Unit 1.

Quantization of the bosonic string

(i) What is string theory?

(ii) classical relativistic string

(iii) quantization in LCG.

Unit 1.

1.1 What is string theory?

1.2 Classical bosonic string.

S_{NG}

e in S_{pp}

S_P

$S_P \to S_{NG}$

Symmetries of S_P (review of Noether)

diff × Weyl + Virasoro constraints

covariant vs lightcone

classical eom: solution before constraints

lightcone reparametrizations

S_{pp} in lightcone gauge

string in lightcone gauge: classical solution with constraints

1.3 quantization in LCG: open
closed

normal-ordering

Lorentz anomaly: representations
commutators

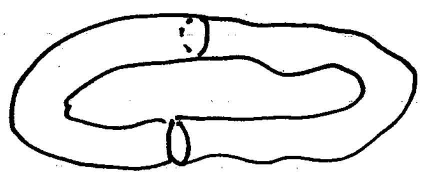
1.1 What is string theory?

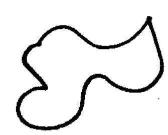
- the quantum theory of one-dimensional objects: strings. 
- defined as a quantum field theory living on the (1+1)-dimensional worldsheet of the string

$$S = \int d^2\sigma \, \mathcal{L}_{string}$$

There are many such QFTs ⇒ many string theories.

Sometimes (ie, for some string theories) the strings themselves arise from wrapped higher-dimensional objects (membranes, ...) and hence have internal structure.



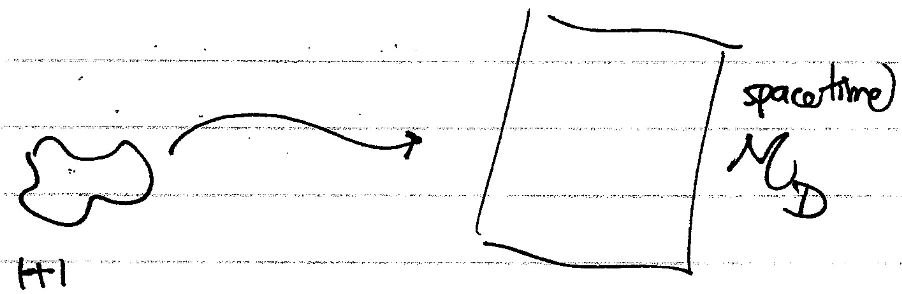
We may consider <u>closed</u> strings 

and <u>open</u> strings 

1.2 As we will see, all closed string theories contain a massless spin-2 excitation.

We will see that quantum consistency of string theory requires

- more than 3+1 dimensions of spacetime



$D=10$ superstrings
$D=26$ bosonic strings
other values for more exotic theories.

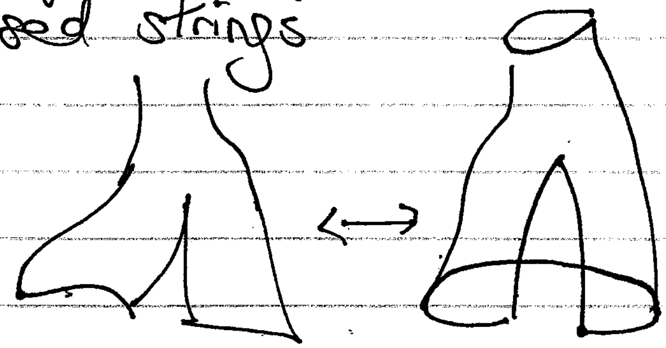
- that the spacetime metric obeys the Einstein equations!

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

check, $D\neq 4$

We will also see that open string theories readily contain nonabelian gauge fields + chiral fermions.

1.3 Furthermore, open string theories always contain closed strings

cartoon: 

Hence, except in exotic cases, we find that

string theory is a theory of quantum gravity

(we will see that it is in fact finite)

Moreover,

string theory naturally exists in $D>4$

and readily contains nonabelian gauge fields + chiral fermions (+ SUSY).

