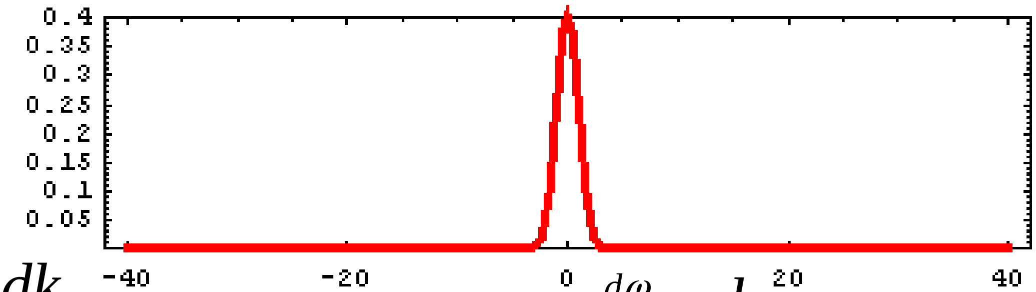
Dispersion of a free particle packet states

$$\Psi(x,0) = \frac{1}{\sqrt{\sqrt{\pi}\sigma}} e^{-\frac{x^2}{2\sigma^2}} = f(x)$$



$$v_{\text{group}} \doteq \frac{d\omega}{dk} \propto k$$

$$\Psi(x,0) = \frac{1}{\sqrt{\sqrt{\pi}\sigma^{-1}}} \int_{-\infty}^{\infty} e^{-\frac{k^2}{2\sigma^{-2}}} \frac{e^{ikx}}{\sqrt{2\pi}} dk$$

$$\Psi(x,t) = \frac{1}{\sqrt{\sqrt{\pi}\sigma^{-1}}} \int_{-\infty}^{\infty} e^{-\frac{k^2}{2\sigma^{-2}}} \frac{e^{i(kx-\omega t)}}{\sqrt{2\pi}} dk, \quad \omega = \frac{\hbar}{2m} k^2$$

$$= \frac{(2\pi)^{-1}}{\sqrt{\sqrt{\pi}\sigma^{-1}}} e^{-\frac{1}{2}\frac{x^2}{\sigma^2 + i\frac{\hbar}{m}t}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(\sigma^2 + i\frac{\hbar}{m}t)(k + i\frac{x}{\sigma^2 + i\frac{\hbar}{m}t})^2} dk = \frac{(2\pi)^{-1}}{\sqrt{\sqrt{\pi}\sigma^{-1}}} e^{-\frac{1}{2}\frac{x^2}{\sigma^2 + i\frac{\hbar}{m}t}} \frac{\sqrt{2}}{\sqrt{\sigma^2 + i\frac{\hbar}{m}t}} \sqrt{\pi}$$

$$|\Psi(x,t)|^2 = \frac{1}{\sqrt{\pi}\sqrt{\sigma^2 + (\frac{\hbar}{m\sigma}t)^2}} e^{-\frac{x^2}{\sigma^2 + (\frac{\hbar}{m\sigma}t)^2}}$$

(Gaussian with width)

$$\Delta x = \frac{1}{\sqrt{2}} \sqrt{\sigma^2 + (\frac{\hbar}{\sigma m} t)^2}$$

Position-momentum uncertainty relation:

$$\Delta x \geq \frac{\sigma}{\sqrt{2}}, \quad \Delta k = \frac{\sigma^{-1}}{\sqrt{2}} \quad \rightarrow \quad \Delta x \Delta p \geq \frac{\hbar}{2}$$

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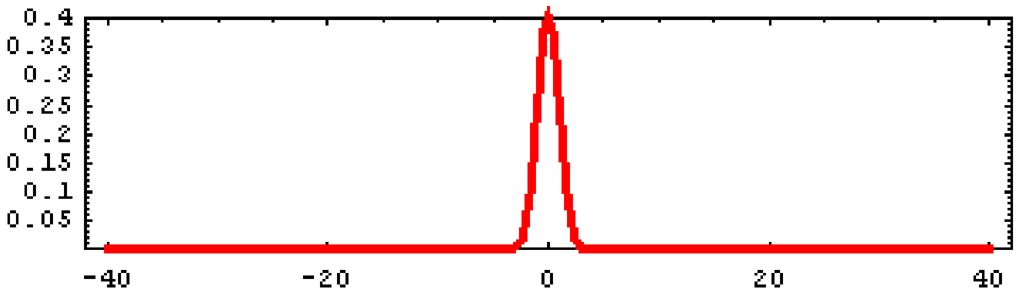
04/04/2005



