Orthogonality for different Energies

It is a feature of the wave functions of any potential $V(x)$ that the stationary wave functions are orthogonal to each other.

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \Phi_E + V(x) \Phi_E = E \Phi_E$$

$$\int_{-\infty}^{\infty} E_2 \Phi_{E_1}^* \Phi_{E_2} \, dx = \int_{-\infty}^{\infty} \Phi_{E_1}^* \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \Phi_{E_2} + V(x) \Phi_{E_2} \right] dx$$

$$\int_{-\infty}^{\infty} \Phi_{E_1}^* \frac{\partial^2}{\partial x^2} \Phi_{E_2} \, dx = \left[ \Phi_{E_1}^* \frac{\partial}{\partial x} \Phi_{E_2} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{\partial}{\partial x} \Phi_{E_1}^* \frac{\partial}{\partial x} \Phi_{E_2} \, dx$$

$$\int_{-\infty}^{\infty} E_2 \Phi_{E_1}^* \Phi_{E_2} \, dx = \int_{-\infty}^{\infty} \left[ \frac{\hbar^2}{2m} \frac{\partial}{\partial x} \Phi_{E_1}^* \frac{\partial}{\partial x} \Phi_{E_2} + \Phi_{E_1}^* V(x) \Phi_{E_2} \right] dx$$

$$\int_{-\infty}^{\infty} E_1 \Phi_{E_1}^* \Phi_{E_2} \, dx = \int_{-\infty}^{\infty} \left[ \frac{\hbar^2}{2m} \frac{\partial}{\partial x} \Phi_{E_1}^* \frac{\partial}{\partial x} \Phi_{E_2} + \Phi_{E_1}^* V(x) \Phi_{E_2} \right] dx$$

$$(E_1 - E_2) \int_{-\infty}^{\infty} \Phi_{E_1}^* \Phi_{E_2} \, dx = 0 \quad \Rightarrow \quad \int_{-\infty}^{\infty} \Phi_{E_n}^* \Phi_{E_m} \, dx = \delta_{nm} = \begin{cases} 0 & \text{if } E_n \neq E_m \\ 1 & \text{if } E_n = E_m \end{cases}$$
6 Photons and quantum states

A electromagnetic plain wave can be uniquely defined by specifying four things:
1. Frequency
2. Direction of propagation
3. Polarization
4. Amplitude of the electric field (and thereby the energy in the field)

Similarly the state of a photon is uniquely defined by specifying three things:
1. Frequency (and therefore the energy of the photon $h\nu$)
2. Direction of motion
3. Polarization state

The energy in the wave is then determined by the number of photons.

The state of a photon completely determines a photon in the following sense:
Everything that can be known about a photon is specified.
Linearly polarized waves

A beam of light is linearly polarized if its electric vector lies in a single plane that includes the beam (the plane of polarization). The polarization axis is parallel to the direction of the electric field.

Any plane wave with wave number $k = \frac{2\pi}{\lambda}$ and angular frequency $\omega = 2\pi v = ck$ can be described by a superposition of two orthogonally polarized waves:

$$\vec{E} = \text{Re}[\vec{a}e^{i(kz-\omega t + \alpha)} + \vec{b}e^{i(kz-\omega t + \beta)} + \vec{c}e^{i(kz-\omega t + \gamma)} + \ldots]$$

$$= \vec{e}_x \text{Re}[(a_1e^{i\alpha} + b_1e^{i\beta} + c_1e^{i\gamma} + \ldots)e^{i(kz-\omega t)}]$$

$$+ \vec{e}_y \text{Re}[(a_2e^{i\alpha} + b_2e^{i\beta} + c_2e^{i\gamma} + \ldots)e^{i(kz-\omega t)}]$$

$$= \text{Re}[\vec{e}_x a_x e^{i\phi_x} e^{i(kz-\omega t)} + \vec{e}_y a_y e^{i\phi_y} e^{i(kz-\omega t)}]$$
Production of linearly polarized states

1) Passing of an arbitrary beam of light through a **polarizer** produces a polarized beam with a polarization axis parallel to the transmission axis.

2) A **calcite crystal** has a different refractive index for different polarization directions and therefore splits an arbitrary beam into two beams with two perpendicular polarization directions.

These polarizers can also be used to determine whether a beam is in a linearly polarized state and the polarization direction can be found.