Some Algebraic Structures

Set: thought of as any collection of objects.

Magma or groupoid: a set $\mathcal{S}$ with any single binary operation. For here we denote it by $+$. It takes $\mathcal{S} \times \mathcal{S} \rightarrow \mathcal{S}$, i.e. $\forall\, a, b \in \mathcal{S},\ \exists\, c \in \mathcal{S}$ so that $c = a + b$.

Semigroup: an associative magma $\mathcal{M}\}$, i.e. $\forall\, a, b, c \in \mathcal{M}\},\ a + (b + c) = (a + b) + c$.

Monoid: a semigroup $\mathcal{SG}$ with an identity element, i.e. $\exists\, a_0 \in \mathcal{SG}$ so that $\forall\, a \in \mathcal{SG}, a + a_0 = a$.

Group: a monoid $\mathcal{M}\wr$ in which every element has an inverse, i.e. $\forall\, a \in \mathcal{M}\wr,\ \exists\, b \in \mathcal{M}\wr$ so that $a + b + c = c\ \forall\, c \in \mathcal{M}\wr$.

Abelian group: a commutative group $\mathcal{G}$, i.e. $\forall\, a, b \in \mathcal{G}, a + b = b + a$.

Ring: an Abelian group $\mathcal{AG}$ with a monoid operation called multiplication, here denoted as $\times$, satisfying distributivity with respect to the binary operation of $\mathcal{AG}$, i.e. the set $\mathcal{S}$ of $\mathcal{AG}$ with $+$ is an Abelian group, $\mathcal{S}$ with $\times$ is a monoid, and furthermore $\forall\, a, b, c \in \mathcal{S}, a \times (b + c) = a \times b + a \times c$.

Commutative ring: a ring $\mathcal{R}$ whose multiplication is commutative, i.e. $\forall\, a, b \in \mathcal{R}, a \times b = b \times a$.

Field: a commutative ring $\mathcal{CR}$ where no element is simultaneously identity element of $+$ and of $\times$, and in which each element has an inverse of $\times$, i.e. $\forall\, a \in \mathcal{CR}$ for which $\exists\, b \in \mathcal{CR}$ so that $a + b \neq a,\ \exists\, c \in \mathcal{CR}$ so that $(a \times c) \times d = d\ \forall\, d \in C\mathcal{R}$.

Module over a ring: an Abelian group $\mathcal{AB}$ with its binary operation, here called $\oplus$, and a ring $\mathcal{R}$ with its operations $+$ and $\times$ which has the following properties: (A) There is an additive unary operation called scalar multiplication for every element of $\mathcal{R}$, i.e. $\forall\, x \in \mathcal{R},\ \exists\, x\cdot : \mathcal{S} \rightarrow \mathcal{S}$ so that $\forall\, a, b \in \mathcal{S}, x \cdot (a \oplus b) = (x \cdot a) \oplus (x \cdot b)$, (B) The scalar multiplication is linked to multiplication in $\mathcal{R}$ by an associativity condition, i.e. $\forall\, x, y, z \in R, a \in \mathcal{S}, x \cdot [(y \times z) \cdot a] = (x \times y) \cdot (z \cdot a)$.

Algebra a module $\mathcal{M}$ over a ring $\mathcal{R}$ together with a binary operation $\otimes$ on $\mathcal{M}$ that is bilinear with respect to the scalar multiplication, i.e. $\forall\, a, b \in \mathcal{M},\ \exists\, c \in \mathcal{M}$ with $c = a \otimes b$ and $\forall\, x, y \in \mathcal{R}, a, b, c \in \mathcal{M}, [(x \cdot a) \oplus (y \cdot b)] \otimes c = [(x \cdot a) \otimes c] \oplus [(y \cdot b) \otimes c]$ and $a \otimes [(x \cdot b) \oplus (y \cdot c)] = [(x \cdot a) \otimes b] \oplus [(y \cdot a) \otimes c]$.

Lie algebra: an algebra $\mathcal{A}$ where the binary operation $\otimes$ satisfies $\forall\, a \in \mathcal{A}, a \otimes a = 0$ and the Jacobi identity, i.e. $\forall\, a, b, c \in \mathcal{A}, [a \otimes (b \otimes c)] \oplus [b \otimes (c \otimes a)] \oplus [c \otimes (a \otimes b)] = 0$.

Associative algebra: an algebra $\mathcal{A}$ where the module's binary operator $\otimes$ is associative, i.e. $\forall\, a, b, c \in \mathcal{A}, a \otimes (b \otimes c) = (a \otimes b) \otimes c$.

Commutative algebra: an associative algebra $\mathcal{AA}$ where the module's multiplication is commutative, it is called a multiplication, i.e. $\forall\, a, b \in \mathcal{AA}, a \otimes b = b \otimes a$.

Differential algebra: a commutative algebra $\mathcal{CA}$ with a differentiation $\partial : \mathcal{CA} \rightarrow \mathcal{CA}$, i.e. $\forall\, a, b \in \mathcal{CA},\ \partial(a \oplus b) = \partial a \oplus \partial b$ and $\partial(a \otimes b) = (a \otimes \partial b) \oplus (b \otimes \partial a)$.