Derivation of black body radiation

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The first quantum property discussed was that of light by Max Plank (1900).

By proposing light quanta he derived the function $f\left(\frac{\nu}{T}\right)$

Derivation:

1. Represent the black body as a black body box.
2. In order to find what radiation escapes from the hole, compute the energy in the box in the frequency interval from ν to ν**+d**ν.
3. Derive the number of radiation modes for this interval in the black body box: $dZ = g(\nu)d\nu$
4. Use the fact from statistical mechanics that for a temperature T, the probability for system to have energy E is given by **exp(-E/kT)** .

5a. Initially: Each radiation mode can have arbitrary intensity

5b. Plank: **Assume the each mode can only have an energy that is a multiple of some ν–dependent quantum** $\varepsilon(\nu)$.

6. Compute the average energy per mode and sum over all modes.

Result: Wien's displacement law and Stefan's law

Max Plank, 1858-1947
Nobel Prize in Physics, 1918

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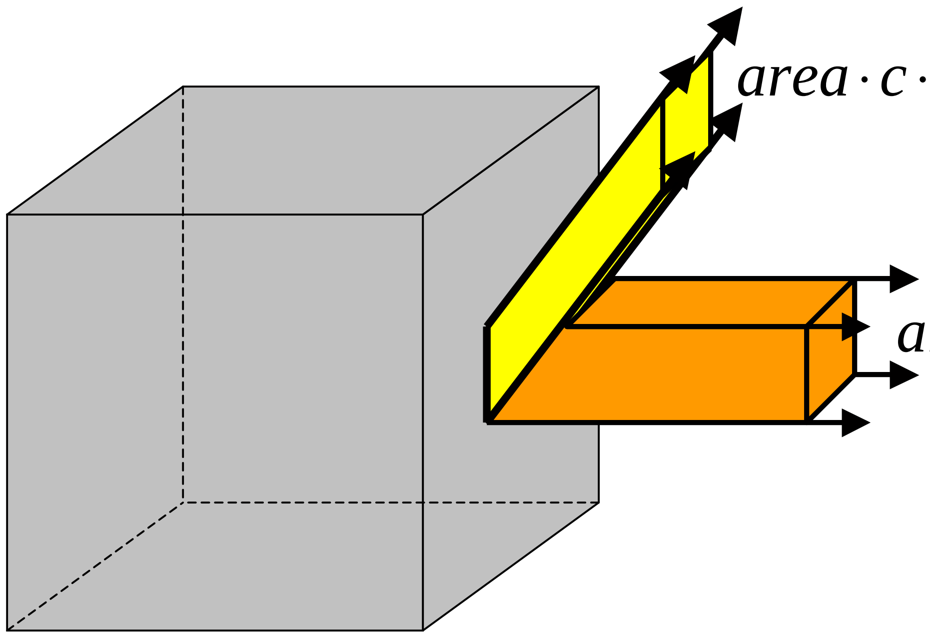
Energy in a black body box and radiation emerging from it

u(θ,ϕ) = Energy density of light with direction in [θ,θ+dθ] and [ϕ,ϕ+dϕ] in frequency interval dν



Assuming isotropy: $u(\vartheta,\varphi) = u(0,0)$

Energy density $$u = \int_0^{2\pi}\int_0^{\pi} u(\vartheta,\varphi)\sin(\vartheta)\,d\vartheta\, d\varphi = 4\pi \cdot u(0,0)$$

Emitted power per area $$R_T = c\int_0^{2\pi}\int_0^{\pi/2} u(\vartheta,\varphi)\cos(\vartheta)\sin(\vartheta)\,d\vartheta\, d\varphi = \tfrac{c}{4}u$$