1.4 Why would we want a theory of QG?

recall in QFT, eg $\lambda\phi^4$,

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \lambda_4\phi^4 - \left(\frac{\lambda_6}{M^2}\right)\phi^6 + \cdots$$

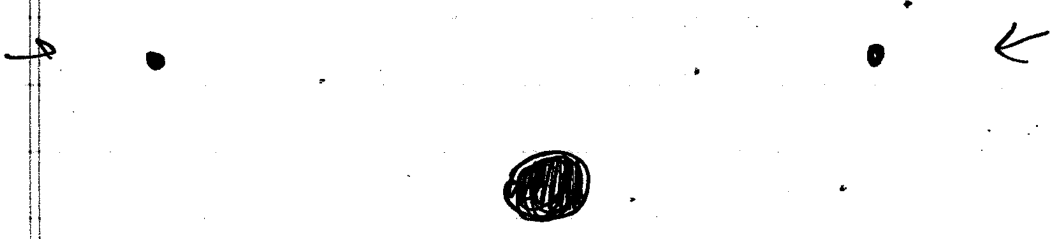
interactions controlled by
the couplings with positive mass dimension are termed superrenormalizable,
the " " vanishing " " " renormalizable
the " " negative nonrenormalizable

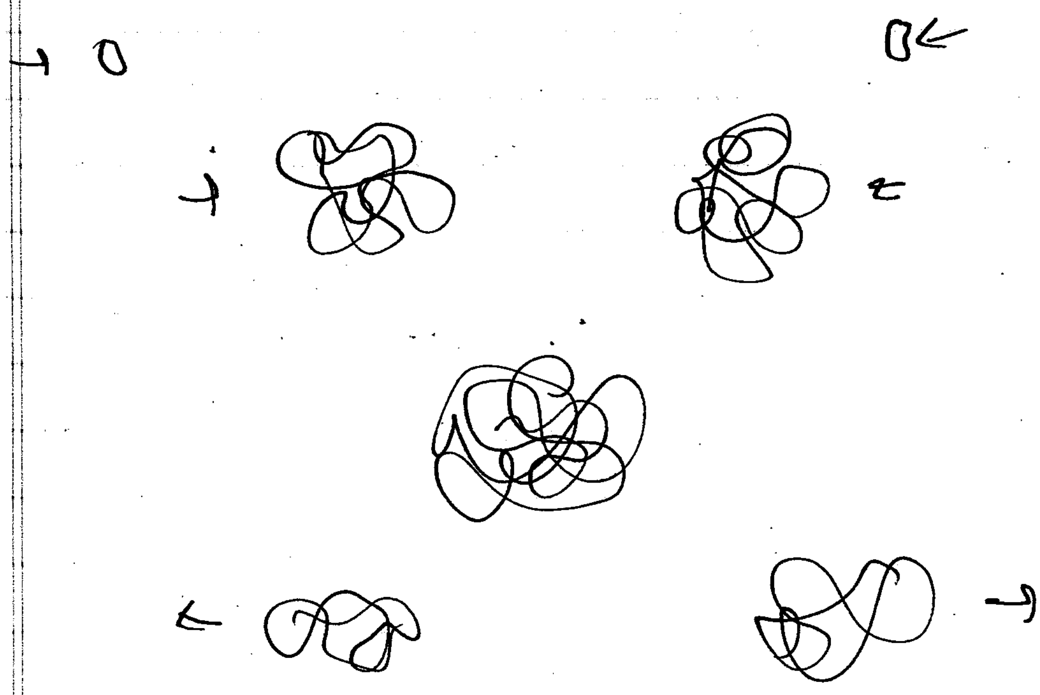
In gravity, the coupling constant is G_N. $\sim L^2$.

$\Rightarrow$ highly nonrenormalizable.

