AdS/CFT and GKPW

Outline

i) AdS/CFT is a field theory dream
ii) AdS/CFT and GKPW
iii) Calculating a 2-point function

References

Aharony, Gubser, Maldacena, Ooguri, Oz : "Large N Field theories, String Theory and Gravity", hep-th/9905111

Maldacena : "The Large N limit of Superconformal Field theories and supergravity", hep-th/9711200

Gubser, Klebanov, Polyakov : "Gauge theory Correlators from non-Critical String theory", hep-th/9802109

Witten : "Anti De Sitter Space and holography", hep-th/9802150

McGreevy : "Holographic Duality with a View Toward many-body physics", hep-th/0909.0518

AdS/CFT

We consider a 4D QFT with fields $\phi(x)$. All observables are encoded in various correlation functions of operators $\mathcal{O}(x)$ in the QFT. We can consider eg $\mathcal{O}(x) = \phi(x)$ we calculating the propagator $\langle \phi(x) \phi(y) \rangle$ or eg $\mathcal{O}(x) = \phi^2(x)$ when calculating the anomalous dimension of the mass operator, see for example eq 12.117 Peskin & Schröder for the case of $\mathcal{O} = \phi^2$ in ϕ^4-theory.

A formal way of calculating correlation functions is to add a background source $J(x)$ to $\mathcal{O}(x)$, thus modifying the partition function $Z_0 \to Z[J]$, $Z_0 = Z[0]$:

$$Z[J] = \int \mathcal{D}[\phi] \exp\left\{ i S[\phi] + i \int d^4x \, J(x) \mathcal{O}(x) \right\}$$

Then say the two-point function is given by

$$\langle \mathcal{O}(x) \mathcal{O}(y) \rangle \equiv \frac{1}{\int \mathcal{D}[\phi] e^{iS}} \cdot \int \mathcal{D}[\phi] \, \mathcal{O}(x) \mathcal{O}(y) \, e^{iS[\phi]}$$

$$= \frac{1}{Z[0]} (-i)^2 \left. \frac{\delta^2 Z[J]}{\delta J(x) \delta J(y)} \right|_{J=0}$$

We now consider an example where we can do this.

Example consider a free scalar $S = \int d^4x \, \frac{1}{2} \phi(x) (\partial^2 - m^2) \phi(x)$

Then the shift $\phi'(x) = \phi(x) + \int d^4y \, G(x,y) J(y)$ where G is the Greens function: $(\partial^2 - m^2) G(x,y) = \delta(x-y)$ let us perform the path integral exactly

$$
\begin{aligned}
Z[J] &= \int \mathcal{D}\phi \, \exp\left(i \int \tfrac{1}{2} \phi (\partial^2 - m^2) \phi + i \int J \phi \right) \\
&= \int \mathcal{D}\phi' \exp\left(i \int \tfrac{1}{2} \phi' (\partial^2 - m^2) \phi' - \tfrac{i}{2} \int dx\, dy \, J(x) G(x,y) J(y) \right) \\
&\equiv Z[0] \exp\left(-\tfrac{1}{2} \int dx\, dy \, J(x) \, i G(x,y) \, J(y) \right) ,
\end{aligned}
$$

Thus we get

$$
\langle \phi(x) \phi(y) \rangle = \frac{1}{Z[0]} (-i)^2 \left. \frac{\delta^2 Z[J]}{\delta J(x) \delta J(y)} \right|_{J=0} = i G(x,y) ,
$$

just as it should be. See eq 9.39 Peskin & Schroeder □

Now the ultimate dream in every QFT is to perform the path integral completely so that everything is determined:

$$
Z[J] \overset{\text{DREAM}}{\equiv} \exp(-i E[J]) ,
$$

where $E[J]$ is known explicitly!

Now AdS/CFT does exactly this for you:

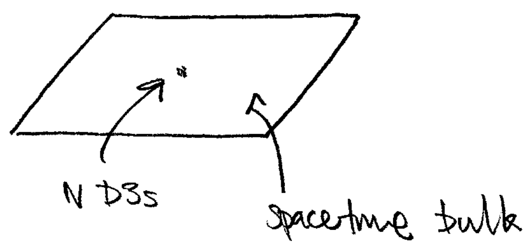
$$
Z_{CFT}[J] \overset{\text{AdS/CFT}}{=} \exp\left(i S_{AdS}[J] \right)
$$

In the following I will explain the right-hand side above.

