Some Algebraic Structures

Set: thought of as any collection of objects.

**Magma or groupoid:** a set $S$ with any single binary operation. For here we denote it by +. It takes $S \times S \rightarrow S$, i.e. $\forall a, b \in S$, $\exists c \in S$ so that $c = a + b$.

**Semigroup:** an associative magma $M$, i.e. $\forall a, b, c \in M$, $a + (b + c) = (a + b) + c$.

**Monoid:** a semigroup $S$ with an identity element, i.e. $\exists a_0 \in S$ so that $\forall a \in S, a + a_0 = a$.

**Group:** a monoid $M$ in which every element has an inverse, i.e. $\forall a \in M$, $\exists b \in M$ so that $a + b + c = c \forall c \in M$.

**Abelian group:** a commutative group $G$, i.e. $\forall a, b \in G, a + b = b + a$.

**Ring:** an Abelian group $AG$ with a monoid operation called multiplication, here denoted as $\times$, satisfying distributivity with respect to the binary operation of $AG$, i.e. the set $S$ of $AG$ with $+$ is an Abelian group, $S$ with $\times$ is a monoid, and furthermore $\forall a, b, c \in S, a \times (b + c) = a \times b + a \times c$.

**Commutative ring:** a ring $R$ whose multiplication is commutative, i.e. $\forall a, b \in R, a \times b = b \times a$.

**Field:** a commutative ring $CR$ where no element is simultaneously identity element of $+$ and of $\times$, and in which each element has an inverse of $\times$, i.e. $\forall a \in CR$, $\exists b \in CR$ so that $a + b \neq a$, $\exists c \in CR$ so that $(a \times c) \times d = d \forall d \in CR$.

**Module over a ring:** an Abelian group $M$ with its binary operation, here called $\oplus$, and a ring $R$ with its operations $+$ and $\times$ which has the following properties: (A) There is an additive unary operation called scalar multiplication for every element of $R$, i.e. $\forall x \in R, \exists x: S \rightarrow S$ so that $\forall a, b \in S, x \cdot (a \oplus b) = (x \cdot a) \oplus (x \cdot b)$, (B) The scalar multiplication is linked to multiplication in $R$ by an associativity condition, i.e. $\forall x, y, z \in R, a \in S, x \cdot [(y \times z) \cdot a] = (x \times y) \cdot (z \cdot a)$.

**Algebra** a module $M$ over a ring $R$ together with a binary operation $\otimes$ on $M$ that is bilinear with respect to the scalar multiplication, i.e. $\forall a, b \in M, c \in M$ with $c = a \otimes b$ and $\forall x, y \in R, a, b, c \in M, [(x \cdot a) \oplus (y \cdot b)] \otimes c = [(x \cdot a) \otimes c] \oplus [(y \cdot b) \otimes c]$ and $a \otimes [(x \cdot b) \oplus (y \cdot c)] = [(x \cdot a) \otimes b] \oplus [(y \cdot a) \otimes c]$.

**Lie algebra:** an algebra $A$ where the binary operation $\otimes$ satisfies $\forall a \in A, a \otimes a = 0$ and the Jacobi identity, i.e. $\forall a, b, c \in A, [a \otimes (b \otimes c)] \oplus [b \otimes (c \otimes a)] \oplus [c \otimes (a \otimes b)] = 0$.

**Associative algebra:** an algebra $A$ where the module’s binary operator $\otimes$ is associative, i.e. $\forall a, b, c \in A, a \otimes (b \otimes c) = (a \otimes b) \otimes c$.

**Commutative algebra:** an associative algebra $AA$ where the module’s multiplication is commutative, it is called a multiplication, i.e. $\forall a, b \in AA, a \otimes b = b \otimes a$.

**Differential algebra:** a commutative algebra $CA$ with a differentiation $\partial: CA \rightarrow CA$, i.e. $\forall a, b \in CA, \partial(a \otimes b) = \partial a \otimes \partial b$ and $\partial(a \otimes b) = (a \otimes \partial b) \oplus (b \otimes \partial a)$.