1.5 Heuristic: point-particle scattering at sufficiently high energies leads to black hole formation

String scattering at high energies is much softer (as we will see quantitatively):





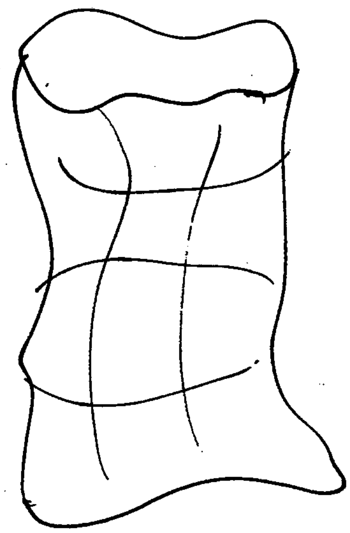
1.6 So is that all? A finite theory of QG, with some prospect of model-building?

No!

The extremely rich structure of S.T. has led to important insights into

- nonperturbative dualities
- gauge theories (eg at strong coupling)
- mathematics (eg in algebraic geometry)
- black holes (eg microscopic counting of SYM, states giving Bekenstein-Hawking S)
- holography [AdS/CFT correspondence]
- theories on branes

How to formulate a classical relativistic string theory?



spacetime $(M, g_{\mu\nu})$

view as embedding $\Phi : \Sigma \to M$

Σ : $(1+1)$-dimensional worldsheet
coordinates ξ^α $\alpha = 1, 2$

point $\xi^\alpha \xrightarrow[\Phi]{} X^\mu(\xi^\alpha)$

by the embedding Φ

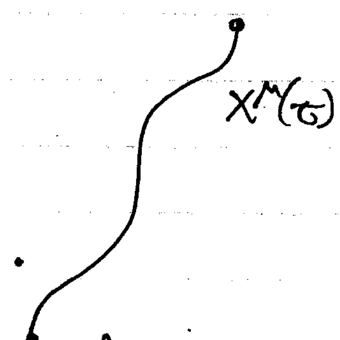
pull-back of spacetime metric:

$$\Phi^* g : \quad \frac{\partial X^\mu}{\partial \xi^\alpha}\frac{\partial X^\nu}{\partial \xi^\beta} g_{\mu\nu} \equiv h_{\alpha\beta}$$

This ~~is~~ leads to a natural measure of the size of an embedded submanifold:

$$\int \sqrt{-\det h_{\alpha\beta}}\; d^{2}\xi$$

cf point particle case: $\xi^\alpha \leftrightarrow \tau$ worldline parameter



$$h_{\alpha\beta} = \frac{\partial X^\mu}{\partial \tau}\frac{\partial X_\mu}{\partial \tau}$$

$$\Rightarrow S_{\text{point}} = \int d\tau \sqrt{\frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau}}$$

More generally, for a theory of relativistic membranes, dimension p+1, we would have

$$h_{\alpha\beta} = \frac{\partial X^\mu}{\partial \xi^\alpha} \frac{\partial X^\nu}{\partial \xi^\beta} g_{\mu\nu}$$ "induced metric"

$$S = \int d^{p+1}\xi \sqrt{-\det h} \; .$$

Actions with $\sqrt{\ }$ are not easy to quantize. ☹

How can we get a simpler-looking but equivalent action, so we can quantize that instead?

First do point particle as a warmup.

$$S = m\int \sqrt{\dot{X}^\mu \dot{X}_\mu}\, d\tau \qquad \dot{} : \frac{d}{d\tau} \quad (\tau \neq X^0)$$

e.o.m: $$\frac{m\dot{X}^\mu}{\sqrt{\dot{X}^\nu \dot{X}_\nu}} = \text{const.}$$

note invariance under $\tau \to \lambda\tau$.

<u>Guess:</u>

$$S' = \int d\tau \left(\dot{X}^\mu \dot{X}_\mu + m^2 \right)$$

<u>not</u> invariant, oops.