The duality

consider IIB string theory. It contains closed and open strings plus D-branes where open strings end.

Now consider a stack of N D3-branes in 10D spacetime



The open strings on the D3s is described by $\mathcal{N}=4$, $U(N)$, super-Yang-Mills (in the low-energy limit)
The closed strings in the bulk are described by supergravity (in the low energy limit).

The action is $S = S_{bulk} + S_{brane} + S_{int}$.

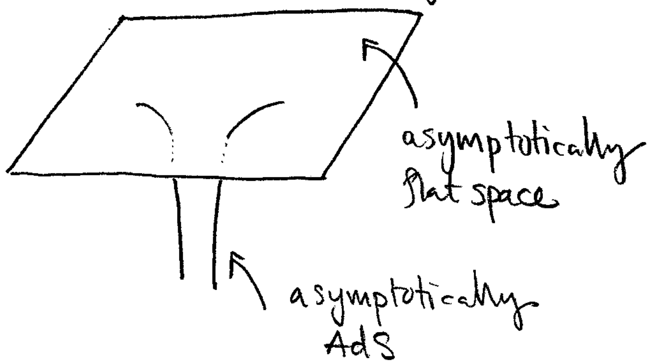
Now take the low energy limit sending $M^{(10)}_{Planck} \to \infty$
Then $S_{bulk} \to S_{\substack{free\\gravity}}$

$S_{brane} \to S_{\substack{pure\\SYM}}$

$S_{int} \to 0$

Thus we have two decoupled systems.

Now consider the back reaction of the N D3-branes of the spacetime geometry



Solving the IIB equations of motion for the fields g_{MN}, B_{MN}, Φ ; C_0, C_2, C_4 one finds

$$ds^2 = \frac{1}{\sqrt{H(r)}}\left(-dt^2 + d\vec{x}^{\,2}\right) + \sqrt{H(r)}\left(dr^2 + r^2 d\Omega_5^2\right)$$

$$F_5 = dC_4 = (1+*)\left(d^4x \wedge d(H^{-1})\right), \quad e^{-2\Phi} = g_s^{-2} = \text{const}$$

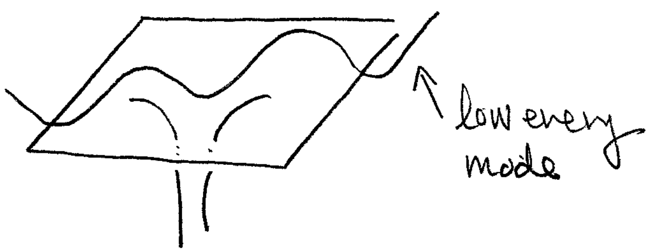
$$H = 1 + \frac{L^4}{r^4}, \quad L^4 = 4\pi g_s (\alpha')^2 N$$

Asymptotically far away we have flat space since $H \to 1$. But in small r region $H \approx L^4/r^4$ and we get $AdS_5 \times S^5$

$$ds^2 \to ds^2_{AdS_5 \times S^5} = \frac{r^2}{L^2}\left(-dt^2 + d\vec{x}^{\,2}\right) + \frac{L^2}{r^2} dr^2 + L^2 d\Omega_5^2$$

We now take the low energy limit of also this theory. In this limit the two asymptotics decouple.

Strings moving in the throat can't make it up the gravitational potential and strings moving in flat space does not see the throat region in the low energy limit, heuristically when $E \to 0$, $\lambda \to \infty$ and a mode does not see the throat region of size $R \sim$ finite.



If we mod out the flat space of the two descriptions we find a description of the same theory, thus we have a duality between

$\{$Strings in $AdS_5 \times S^5\}$ and $\{\mathcal{N}=4$, SYM in 4D$\}$.

The duality can be made precise, matching Hilbert space, correlation functions etc. Above I followed MAG00.

The matching of correlation functions was done by GKPW, which we now turn to explaining.

