The harmonic oscillator:

The classical oscillator often occurs in nature and is often a good approximation for small oscillations.

\[ V(x) = \frac{1}{2} Cx^2 \quad \rightarrow \quad m\ddot{x} = -Cx \quad \rightarrow \quad \text{classical oscillation with } \omega_0 = \sqrt{\frac{C}{m}} \]

Maximum oscillation amplitude: \( x_{\text{max}}^2 = \frac{2E}{m\omega_0^2} \)

\[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \Phi(x) + \frac{1}{2} Cx^2 \Phi(x) = E \Phi(x)\]

\[ \frac{\hbar\omega_0}{2} \left\{ -\frac{\hbar}{m\omega_0} \frac{\partial^2}{\partial x^2} + \frac{m\omega_0}{\hbar} x^2 \right\} \Phi(x) = E \Phi(x) \]

Simplification: \( a = \sqrt{\frac{\hbar}{m\omega_0}} \), \( \xi = \frac{x}{a} \)

\[ \frac{\hbar\omega_0}{2} \left\{ -\frac{\partial^2}{\partial \xi^2} + \xi^2 \right\} \Phi(x) = E \Phi(x) \]

\[ \Phi_n (x) = A_n f_n (\xi) e^{-\frac{1}{2} \xi^2} \quad \Rightarrow \quad \frac{\hbar\omega_0}{2} \left\{ -\frac{\partial^2}{\partial \xi^2} + 2\xi \frac{\partial}{\partial \xi} + 1 \right\} f_n (\xi) = E_n f_n (\xi) \]

\[ f_0 (\xi) = 1, \quad \Phi_0 (x) = A e^{-\frac{1}{2} \xi^2}, \quad E_0 = \frac{\hbar\omega_0}{2} \]

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Ground states

Gaussian function:

\[ f_0(\xi) = 1 , \quad \Phi_0(x) = Ae^{-\frac{1}{2}\xi^2} , \quad E_0 = \frac{\hbar \omega_0}{2} \]

The falloff for large \( x \) is even faster than for the finite potential well where it was \( e^{-\sqrt{\frac{2m}{\hbar}} \sqrt{V_0 - E} x} \)

This is due to the fact that now the potential increases with \( x \): \( V(x) \propto x^2 \)

All wave functions are dominated by the same form at large \( x \).

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Exited states:

\[ a = \sqrt{\frac{\hbar}{m\omega_0}} , \quad \xi = x/a \]

\[ \frac{\hbar\omega_0}{2} \left\{ -\frac{\partial^2}{\partial \xi^2} + \xi^2 \right\} \Phi(x) = E \Phi(x) \]

\[ \Phi_n(x) = A_n f_n(\xi) e^{-\frac{1}{2}\xi^2} \quad \Rightarrow \quad \frac{\hbar\omega_0}{2} \left\{ -\frac{\partial^2}{\partial \xi^2} + 2\xi \frac{\partial}{\partial \xi} + 1 \right\} f_n(\xi) = E_n f_n(\xi) \]

\[ f_n(\xi) = \sum_{j=0}^{\infty} c_j \xi^j \quad \Rightarrow \quad \sum_{j=0}^{\infty} \left\{ -c_{j+2} \frac{(j+2)(j+1)}{2} + c_j \left( j + \frac{1}{2} \right) \right\} \xi^j = \frac{E_n}{\hbar\omega_0} f_n(\xi) \]

\[ f_n(\xi) = \sum_{j=0}^{\infty} c_j \xi^j \quad \Rightarrow \quad c_{j+2} \frac{(j+2)(j+1)}{2} = c_j \left( j + \frac{1}{2} - \frac{E_n}{\hbar\omega_0} \right) \]

The series terminates when \( \frac{E_n}{\hbar\omega_0} \) equals \( n + \frac{1}{2} \) for some integer \( n \), yielding an nth order polynomial \( f_n \).

This leads to a wave function \( \Phi_n(x) = A_n f_n(\xi) e^{-\frac{1}{2}\xi^2} \) with \( n \) nodes.

Therefore this leads to all possible wave functions.

\[ E_n = \hbar\omega_0 \left( n + \frac{1}{2} \right) \]
Hermit polynomials

Stationary states of the harmonic oscillator:

\[ \Phi_n(x) = \frac{A}{\sqrt{n!}} \frac{1}{\sqrt{2^n}} e^{-\frac{1}{2} \xi^2} H_n(\xi) \]

Hermite polynomials: \( H_n(\xi) = (-1)^n e^{\xi^2} \frac{\partial^n}{\partial \xi^n} e^{-\xi^2} \)

Normalization: \( \int_{-\infty}^{\infty} |\Phi(x)|^2 \, dx = 1 \) leads to the constant \( A \).
Lowest Eigenvalue:

The chain of eigenfunctions $\Phi_n$ where $n$ is a positive integer.

The Schrödinger equation of an harmonic oscillator has eigenvalues

$$E_n = \hbar \omega_0 (n + \frac{1}{2})$$

With the lowest possible energy or ground state energy

$$E_0 = \frac{1}{2} \hbar \omega_0$$
Probability amplitudes for eigenstates

\[ \Phi_n = \frac{A}{\sqrt{n!}} \frac{1}{\sqrt{2^n}} e^{-\frac{1}{2} \xi^2} H_n(\xi), \quad \xi = x / a, \quad a = \sqrt{\frac{\hbar}{m \omega_0}}, \quad E_n = \hbar \omega_0 (n + \frac{1}{2}) \]

- \( H_0(\xi) = 1 \)
- \( H_1(\xi) = 2\xi \)
- \( H_2(\xi) = 4\xi^2 - 2 \)
- \( H_3(\xi) = 8\xi^3 - 12\xi \)
Time dependent states in the square potential

\[ \Psi(x,t) \propto \Phi_0(x)e^{-i\frac{E_0}{\hbar}t} + \frac{1}{2} \Phi_1(x)e^{-i\frac{E_1}{\hbar}t} \]

\[ |\Psi(x,t)|^2 \]

\[ E_0 \quad E_1 \]
Time dependent states in the square potential

\[ \Psi(x, t) \propto \Phi_0(x) e^{-\frac{i E_0}{\hbar} t} + \frac{1}{2} \Phi_1(x) e^{-\frac{i E_1}{\hbar} t} \]

\[ \Psi(x, t) = \sum_{n=0}^{30} A_n \Phi_n(x) e^{-\frac{i E_n}{\hbar} t} \]

\[ |\Psi(x, t)|^2 \]