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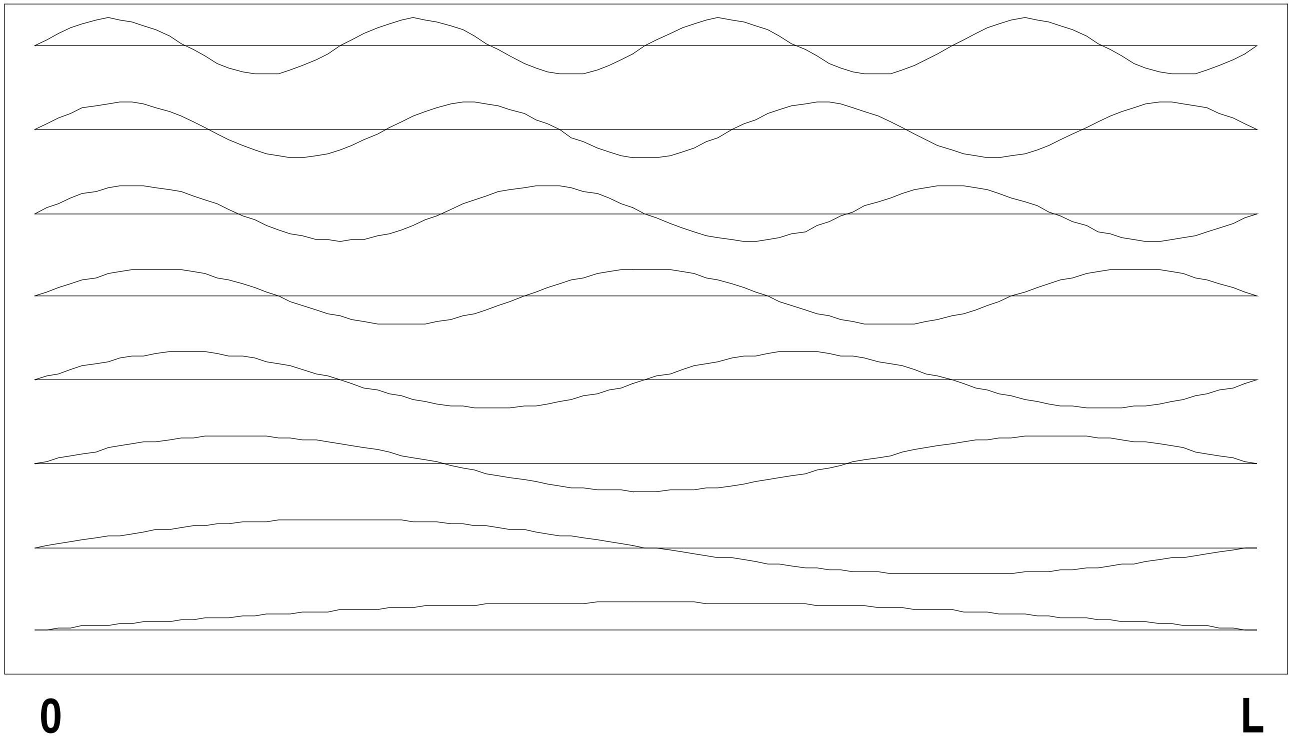
Electromagnetic modes in a black body box



For any polarization direction $\vec{E}_{pol}$, there is a mode for the standing electric wave:

$$\vec{E} = \vec{E}_{pol} \sin(n_x \tfrac{\pi}{L} x) \sin(n_y \tfrac{\pi}{L} y) \sin(n_z \tfrac{\pi}{L} z) \sin(2\pi\nu \cdot t)$$

Any field direction can be produced from 2 perpendicular polarization states

Result: $dZ = 8\pi(\tfrac{L}{c})^3 \nu^2 d\nu$

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Average energy in each mode for a classical wave

In statistical mechanics one learns that at temperature T, the probability P for a system to have energy H is:

$$P \propto e^{-\frac{H}{kT}}\,, \quad k = 1.381 \cdot 10^{-23}\,\tfrac{J}{K}$$

Since P is normalized to 1, $P = e^{-\frac{H}{kT}} \dfrac{1}{\sum\limits_{possible.H} e^{-\frac{H}{kT}}}$

Average energy for each radiation mode can now be computed and turns out to be independent of frequency ν. $< H >= kT$

This hints at a self-consistent picture, since the electrons in the walls that produce the electromagnetic fields can be modeled as harmonic oscillators. And in an array of harmonic oscillators each one also has an average energy proportional to **kT** independent of ν.

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Ultra-violet catastrophe of black-body radiation

Number of modes in a frequency interval: $dZ = 8\pi(\frac{L}{c})^3 \nu^2 d\nu$

Average energy density in a frequency interval: $u(\nu)d\nu = 8\frac{\pi}{c^3}\nu^2 kTd\nu$

$$R_T(\nu) = u(\nu)\frac{c}{4} = \frac{2\pi}{c^2}k(\frac{\nu}{T})^2 T^3 = T^3 f(\frac{\nu}{T}), \quad \int_0^\infty R_T(\nu)d\nu \to \infty$$

This is the **Rayleigh-Jeans formula**: **it cannot be correct**

John W. Strutt Lord Rayleigh,
(1842-1919, UK)
Nobel Prize in Physics, 1904
For the discovery of Argon

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