Now the theories must have the same partition functions

$$\boxed{Z_{CFT}[J] = Z_{AdS}[\phi(J)]}$$

From the fact that the gravity theory is classical in this limit we consider we have $Z_{AdS}[\phi] = e^{iS[\phi_{cl}]}$

Thus we have found exactly what we want, our dream,

$$\boxed{Z_{CFT}[J] = e^{i S_{AdS}[\phi_{cl}(J)]} + \mathcal{O}\left(\tfrac{1}{N^2}\right)}$$

Now I just have to tell you how we match operators between the two theories.

GKPW:

For an operator $\mathcal{O}$ in the CFT we have a field ϕ_{cl} in AdS with $\boxed{\lim_{\text{boundary}} \phi_{cl} = J.}$

I will now give examples of operators and fields. First consider the IIB action S_{IIB}. Now make a Kaluza-Klein reduction on the S^5 (for the experts $S^5 = SO(6)/SO(5)$). Then for the zero-modes we get a 5D effective action

$$S_{5D} = \frac{N^2}{8\pi^2 L^3} \int d^5x \left[R - \frac{1}{2}(\nabla \Phi)^2 - \frac{L^2}{8} \operatorname{tr} F^2 + \dots \right]$$

R gravity, see KRN. Φ dilaton, $F^a_{\mu\nu}$ $SO(6)$ field strength,

The matching is:

CFT $\mathcal{O}$	AdS field	
$\operatorname{tr} F^2$	Φ	
J^μ_a	A^μ_a	, J^μ_a is the $SU(4) \cong SO(6)$ R-current
$T_{\mu\nu}$	$g_{\mu\nu}$	

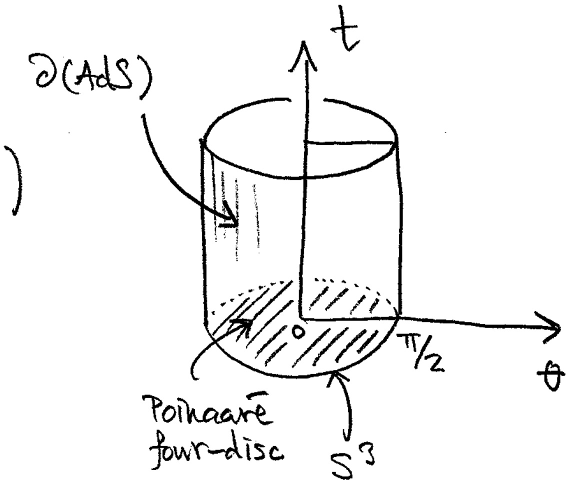
I will now consider a toy model with some operator $\mathcal{O}$ in the CFT dual to a scalar field ϕ in AdS and then calculate $\langle \mathcal{O}\mathcal{O} \rangle$ using the GKPW prescription.
I will follow McGreevy.

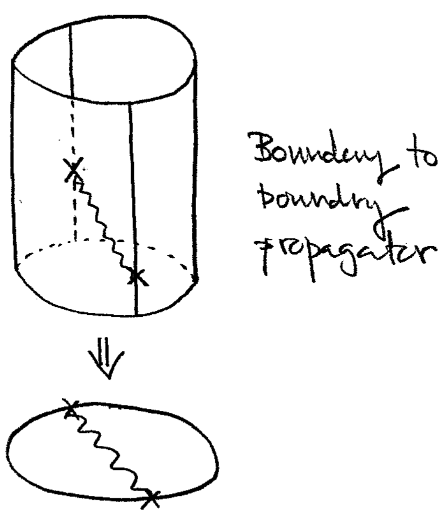
First I will remind you a little bit about the AdS_5 geometry.

