The stationary Schrödinger equation

$$-i\frac{\partial}{\partial x}\Psi = k\Psi$$

Conclusion: whenever $k\,\Psi$ needs to be computed, one can use $-i\frac{\partial}{\partial x}\Psi$

$$\frac{\partial^2}{\partial x^2}\Psi = i(\tfrac{\partial}{\partial x}k)\Psi + ik\frac{\partial}{\partial x}\Psi$$

A wave function changes significantly over a wavelength.

Whenever the potential does not change much over a wavelength,

$$\frac{\Delta\Psi}{\Psi} >> \frac{\Delta k}{k} \quad\Rightarrow\quad k\frac{\Delta\Psi}{\Delta x} >> \frac{\Delta k}{\Delta x}\Psi \quad\Rightarrow\quad k\frac{\partial}{\partial x}\Psi >> (\tfrac{\partial}{\partial x}k)\Psi$$

$$\frac{\partial^2}{\partial x^2}\Psi = i(\tfrac{\partial}{\partial x}k)\Psi + ik\frac{\partial}{\partial x}\Psi \approx -k^2\Psi$$

The correspondence principle has to hold in the classical limit where the potential always changes little over the very small wavelength.

$$\Psi(x,t) = \Phi(x)e^{-i\omega t} \quad\rightarrow\quad -\frac{\hbar^2}{2m}\frac{\partial}{\partial x}\frac{\partial}{\partial x}\Phi(x) + V(x)\Phi(x) = E\,\Phi(x)$$

Georg.Hoffstaetter@Cornell.edu

The time dependent Schrödinger equation

Linearity for equal energies:

If $\boldsymbol{\Psi_1}$ and $\boldsymbol{\Psi_2}$ are solutions of

$$-\frac{\hbar^2}{2m}\frac{\partial}{\partial x}\frac{\partial}{\partial x}\Psi(x,t)+V(x)\Psi(x,t)=\hbar\omega\Psi(x,t)$$

for a common energy **E**, then also $\boldsymbol{\Psi = \Psi_1+\Psi_2}$ is a solution.

Linearity for different energies:

$$\Psi_\omega(x,t)=\Phi_\omega(x)e^{-i\omega t} \quad\rightarrow\quad i\hbar\frac{\partial}{\partial t}\Psi_\omega(x,t)=\hbar\omega\Psi_\omega(x,t)$$

If $\boldsymbol{\Psi_{\omega 1}}$ and $\boldsymbol{\Psi_{\omega 2}}$ are solutions for different energies **E_1 and E_2**,

$$-\frac{\hbar^2}{2m}\frac{\partial}{\partial x}\frac{\partial}{\partial x}\Psi_{\omega_1}(x,t)+V(x)\Psi_{\omega_1}(x,t)=\hbar\omega_1\Psi_{\omega_1}(x,t)=i\hbar\frac{\partial}{\partial t}\Psi_{\omega_1}(x,t)$$

$$-\frac{\hbar^2}{2m}\frac{\partial}{\partial x}\frac{\partial}{\partial x}\Psi_{\omega_2}(x,t)+V(x)\Psi_{\omega_2}(x,t)=\hbar\omega_2\Psi_{\omega_2}(x,t)=i\hbar\frac{\partial}{\partial t}\Psi_{\omega_2}(x,t)$$

then also $\boldsymbol{\Psi = \Psi_{\omega 1}+\Psi_{\omega 2}}$ a solution of

$$-\frac{\hbar^2}{2m}\frac{\partial}{\partial x}\frac{\partial}{\partial x}\Psi(x,t)+V(x)\Psi(x,t)=i\hbar\frac{\partial}{\partial t}\Psi(x,t)$$

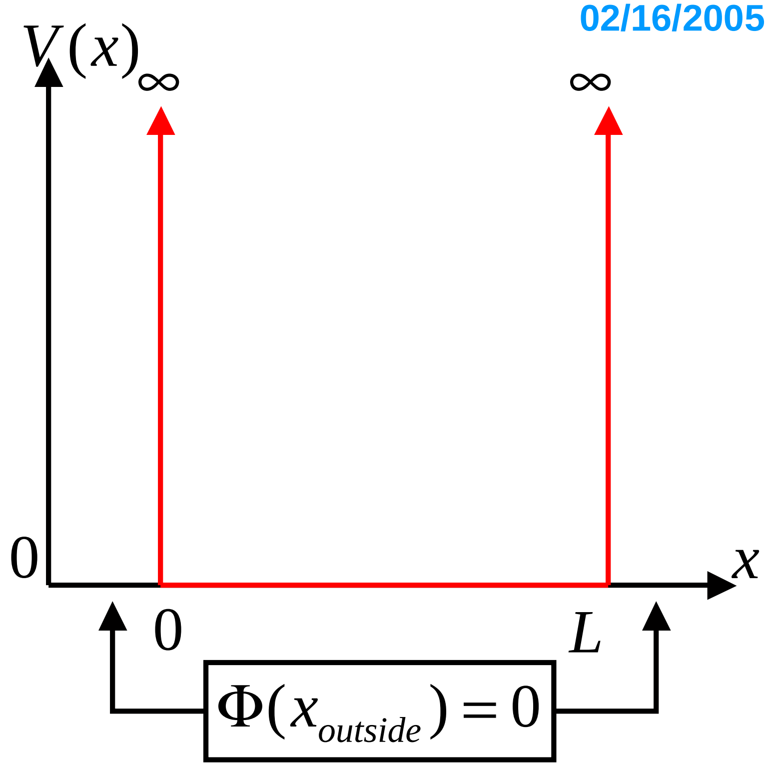
This holds for an arbitrary superposition of waves:

$$\Psi(x,t)=\sum_{n=0}^{\infty}\Phi_{\omega_n}(x)e^{-i\omega_n t} \quad\rightarrow\quad \int_{-\infty}^{\infty}\Phi(\omega,x)e^{-i\omega t}d\omega$$

CORNELL

Georg.Hoffstaetter@Cornell.edu

Particle in a one dimensional box



Stationary state:

$$\Psi_\omega(x,t) = \Phi(x)e^{-i\omega t} = \Phi(x)e^{-i\frac{E}{\hbar}t}$$

$$-\frac{\hbar^2}{2m}\frac{\partial}{\partial x}\frac{\partial}{\partial x}\Phi(x) + V(x)\Phi(x) = E\,\Phi(x)$$

Inside: $\frac{\partial}{\partial x}\frac{\partial}{\partial x}\Phi(x) = -\frac{2mE}{\hbar^2}\Phi(x)$

$$\Phi(x) = C_1 e^{ikx} + C_2 e^{-ikx}\,, \quad k = \frac{\sqrt{2mE}}{\hbar}$$

Boundary condition:

$$\Phi(0_+) = \Phi(0_-) = 0 \rightarrow C_1 = -C_2\,, \quad \Phi(x) = B\sin(kx)$$

$$\Phi(L_-) = \Phi(L_+) = 0 \rightarrow kL = n\pi$$

Quantized energies:

$$k_n = n\frac{\pi}{L} \rightarrow E_n = \frac{\hbar^2 k_n^2}{2m} = n^2\,\frac{\hbar^2\pi^2}{2mL^2} \quad \rightarrow \quad \Psi(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\frac{\pi}{L}x)\,e^{-i\frac{E_n}{\hbar}t}$$

CORNELL

Requiring **boundary conditions** around particles with wave character leads to **energy quantization.**

Georg.Hoffstaetter@Cornell.edu

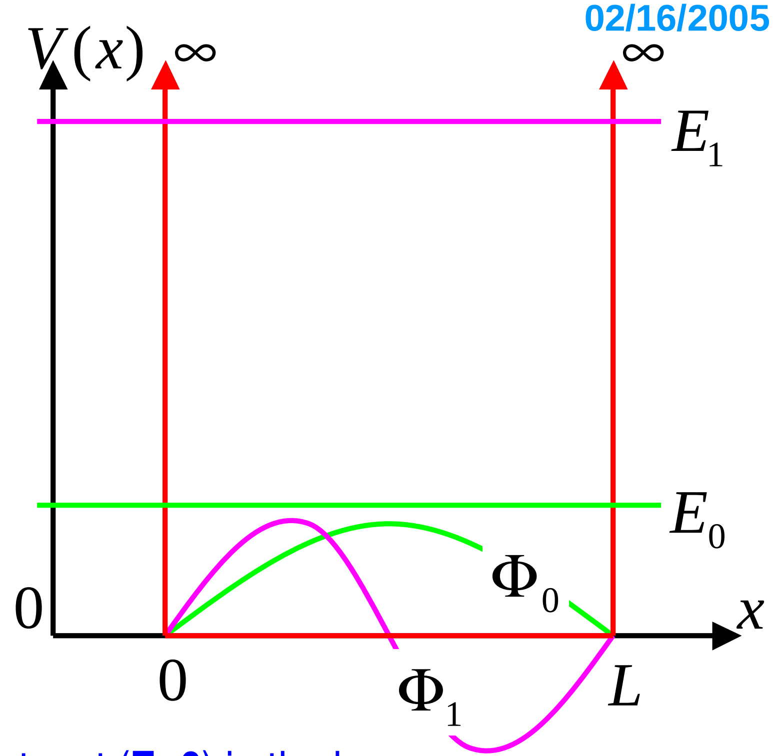
Ground state and classical limit

The state with the lowest possible energy is called the **ground state**. To start with n=0,