Introduce a 'compensator' e,

$$\tau \to \lambda \tau$$
$$e \to \lambda^{-1} \tau$$

and write

$$S'' = \int d\tau \left(e^{-1} \dot{X}^\mu \dot{X}_\mu + e m^2 \right)$$

e.o.m: $\quad e: \quad -\frac{1}{e^2} \dot{X}^\mu \dot{X}_\mu \mp m^2 = 0$

$$\dot{X}^\mu \dot{X}_\mu = e^2 m^2$$

$$e = \frac{1}{m} \sqrt{\dot{X}^\mu \dot{X}_\mu}$$

$$X^\mu: \quad \frac{d}{d\tau} \left(e^{-1} \cdot \dot{X}^\mu \right) = 0$$

$$\frac{d}{d\tau} \left(\frac{m \dot{X}^\mu}{\sqrt{\dot{X}^\nu \dot{X}_\nu}} \right) = 0 \quad \checkmark$$

Can view e as an independent 'metric' on the line, $e\,d\tau \leftrightarrow d\hat{\tau}$.

Let's apply a similar method to our proposed string action

$$S = -T\int d^2\xi \sqrt{-\det h}$$

$$h_{\alpha\beta} = \frac{\partial X^\mu}{\partial \xi^\alpha}\frac{\partial X^\nu}{\partial \xi^\beta}\eta_{\mu\nu}$$

$$T = \text{tension}$$

(this is the 'Nambu-Goto action).

$$-m\int d\tau \sqrt{\frac{dX^\mu}{d\tau}\frac{dX_\mu}{d\tau}} \to \int d\tau \left(e^{-1}\frac{dX^\mu}{d\tau}\frac{dX_\mu}{d\tau} + em^2 \right)$$

(independent metric on worldline)

$$-T\int d^2\xi \sqrt{-\det h} \to -\frac{T}{2}\int d^2\xi \sqrt{-\gamma}\,\gamma^{\alpha\beta}\underbrace{\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}}_{h_{\alpha\beta}}$$

$$S_{NG} \to S_P \text{ (Polyakov)}.$$

Here $\gamma_{\alpha\beta}$ is a genuine ~~dynamical~~ physical field, i.e. the e.o.m. are

$$\begin{cases} \dfrac{\delta S_P}{\delta \gamma_{\alpha\beta}} = 0 \,; \\[2ex] \dfrac{\delta S_P}{\delta X^\mu} = \partial_\alpha \left(\dfrac{\delta S_P}{\partial_\alpha X^\mu} \right) . \end{cases}$$

If we have a (1+1)d metric as a physical field, does this mean we are doing (1+1)d GR coupled to fields X^μ?

No: no Einstein-Hilbert term $S_{EH} = \frac{1}{2\kappa^2} \int d^2\xi \sqrt{-\gamma}\, R$

and in fact, GR in (1+1)d is nearly trivial.

Aside: GR in (1+1)d.

from $S_{EH} = \frac{1}{2\kappa^2}\int d^2\xi \sqrt{-\gamma}\, R$

we get the Einstein equations

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = T_{\alpha\beta} \quad (=0 \text{ in vacuum}$$

but $R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma} = R_{\gamma\delta\alpha\beta}$

$\Rightarrow$ since $\alpha\beta\gamma\delta \in \{0,1\}$, can write

$$\cancel{R_{\alpha\beta\gamma\delta} = C\left(g_{\alpha\gamma} g_{\beta\delta} - g_{\beta\gamma} g_{\alpha\delta}\right)}$$

$$R_{\alpha\beta\gamma\delta} = f \cdot \epsilon_{\alpha\beta}\epsilon_{\gamma\delta}$$

so $R = f \epsilon_{\alpha\beta}\epsilon_{\gamma\delta} g^{\alpha\gamma} g^{\beta\delta}$

$$= f g^{\alpha\gamma}\left(\epsilon_\alpha{}^\delta \epsilon_{\gamma\delta}\right) = f \epsilon^{\gamma\delta}\epsilon_{\gamma\delta} = 2f$$