AdS

Global coordinates

$$ds^2 = \frac{L^2}{\cos^2\theta}\left(-dt^2 + d\theta^2 + \sin^2\theta \, d\Omega_3^2\right)$$

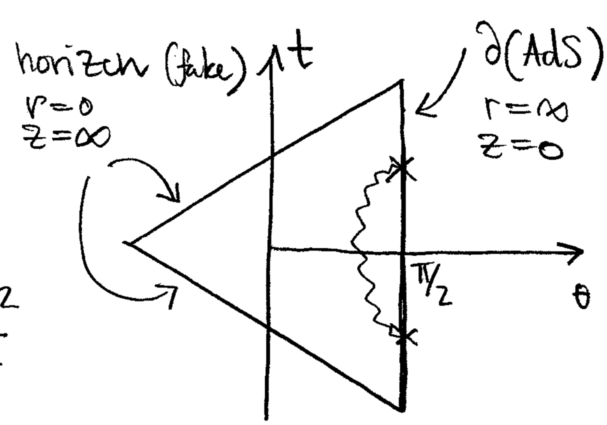




Poincaré coordinates

$$ds^2 = \frac{r^2}{L^2}\left(-dt^2 + d\vec{x}^2\right) + \frac{L^2}{r^2} dr^2$$

$$ds^2 = \frac{L^2}{z^2}\left(-dt^2 + d\vec{x}^2 + dz^2\right), \quad z = \frac{L^2}{r}$$



$\langle OO \rangle$

Consider now a scalar field in the AdS_{d+1} bulk with $d=4$ in mind

$$S = -\frac{N^2}{2}\int d^{d+1}x\sqrt{-g}\left[g^{AB}\nabla_A\phi\nabla_B\phi + m^2\phi^2 + \text{interactions}\right],$$

we neglect the interactions when calculating the 2-point function.

We now have to solve the eom to find ϕ_{cl} (with $\left.\frac{\delta S}{\delta\phi}\right|_{\phi=\phi_{cl}}=0$) and then evaluate the action at ϕ_{cl} to obtain:

$$Z[J] = \exp\left(iS[\phi_{cl}]\right).$$

Now we have

$$\sqrt{-g}\,g^{AB}\partial_A\phi\partial_B\phi = \partial_A\left[\sqrt{-g}\,g^{AB}\phi\partial_B\phi\right] - \phi\partial_A\left[\sqrt{-g}\,g^{AB}\partial_B\phi\right]$$
$$\equiv \sqrt{-g}\left(\nabla_A j^A - \phi\Box\phi\right),$$

where $j^A = g^{AB}\phi\partial_B\phi$, $\quad \Box = \frac{1}{\sqrt{-g}}\partial_A\left(\sqrt{-g}\,g^{AB}\partial_B\right)$

$$\therefore\quad S = -\frac{N^2}{2}\int d^{d+1}x\sqrt{-g}\left\{\phi(-\Box+m^2)\phi + \nabla_A j^A\right\}$$

Notice that the eom is $(-\Box+m^2)\phi$ such that on-shell

$$S[\phi_{cl}] = -\frac{N^2}{2}\int_{AdS} d^{d+1}x\sqrt{-g}\,\nabla_A j^A$$
$$= -\frac{N^2}{2}\int_{\partial(AdS)} d^dx\sqrt{-h}\,n_A j^A$$

That is, $S[\phi_{cl}]$ only depends on the value of $\phi_{cl}(x,z)$ at the boundary $\boxed{\phi_{cl}(x,0) = \phi^0_{cl}(x) \equiv J(x)}$ thus it is easy to take functional derivatives $\frac{\delta}{\delta J(x)}$ of $e^{iS[\phi_{cl}]}$.

In fact we have

$$\int_{AdS} d^{d+1}x \sqrt{g}\, \nabla_A j^A \equiv \int_{AdS} d^{d+1}x\, \partial_A (\sqrt{g}\, j^A) = \int_{\partial(AdS)} d^d x\, \hat{n}_A \sqrt{g}\, j^A ,$$

where $\hat{n}_A = \delta^z_A$

We write this in a more covariant form

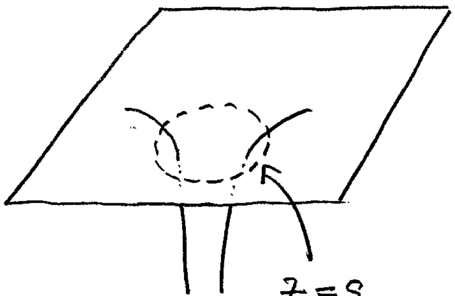
$$\hat{n}_A \sqrt{g} = \hat{n}_A \cdot \sqrt{g_{zz}} \cdot \sqrt{-h} \equiv n_A \sqrt{-h} ,$$

where $n_A = \sqrt{g_{zz}}\, \delta^z_A$ and h is the metric on the boundary $ds^2_{\partial(AdS)} = ds^2 \big|_{z=0}$. Notice that $n^2 = 1$.

$$\therefore \int_{AdS} d^{d+1}x \sqrt{g}\, \nabla_A j^A = \int_{\partial(AdS} d^d x \sqrt{-h}\, n_A j^A ,$$

where $n \cdot J = \phi (n \cdot \nabla) \phi$.

