Average energy in each mode for light quanta

In statistical mechanics one learns that at temperature T, the probability P for a state to have energy E is:

$$P \propto e^{-\frac{H}{kT}}\ , \quad k = 1.381 \cdot 10^{-23}\ \tfrac{J}{K}$$

Since P is normalized to 1, $P = e^{-\frac{H}{kT}} \frac{1}{\sum_{all\ H} e^{-\frac{H}{kT}}}$

Average energy in this mode, $< H >= \sum_{all\ H} H \cdot e^{-\frac{H}{kT}} \frac{1}{\sum_{all\ H} e^{-\frac{H}{kT}}}$

Energy of the mode is $H = n \cdot \varepsilon(\nu)$ for n light quanta, each with energy $\varepsilon(\nu)$

$$\sum_{all\ H} e^{-\frac{H}{kT}} = \sum_{n=0}^{\infty} [e^{-\frac{\varepsilon(\nu)}{kT}}]^n = \frac{1}{1 - e^{-\frac{\varepsilon(\nu)}{kT}}}\ , \quad \sum_{all\ H} H e^{-\beta H} = -\frac{d}{d\beta} \sum_{all\ H} e^{-\beta H} = \frac{\varepsilon\, e^{-\beta\varepsilon}}{(1 - e^{-\beta\varepsilon})^2}$$

Average energy in each mode:

$$< H >= \frac{\varepsilon\, e^{-\beta\varepsilon}}{1 - e^{-\beta\varepsilon}} = \frac{\varepsilon(\nu)}{e^{\varepsilon(\nu)/kT} - 1}$$

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The energy of light quanta and Plank's constant

Number of modes in a frequency interval: $dZ(\nu) = 8\pi(\frac{L}{c})^3 \nu^2 d\nu$

Average energy density in that interval: $u(\nu)d\nu = 8\frac{\pi}{c^3}\nu^2 \frac{\varepsilon(\nu)}{e^{\varepsilon(\nu)/kT}-1} d\nu$

Wien's displacement law: $u(\nu)/T^3$ only depends on ν/T

- $\varepsilon(\nu) = h\nu$ and $$R_T(\nu) = \frac{c}{4}u = T^3 \frac{2\pi h}{c^2} \frac{(\nu/T)^3}{\exp(\frac{h}{k}\frac{\nu}{T})-1}$$

- There is no longer an ultra-violet catastrophe

$$\int_0^\infty R_T(\nu)d\nu = T^4 \frac{2\pi}{c^2}\frac{k^4}{h^3}\int_0^\infty \frac{(\frac{h\nu}{kT})^3}{\exp(\frac{h\nu}{kT})-1}\frac{h}{kT}d\nu = T^4 \frac{2\pi}{c^2}\frac{k^4}{h^3}\int_0^\infty \frac{x^3}{\exp(x)-1}dx = T^4 \frac{2\pi}{c^2}\frac{k^4}{h^3}\frac{\pi^4}{15}$$

$$\int_0^\infty R_T(\nu)d\nu = T^4\sigma \quad \Rightarrow \quad h = \sqrt[3]{\frac{15c^2}{2\pi^5 k^4}\sigma}$$

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- Plank's constant: $h = 6.626 \cdot 10^{-34}\, Js$

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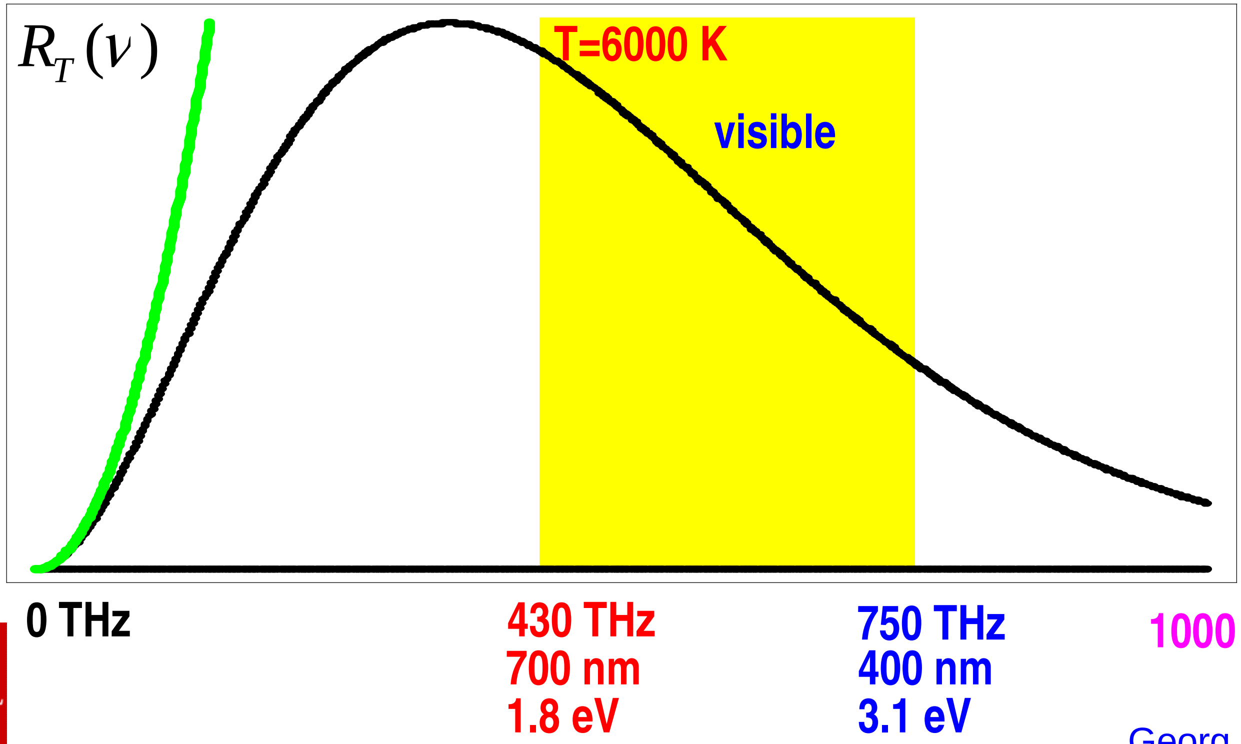
Interpretation

A electromagnetic wave with frequency ν contains light quanta (photons) with the energy **h**ν.
The energy of the wave determines the number of such photons that make up the wave.

- For small ν, the wave can have nearly all energies **nh**ν and one obtains the classical limit

$$R_T(\nu) = T^3 \frac{2\pi h}{c^2} \frac{(\nu/T)^3}{\exp(\frac{1}{k}\frac{\nu}{T})-1} \approx \frac{2\pi h}{c^2} \frac{\nu^3}{h\nu/kT} + O(\tfrac{h\nu}{kT}) = \frac{2\pi}{c^2} kT\nu^2 + O(\tfrac{h\nu}{kT})$$



$$\nu_{max} = 0.06 \tfrac{\text{THz}}{\text{K}} T$$

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