_r} \varepsilon_{\nu_1 \dots \nu_m}$

Then if $\omega = \frac{1}{r!} \omega_{\mu_1 \dots \mu_r} dx^{\mu_1} \wedge \dots dx^{\mu_r}, \quad \in \Omega^r(M)$

$$*\omega = \frac{\sqrt{g}}{r!\,(m-r)!} \omega_{\mu_1 \dots \mu_r} \varepsilon^{\mu_1 \dots \mu_r}{}_{\nu_{r+1} \dots \nu_m} dx^{\nu_{r+1}} \wedge \dots \wedge dx^{\nu_m}$$
$$\in \Omega^{m-r}(M).$$

Can prove $**\omega = (-1)^{r(m-r)} \omega$.

Now, given $\omega, \eta \in \Omega^r(M)$,

$$\omega \wedge * \eta = \frac{1}{r!}\frac{1}{r!}\,\omega_{\mu_1\dots\mu_r}\,\eta_{\nu_1\dots\nu_r}\,\frac{\sqrt{g}}{(m-r)!}\,\varepsilon^{\nu_1\dots\nu_r}{}_{\mu_{r+1}\dots\mu_m} \times$$

$$\times\, dx^{\mu_1}\wedge\dots\, dx^{\mu_m}$$

and is $\in \Omega^m(M)$.

why sum? now ε has downstairs indices

$$= \sum_{\mu\nu} \frac{1}{r!}\,\omega_{\mu_1\dots\mu_r}\,\eta^{\nu_1\dots\nu_r}\,\frac{1}{r!\,m-r!}\,\varepsilon_{\nu_1\dots\nu_r\mu_{r+1}\dots\mu_m}\sqrt{g}\, \times$$

$$\times\, \varepsilon_{\mu_1\dots\mu_r\mu_{r+1}\dots\mu_m}\, dx^1\wedge\dots\, dx^m$$

$$= \underbrace{m-r!}_{\text{from } \sum \mu_{r+1}\dots\mu_m}\;\underbrace{r!}_{\text{from } \sum \nu_1\dots\nu_r}\,\frac{1}{r!}\,\omega_{\mu_1\dots\mu_r}\,\eta^{\mu_1\dots\mu_r}\,\frac{1}{r!\,m-r!}\sqrt{g}\, dx^1\wedge\dots dx^m$$

$$= \frac{1}{r!}\,\omega_{\mu_1\dots\mu_r}\,\eta^{\mu_1\dots\mu_r}\sqrt{g}\, dx^1\wedge\dots\, dx^m$$

So $$\int \omega\wedge *\eta = \frac{1}{r!}\int_M \omega_{\mu_1\dots\mu_r}\,\eta^{\mu_1\dots\mu_r}\sqrt{g}\, dx^1\wedge\dots dx^m$$

So the Maxwell action

$$S = -\frac{1}{4g^2}\int_M F_{\mu\nu}F^{\mu\nu}\sqrt{g}\, dx^0 \wedge \dots dx^3$$

$$= -\frac{1}{2g^2}\int_M F \wedge * F$$

Moreover $F = F_{\mu\nu}\, dx^\mu\, dx^\nu$

$$= dA$$

$$A = A_\mu dx^\mu \qquad \text{(Abelian case)}$$

The Maxwell eqns are then (in vacuum)

$$\begin{cases} dF = 0 \\ \cancel{**}\, d*F = 0 \end{cases}$$

with a current 1-form $j = \eta_{\mu\nu} j^\nu dx^\mu$

$$= -\rho\, dt + \vec{j}\cdot d\vec{x}$$

we have $\quad dF = 0 \quad \Leftrightarrow \quad \nabla\cdot B = 0 \quad \nabla\times E = -\frac{\partial B}{\partial t}$

$$\cancel{**}\, d*F = *j \quad \Leftrightarrow \quad \nabla\cdot E = \rho \quad \nabla\times B = \frac{\partial E}{\partial t} + \vec{j}$$

Even better, in "Lorentz gauge" $d*A=0$,

we may write $(F = dA)$

$$\left\{\begin{array}{c} d^2 A = 0 \\ *d*dA = j \end{array}\right\}.$$

and $S = -\frac{1}{2g^2}\int_M dA \wedge * dA$

For a p-form potential, A_p, ($p = 2 \Leftrightarrow$ NS B-field, $p=1 \Leftrightarrow$ Maxwell)

$$F_{p+1} \equiv dA_p$$

enjoys invariance:

$$A_p \to A_p + d\Lambda_{p-1}$$

$$F \to dA_p + d^2\Lambda_{p-1} = dA_p \quad \text{inv.}$$

Then $S_p = \frac{-1}{2(p+1)!}\int_M dA_p \wedge * dA_p = \frac{-1}{2(p+1)!}\int F_{p+1} \wedge * F_{p+1}$

is gauge-invariant ((p−1)-form gauge invariance).

Note $\int \sqrt{g}\, F_{\mu_1 \ldots \mu_p} F^{\mu_1 \ldots \mu_p} \cdot \frac{1}{p!} \equiv \int \sqrt{g}\, |F_p|^2 = \int F_p \wedge * F_p$