$\Rightarrow \epsilon_\alpha{}^\delta \epsilon_{\gamma\delta} = g_{\alpha\gamma}$

$$\Rightarrow R_{\alpha\beta\gamma\delta} = \frac{R}{2}\epsilon_{\alpha\beta}\epsilon_{\gamma\delta}$$

check $\longrightarrow$ $\cancel{\left[= \frac{R}{2}\left(g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}\right)\right]}$ don't use

$$\Rightarrow R_{\alpha\gamma} \equiv R_{\alpha\beta\gamma\delta} g^{\beta\delta} = \frac{R}{2}\left(\cancel{2g_{\alpha\gamma} - g_{\alpha\gamma}}\right) = \frac{1}{2} R g_{\alpha\gamma}$$

$\Rightarrow$ vacuum E.E. is an identity.

Back to our task.

$$S = -\frac{T}{2}\int d^2\xi \sqrt{-\gamma}\, \gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

need to know $\dfrac{\delta}{\delta \gamma_{\alpha\beta}} \sqrt{-\gamma}$.

$$\frac{\delta S}{\delta \gamma_{\gamma\delta}} = -\frac{T}{2}\int d^2\xi \left\{ \frac{\delta\sqrt{-\gamma}}{\delta \gamma_{\gamma\delta}} \gamma^{\alpha\beta} h_{\alpha\beta} + \sqrt{-\gamma}\, h_{\alpha\beta} \frac{\delta \gamma^{\alpha\beta}}{\delta \gamma_{\gamma\delta}} \right\}$$

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For a general metric g_{ab}, compute

$$\frac{\delta \det g}{\delta g_{ab}}.$$

Use $\det M = \exp \mathrm{Tr} \ln M$ $\quad M =$ matrix

and $(M + \delta M)_{ab} = M_{ac}\left(\delta^c{}_b + (M^{-1})^{cd}\, \delta M_{db}\right)$

where $M_{ac} (M^{-1})^{cd} \equiv \delta_a{}^d$

then

$$\det (M + \delta M) = \det M \det (\mathbb{1} + M^{-1} \delta M)$$

check $\det AB = \det A \det B$
it's true
Hoffman + Kunze.

$$= \det M \exp \mathrm{Tr} \ln (\mathbb{1} + M^{-1} \delta M)$$

$$= \det M \left(1 + \mathrm{Tr}\, M^{-1} \delta M \cdots\right)$$

$$\Rightarrow \delta \det M = \det M \,\mathrm{Tr}\, M^{-1} \delta M$$

$$= \det M \,(M^{-1})^{ab}\, \delta M_{ab}$$

for $M = g$,

$$\boxed{\delta g = g \cdot g^{ab}\, \delta g_{ab}}$$

so $\delta\sqrt{-g} = \frac{1}{2}\frac{1}{\sqrt{-g}}(-1)\,\delta g$

$$= +\frac{1}{2}\sqrt{-g}\, g^{ab}\,\delta g_{ab}$$

but also using

$$0 = \delta\left(g_{ab}\, g^{ab}\right) = g^{ab}\,\delta g_{ab} + g_{ab}\,\delta g^{ab}$$

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{ab}\,\delta g^{ab}$$

$$\Rightarrow \delta S = -\frac{T}{2}\int d^2\xi \left\{ -\frac{1}{2}\sqrt{-\gamma}\,\gamma_{\gamma\delta}\,\delta\gamma^{\gamma\delta}\,\gamma^{\alpha\beta} h_{\alpha\beta} + \sqrt{-\gamma}\, h_{\gamma\delta}\,\delta\gamma^{\gamma\delta} \right\}$$