$$\Phi_n(x) = B\sin([n+1]\tfrac{\pi}{L}x)$$

There is no wave function for **n=-1**, so the lowest possible energy is

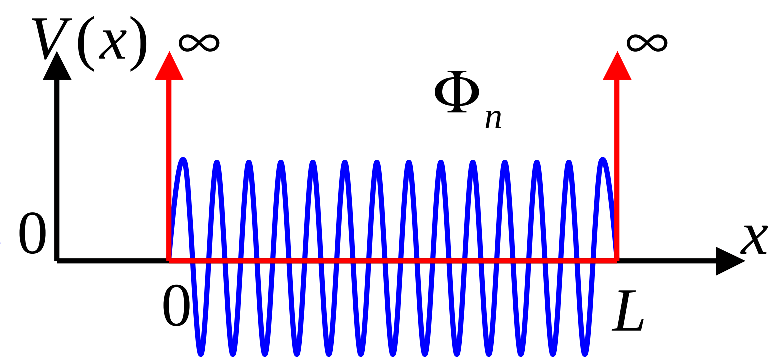
$$E_0 = \frac{\hbar^2\pi^2}{2mL^2}, \quad k_0 = \frac{\pi}{L}$$



There is no wave function that corresponds to a particle at rest (**E=0**) in the box.

Correspondence principle:

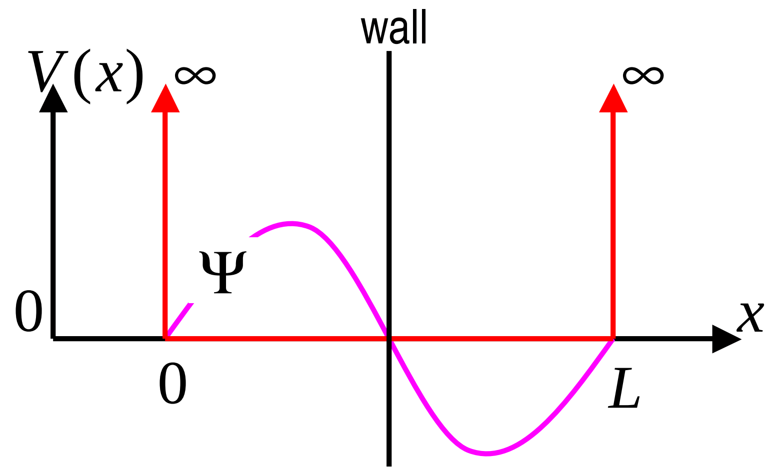
- A **classical particle** in a box is found equally often at any place.
- For any given measurement precision **Dx**, there is a **large state number n** for which the particle is found equally likely at any place in the box.



Georg.Hoffstaetter@Cornell.edu

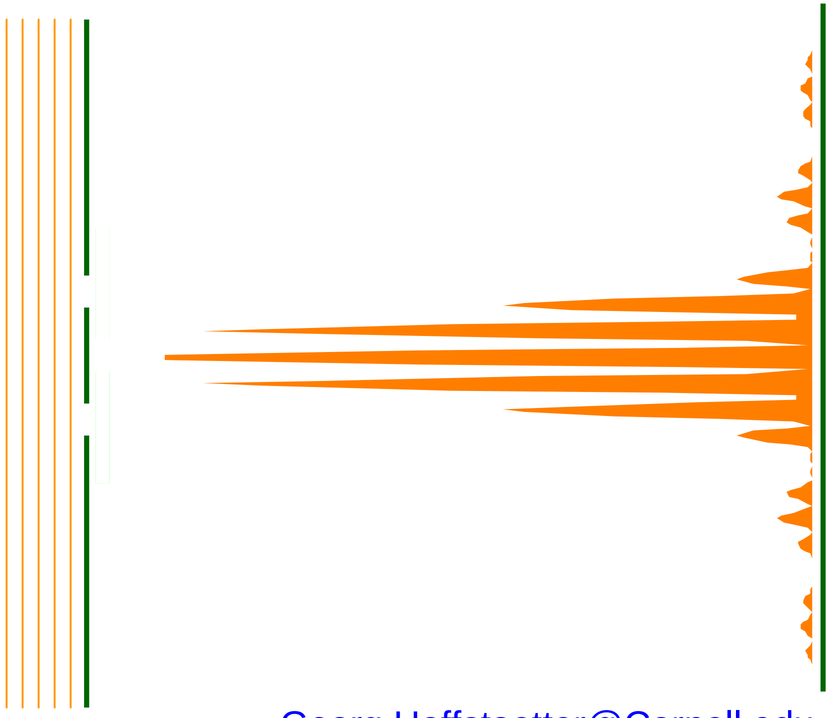
Wave function and probability amplitude

After a wall has been inserted in the center of the box, the particle can only be detected either in the right or the left half.



The wall cannot split the particle in two and **Ψ(x,t)** therefore cannot be a particle density.

To allow interference of particle waves, **Ψ(x,t)** also cannot be a probability distribution.



> **Ψ(x,t)** is a **probability amplitude** with **| Ψ(x,t) |²** being the probability to find a particle in the interval **[x,x+dx]** at time t.

The probability for **stationary states**:

$$|\Psi_\omega(x,t)|^2 = |\Phi(x)e^{-i\omega t}|^2 = |\Phi(x)|^2$$

Normalization: $$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 \, dx = 1$$

Georg.Hoffstaetter@Cornell.edu