We now discuss the boundary of AdS. The boundary is located at $z \to 0$, $r \to \infty$. But way before r reaches infinity, the approximation $H = 1 + \frac{L^4}{r^4} \sim \frac{L^4}{r^4}$ breaks down, that is, the AdS space is cut off at some finite $r \sim L$ where flat space takes over. We denote the cut-off by $z = \varepsilon$.



Solving the eom

The eom is $(-\Box + m^2)\phi = 0$.

We now consider the following ansatz for the modes

$$\phi(x,z) = e^{ik_\mu x^\mu} \cdot f_k(z) \,, \quad k_\mu x^\mu = -\omega t + \vec{k}\cdot\vec{x}$$

Now we have

$$\Box = \frac{1}{\sqrt{-g}} \partial_A \left(\sqrt{-g}\, g^{AB} \partial_B\right) = \frac{1}{\sqrt{-g}} \partial_z \left(\sqrt{-g}\, g^{zz} \partial_z\right) + g^{\mu\nu} \partial_\mu \partial_\nu$$

with $g^{\mu\nu} = \frac{z^2}{L^2} \eta^{\mu\nu}$, $g^{zz} = \frac{z^2}{L^2}$, $\sqrt{-g} = \left(\frac{L}{z}\right)^{d+1}$ gives now

$$\frac{1}{L^2}\left[+ z^2 k^2 - z^{d+1} \partial_z \left(z^{-d+1} \partial_z \right) + m^2 L^2 \right] f_k(z) = 0$$

We will soon determine the full solutions but let us first consider solutions for $z \sim 0$.

consider ansatz $f_k(z) = z^{\Delta}$ then

$$z^{d+1}\partial_z\left(z^{-d+1}\partial_z f_k\right) = z^{d+1}\partial_z\left(z^{-d+1}\Delta z^{\Delta-1}\right)$$

$$= z^{d+1}\cdot\Delta(\Delta-d)\cdot z^{\Delta-d-1} = \Delta(\Delta-d)f_k$$

$$\therefore \quad \left(z^2k^2 - \Delta(\Delta-d) + m^2L^2\right)f_k = 0$$

In the limit $z \sim 0$ we get

$$\boxed{\Delta(\Delta-d) = m^2L^2}$$

Solving for Δ we get

$$\boxed{\Delta_\pm = \frac{d}{2} \pm \left(\left(\frac{d}{2}\right)^2 + m^2L^2\right)^{1/2}}$$

Now since $\Delta_+ > 0$ this mode always decays in the limit $z \to 0$, thus the behaviour at $z \sim 0$ is determined by the mode Δ_-.

With this in mind we renormalize our fields

$$\boxed{\phi_{cl}^{\epsilon}(x) \equiv \phi_{cl}(x, z=\epsilon) \equiv \epsilon^{\Delta_-}\phi_{cl}^{Ren}(x)},$$

such that $\phi_{cl}^{Ren}(x)$ is finite.

Now for a boundary action of the form

$$S \propto \int_{\partial(AdS)} d^d x \sqrt{-h}\, \phi_{cl}^{\epsilon}\, \mathcal{O}_\epsilon \propto \int_{\partial(AdS)} d^d x\, \epsilon^{-d} \phi_{cl}^{\epsilon}\, \mathcal{O}_\epsilon$$

$$\equiv \int_{\partial(AdS)} d^d x\, \phi_{cl}^{Ren}\, \mathcal{O}^{Ren} ,$$

where we have that

$$\mathcal{O}^{Ren} = \epsilon^{-d} \epsilon^{+\Delta_-} \mathcal{O}_\epsilon = \epsilon^{\Delta_- - d} \mathcal{O} = \epsilon^{-\Delta_+} \mathcal{O}$$

$$\boxed{\mathcal{O}^{Ren} = \epsilon^{-\Delta_+} \mathcal{O}_\epsilon}$$

Now interpreting ϵ as the energy scale we find that $\mathcal{O}$ has scaling dimension Δ_+.