$$\frac{\delta S}{\delta\gamma^{\gamma\delta}} = -\frac{T}{2}\int d^2\xi\,\sqrt{-\gamma}\left\{ h_{\gamma\delta} - \frac{1}{2}\gamma_{\gamma\delta}\,\gamma^{\alpha\beta} h_{\alpha\beta} \right\}$$

worth noting:

e.o.m. for WS metric $\gamma_{\alpha\beta}$:

$$\frac{\delta S_p}{\delta \gamma_{\alpha\beta}} = 0$$

$$\Rightarrow \boxed{h_{\alpha\beta} - \frac{1}{2}\gamma_{\alpha\beta}\gamma^{\gamma\delta}h_{\gamma\delta} = 0.}$$

$$\cancel{\Rightarrow \gamma^{\alpha\beta}h_{\alpha\beta} = \frac{1}{2}\gamma^{\alpha\beta}\gamma_{\alpha\beta}\gamma^{\gamma\delta}h_{\gamma\delta}} \quad \text{no content}$$

$$\Rightarrow \det h = \left(\frac{1}{2}\right)^2 \left(\gamma^{\gamma\delta}h_{\gamma\delta}\right)^2 \det \gamma$$

$$\sqrt{-h} = \frac{1}{2}\gamma^{\gamma\delta}h_{\gamma\delta}\sqrt{-\gamma}$$

$$\Rightarrow \frac{h_{\alpha\beta}}{\sqrt{-h}} = \frac{\gamma_{\alpha\beta}\quad \gamma^{\gamma\delta}h_{\gamma\delta}\cdot\frac{1}{2}}{\sqrt{-\gamma}\;\frac{1}{2}\gamma^{\gamma'\delta'}h_{\gamma'\delta'}} = \frac{\gamma_{\alpha\beta}}{\sqrt{-\gamma}}$$

$$\Rightarrow \gamma_{\alpha\beta} \propto h_{\alpha\beta}; \quad \gamma^{\alpha\beta}h_{\alpha\beta} = \frac{\sqrt{-h}}{\sqrt{-\gamma}}\gamma^{\alpha\beta}\gamma_{\alpha\beta} = \frac{2\sqrt{-h}}{\sqrt{-\gamma}}$$

$$\cancel{\gamma^{\alpha\beta} = h^{\alpha\beta}\frac{\sqrt{-\gamma}}{\sqrt{-h}}}$$

plug this into S_p

$$S_p = -\frac{T}{2}\int d^2\xi \sqrt{-\gamma}\,\gamma^{\alpha\beta} h_{\alpha\beta}$$

$$= -\frac{T}{2}\int d^2\xi \sqrt{-\gamma}\cdot \frac{2\sqrt{-h}}{\sqrt{-\gamma}} = -T\int d^2\xi \sqrt{-h}. = S_{NG}.$$

success.

Let's now think about this action before we study the full classical eom.

Symmetries:

(1) $X^\mu(\tau,\sigma) \to \Lambda^\mu{}_\nu X^\nu(\tau,\sigma) + a^\mu$ Poincaré

(2) reparametrization / general coordinate / diffeomorphism

$\tau \to \tau'(\tau,\sigma)$
$\sigma \to \sigma'(\tau,\sigma)$

$$\frac{\partial\sigma'^\gamma}{\partial\sigma^\alpha}\frac{\partial\sigma'^\delta}{\partial\sigma^\beta}\gamma'_{\gamma\delta}(\tau',\sigma') = \gamma_{\alpha\beta}(\tau,\sigma)$$

usual inv. of GR.

NB $X'^\mu(\tau',\sigma') = X^\mu(\tau,\sigma)$ μ index does not transform.

(3)* in S_P but not in S_{NG},

$$\gamma'_{\alpha\beta}(\tau,\sigma) = e^{2\omega(\tau,\sigma)}\gamma_{\alpha\beta}(\tau,\sigma)$$

local dilation (additional redundancy)

Weyl invariance.

Note that Poincaré is an 'internal' (also s/w global) symmetry,

diff, Weyl are local symmetries.