$$\epsilon \frac{d}{d\epsilon} \mathcal{O}_\epsilon = \epsilon \frac{d}{d\epsilon}\left(\mathcal{O}^{Ren} \epsilon^{+\Delta_+}\right) = \Delta_+ \mathcal{O}^{Ren} \epsilon^{\Delta_+} = \Delta_+ \mathcal{O}_\epsilon$$

$$\therefore \quad \frac{d}{d\ln\epsilon} \mathcal{O}_\epsilon = \Delta_+ \mathcal{O}_\epsilon , \qquad \dim \mathcal{O}_\epsilon = \Delta_+ .$$

We will soon show that this is in fact true by actually showing that $\langle \mathcal{O}(x)\mathcal{O}(0)\rangle \sim \frac{1}{|x|^{2\Delta_+}}$.

Now put $f_k = z^{d/2} \cdot g_k$ then we have

$$\left(z^{d+1}\partial_z\left(z^{-d+1}\partial_z\right) - \left[z^2k^2+m^2L^2\right]\right)\left(z^{d/2}g_k\right) = 0$$

$$z^{d+1}\partial_z\left(z^{-d+1}\partial_z\left(z^{d/2}g_k\right)\right) = z^{d+1}\partial_z\left(\frac{d}{2}z^{\underbrace{-d+1+\frac{d}{2}-1}_{-\frac{d}{2}}}g_k + z^{\underbrace{-d+1+\frac{d}{2}}_{-\frac{d}{2}+1}}g_k'\right)$$

$$= z^{d+1}\left(-\left(\frac{d}{2}\right)^2 z^{-\frac{d}{2}-1}g_k + \frac{d}{2}z^{-\frac{d}{2}}g_k' + \left(-\frac{d}{2}+1\right)z^{-\frac{d}{2}}g_k' + z^{-\frac{d}{2}+1}g_k''\right)$$

$$= -\left(\frac{d}{2}\right)^2 z^{\frac{d}{2}}g_k + z^{\frac{d}{2}+1}g_k' + z^{\frac{d}{2}+2}g_k''$$

$$= z^{d/2}\left(-\left(\frac{d}{2}\right)^2 g_k + zg_k' + z^2g_k''\right)$$

This implies that

$$z^2g_k'' + zg_k' - \left[z^2k^2 + \left(m^2L^2 + \left(\frac{d}{2}\right)^2\right)\right]g_k = 0$$

This is Bessel's equation

$$z^2g_k'' + zg_k' - \left[z^2k^2 + \nu^2\right]g_k = 0,$$

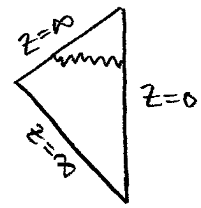
with $\nu = \left(m^2L^2 + \left(\frac{d}{2}\right)^2\right)^{1/2}$.

$\underline{k^2 > 0}$ (spacelike)

$$g_k = A_K K_\nu(kz) + A_I I_\nu(kz) \quad , \quad K, I \text{ modified Bessel functions}$$

Now $I_\nu(kz) \sim e^{kz}, \; z \to \infty$

$K_\nu(kz) \sim e^{-kz}, \; z \to \infty$



$\Rightarrow A_I = 0$ by demanding regularity at $z \to \infty$.