Back to our story.

$$\frac{\delta S_p}{\delta \gamma^{\alpha\beta}} = -\frac{T}{2}\int d^2\xi \sqrt{-\gamma}\left\{ h_{\alpha\beta} - \frac{1}{2}\gamma_{\alpha\beta}\gamma^{\gamma\delta}h_{\gamma\delta}\right\}$$

$$\frac{\delta S_p}{\delta X^\mu} = 0; \quad \frac{\delta S_p}{\delta \partial_\alpha X^\mu} = -\frac{T}{2}\sqrt{-\gamma}\cdot 2\gamma^{\alpha\beta}\partial_\beta X^\nu \eta_{\mu\nu}$$

$$\text{E-L} \Rightarrow \quad \partial_\alpha\left(\sqrt{-\gamma}\,\gamma^{\alpha\beta}\partial_\beta X_\mu\right) = 0$$

but generally

$$\frac{1}{\sqrt{g}}\partial_a\left(\sqrt{g}\,g^{ab}\partial_b f\right) = \nabla^2 f$$

$$\Rightarrow \text{E-L} \Rightarrow \quad \sqrt{-\gamma}\,\nabla^2 X^\mu = 0$$

(nb $\eta_{\mu\nu}$ commutes past ∇_α)

$$\gamma^{\alpha\beta}\nabla_\alpha\nabla_\beta = \gamma^{\alpha\beta}\nabla_\alpha\partial_\beta$$

boundary terms:

$$S_p = -\frac{T}{2}\int d\tau\, d\sigma\, \sqrt{-\gamma}\; \gamma^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X_\mu$$

$$\delta S_p = -\frac{T}{2}\cdot 2\int d\tau \int d\sigma \sqrt{-\gamma}\; \gamma^{\alpha\beta}\, \partial_\alpha X^\mu\, \partial_\beta(\delta X_\mu)$$

now $$\int d\tau \cdot \theta \cdot \partial_\tau\, \delta X_\mu = -\int d\tau\, (\partial_\tau \theta)\, \delta X_\mu + \theta\, \delta X_\mu \Big|_{\tau_i}^{\tau_f}$$

$$\int d\sigma\, \theta\, \partial_\sigma\, \delta X_\mu = -\int d\sigma\, (\partial_\sigma \theta)\, \delta X_\mu + \theta\, \delta X_\mu \Big|_{\sigma=0}^{\sigma=\pi}$$

but can set $\delta X_\mu \Big|_{\tau_i} = \delta X_\mu \Big|_{\tau_f} = 0.$

$$\Rightarrow \delta S_p = +T\int d\tau\, d\sigma \left(\partial_\beta \sqrt{-\gamma}\; \gamma^{\alpha\beta}\, \partial_\alpha X^\mu\right)\delta X_\mu$$

$$- T\int d\tau \sqrt{-\gamma}\; \underbrace{\gamma^{\alpha\beta}\, \partial_\alpha X^\mu}_{\partial^\sigma X^\mu}\; \delta X_\mu \Big|_{\sigma=0}^{\sigma=\pi} \qquad (\beta \to \sigma)$$

bdry term: $$-T\int d\tau \sqrt{-\gamma}\; \partial^\sigma X^\mu\, \delta X_\mu \Big|_{\sigma=0}^{\sigma=\pi}$$

15.75

bdry term vanishes if:

(A) $\partial^\sigma X^\mu(\tau,0) = \partial^\sigma X^\mu(\tau,\pi) = 0$ (Neumann)

(B) $X^\mu(\tau,0) = X_0^\mu$ constants (Dirichlet)
$X^\mu(\tau,\pi) = X_\pi^\mu$

(C) $X^\mu(\tau,0) = X^\mu(\tau,\pi)$
$\partial^\sigma X^\mu(\tau,0) = \partial^\sigma X^\mu(\tau,\pi)$ (periodic)
$\sqrt{-\gamma}(\tau,0) = \sqrt{-\gamma}(\tau,\pi)$

to restate part of our result,

recall definitions of energy-momentum tensor

GR $\quad T^{\alpha\beta} = \frac{c}{\sqrt{-g}} \frac{\delta S}{\delta g_{\alpha\beta}}$ $\quad$ c usu. -2

we take $c = -4\pi$.