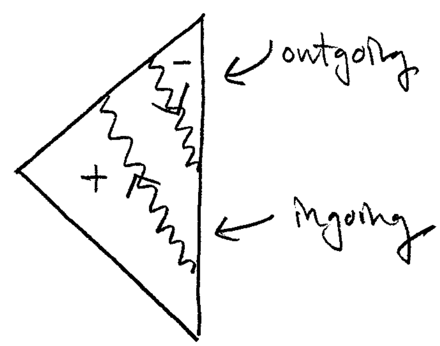
$\underline{k^2 < 0}$ (timelike)

let $q^2 = -k^2$ then $k^2 z^2 = -q^2 z^2 = (iqz)^2$ such that

$$g_k = B_K K(iqz) + B_I I(iqz)$$

Now $I_\nu(iqz) \to e^{iqz} \quad , \; z \to \infty$

$K_\nu(iqz) \to e^{-iqz} \quad , \; z \to \infty$



Phase of wave packet:

$$i\varphi(k,w) = i k_\mu x^\mu \pm i q z = -i\left[wt - \vec{k}\cdot\vec{x} \mp qz\right]$$

consider fixed $\vec{x}$ but moving in z then

$$0 = w\,dt \mp q\,dz \quad , \quad \frac{dz}{dt} = \pm \frac{w}{q}$$

$\therefore$ + infalling , − outgoing

Now specifying what combinations of + and − modes we take we will get different Greens functions for $\langle \Theta \Theta \rangle$.

Taking only + modes gives us the retarded Greens function iG^R corresponding to forward propagation in time, while − modes gives us the advanced Greens function iG^A corresponding to backward propagation in time.

We consider only + modes.

Normalizing the mode $f_k(z=1)=1$ gives us

$$f_k(z) = \frac{z^{d/2} K_\nu(kz)}{\epsilon^{d/2} K_\nu(k\epsilon)}$$

and we find a general solution

$$\phi_{cl}(x,z) = \int \frac{d^d k}{(2\pi)^d} \phi^{\epsilon}_{cl}(k) \cdot e^{ikx} \cdot f_k(z),$$

for $k^2 > 0$ and replacing k with iq for $k^2 < 0$.

The last step is now evaluating the action at ϕ_{cl}

$$S[\phi_{cl}] = -\frac{N^2}{2} \int_{\partial(AdS)} d^d x \sqrt{-h}\, \phi_{cl} (n \cdot \nabla) \phi_{cl}$$

and then varying twice with respect to $J(x) = \phi^{Ren}_{cl}(x)$.

Now we have that $n \cdot \nabla = n^z \partial_z = \frac{z}{L} \partial_z$ so that

$$\phi_{cl}(n\cdot\nabla)\phi_{cl} = \frac{z}{L}\phi_{cl}\partial_z\phi_{cl}$$

$$= \frac{z}{L} \cdot \int \frac{d^dk_1}{(2\pi)^d}\int\frac{d^dk_2}{(2\pi)^d}\,\phi^{\epsilon}_{cl}(k_1)\phi^{\epsilon}_{cl}(k_1)\,e^{i(k_1+k_2)x}\, f_{k_1}(z)\,\partial_z f_{k_2}(z)\Big|_{z=\epsilon}$$

Now we use that $\sqrt{h} = \left(\frac{L}{z}\right)^d$ and perform the x integral

$$S[\phi_{cl}] = -\frac{N^2}{2}\int\frac{d^dk}{(2\pi)^d}\cdot\left(\frac{L}{z}\right)^{d-1}\cdot\phi^{\epsilon}_{cl}(-k)\,\phi^{\epsilon}_{cl}(k)\,f_{-k}(z)\,\partial_z f_k(z)\Big|_{z=\epsilon}$$

$$\equiv +\frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\,\phi^{Ren}_{cl}(-k)\,\mathcal{F}(k)\,\phi^{Ren}_{cl}(k),$$

where $\mathcal{F}(k) = -N^2 L^{d-1}\cdot\epsilon^{-d+1+2\Delta_-}\, f_{-k}(z)\,\partial_z f(z)\Big|_{z=\epsilon}$

Now $\mathcal{F}$ is calculated in McGreevy with the result

$$\mathcal{F}(x) = \int\frac{d^dk}{(2\pi)^d}e^{-ikx}\mathcal{F}(k) \propto \frac{1}{x^{2\Delta_+}}$$

Thus we find

$$\langle\mathcal{O}(-k)\,\mathcal{O}(k)\rangle = \mathcal{F}(k)$$

or equivalently

$$\langle\mathcal{O}(x)\,\mathcal{O}(0)\rangle = \mathcal{F}(x) \propto \frac{1}{x^{2\Delta_+}},$$

confirming $\dim\mathcal{O} = \Delta_+$.