There is also a Noether-type (canonical) definition. Since we'll need Noether's thm, let's recall it now.

~~[illegible]~~

recall Noether procedure

if under some variation $q \to q + \delta q$

the Lagrangian is invariant,

so that

$$\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}} \underbrace{\delta(\dot{q})}_{\frac{d}{dt}\delta q} = 0$$

then

$$\frac{\partial L}{\partial \dot{q}}\delta q \quad \text{is a conserved charge}$$

as can be checked using E-L

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}.$$

more generally,

for $\mathcal{L}(\phi^a, \partial_\alpha \phi^a)$, under $\phi^a \to \phi^a + \delta\phi^a(\xi^\alpha)$

if $\delta\mathcal{L}(\phi^a, \partial_\alpha \phi^a) = \partial_\alpha V^\alpha_a$ ← labels transformation $\delta\phi^a$

then

(1) E-L $\quad \partial_\alpha\left(\dfrac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi^a)}\right) = \dfrac{\partial\mathcal{L}}{\partial\phi^a}$

(2) $$\partial_\alpha V^\alpha_a = \delta_a\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi^a}\delta\phi^a + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi^a)}\underbrace{\delta(\partial_\alpha\phi^a)}_{\partial_\alpha\delta\phi^a}$$

$$= \partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi^a)}\right)\delta\phi^a + \left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi^a)}\right)\partial_\alpha\delta\phi^a$$

$$= \partial_\alpha\left[\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi^a)}\delta\phi^a\right]$$

$$\Rightarrow \quad 0 = \partial_\alpha\left[\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi^a)}\delta\phi^a - V^\alpha_a\right] \equiv \partial_\alpha j^\alpha_a$$

18.5

if $\mathcal{L}$ does not depend explicitly on ξ'^{α}, then

under $\xi'^{\alpha} = \xi'^{\alpha} + \epsilon^{\alpha}$

$$\phi^a(\xi'^{\alpha} + \epsilon^{\alpha}) \approx \phi^a(\xi'^{\alpha}) + \epsilon^{\alpha} \partial_{\alpha} \phi^a$$

can show

$$\delta\mathcal{L} = \epsilon^{\beta} \frac{\partial}{\partial \xi'^{\alpha}} \left(\delta^{\alpha}{}_{\beta} \mathcal{L} \right)$$ total derivative

$\Rightarrow \exists$ conserved currents

$$j^{\alpha}{}_{\beta} = \frac{\partial \mathcal{L}}{\partial(\partial_{\alpha}\phi^a)} \partial_{\beta}\phi^a - \delta^{\alpha}_{\beta}\mathcal{L}$$

and $j^{\alpha\beta} \Leftrightarrow T^{\alpha\beta}$.

and, GR + Noether definitions agree.

18.75

Also useful to notice that under constant spacetime translations

$$X^\mu \to X^\mu + \epsilon^\mu \qquad (\delta X^\mu(\tau,\sigma) = \epsilon^\mu)$$

the action is invariant.

here $V^\mu = 0$

$$\Rightarrow \epsilon^\mu j^\alpha_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\mu)} \delta X^\mu = \epsilon^\mu \frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\mu)}$$

$$\Rightarrow \quad j^\alpha_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\mu)}$$

$$j^\tau_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu}$$

$$j^\sigma_\mu = \frac{\partial \mathcal{L}}{\partial X'^\mu}$$

space integral of time component gives charge:

$$Q = \int d^{d-1}x \; j^0$$

here $Q = \int d\sigma \; j^\mu_\tau$ and, Q = spacetime momentum.