Position-velocity uncertainty relation:

$$\Delta x \Delta v \geq \frac{\hbar}{2m} \approx \begin{cases} 1\text{cm} \cdot 1\frac{\text{m}}{\text{s}} = 1\text{Å} \cdot 10^{8}\frac{\text{m}}{\text{s}} \text{ for an electron} \\ 1\mu\text{m} \cdot 10^{-26}\frac{\text{cm}}{\text{s}} = 1\mu m \cdot 0.1\frac{\text{Å}}{14\text{Gyears}} \text{ for a 0.1g pea} \end{cases}$$

Conclusion:

In the microscopic world of the hydrogen atom, electron orbits cannot be determined since the uncertainty in velocity is as large as the electron velocity itself.

In the macroscopic world, the uncertainty of quantum mechanics is not recognizable.

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Projection amplitudes and ket vectors for free states

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Gaussian probability distribution at t=0: $\Psi(x,0) = \frac{1}{\sqrt{\sqrt{\pi}\sigma}} e^{-\frac{x^2}{2\sigma^2}}$

$$\left|\Psi\right\rangle = \sum_{\text{all } x} \left|x\right\rangle\left\langle x\middle|\Psi\right\rangle = \sum_{\text{all } x} \left|x\right\rangle \frac{1}{\sqrt{\sqrt{\pi}\sigma}} e^{-\frac{x^2}{2\sigma^2}} \qquad \left|\Psi\right\rangle = \sum_{\text{all } k} \left|k\right\rangle \frac{1}{\sqrt{\sqrt{\pi}\sigma^{-1}}} e^{-\frac{k^2}{2\sigma^{-2}}}$$

$$\left\langle x\middle|\Psi\right\rangle = \sum_{\text{all } k} \left\langle x\middle|k\right\rangle\left\langle k\middle|\Psi\right\rangle \rightarrow \left\langle x\middle|k\right\rangle = \frac{e^{ikx}}{\sqrt{2\pi}}$$

$$\left\langle k\middle|\Psi\right\rangle = \sum_{\text{all } x} \left\langle k\middle|x\right\rangle\left\langle x\middle|\Psi\right\rangle \rightarrow \left\langle k\middle|x\right\rangle = \frac{e^{-ikx}}{\sqrt{2\pi}}$$

Satisfies the requirements for projection amplitudes: $\left\langle x\middle|k\right\rangle = \left\langle k\middle|x\right\rangle^*$

And $\Phi_k(x) = \left\langle x\middle|k\right\rangle$ is a stationary state of the free Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Phi_k(x) = E\Phi_k(x), \quad E = \frac{\hbar^2 k^2}{2m}$$

While these stationary basis states cannot be normalized to 1,
any wave function Ψ is normalized to 1.

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The uncertainty principle and ground-state energies

Area in which the ground state wave function has to be confined Δx

Momentum uncertainty in all three dimensions $\Delta p_x \geq \frac{\hbar}{2}\frac{1}{\Delta x}$

$$E_0 \approx \frac{3}{2m}\left(\frac{\hbar}{2\Delta x}\right)^2$$

Example: Hydrogen atom

$$\left.\begin{aligned} E_0 &= \frac{Ze^2}{8\pi\varepsilon_0}\frac{1}{a_0} \\ a_0 &= \frac{4\pi\varepsilon_0\hbar^2}{Ze^2 m} \end{aligned}\right\} \rightarrow E_0 = \frac{\hbar^2}{2ma_0}\frac{1}{a_0} = \frac{4}{2m}\left(\frac{\hbar}{2a_0}\right)^2$$

This provides a quick procedure of estimating ground state energies.

Example: Hydrogen: $\hbar \approx 10^{-34}\,\mathrm{Js}, \quad m_e \approx 10^{-30}\,\mathrm{kg}, \quad \Delta x \approx 10^{-10}\,\mathrm{m}$

$$\mathrm{E}_0 \approx \frac{3}{2\,10^{-30}\,\mathrm{kg}}\left(\frac{10^{-34}\,\mathrm{Js}}{10^{-10}\,\mathrm{m}}\right)^2 = 1.5\,10^{-18}\,\mathrm{J} = \frac{1.5}{1.6}10\mathrm{eV}$$

Neutron or proton confined in the nucleus:

$$m_p \approx 10^{-27}\,\mathrm{kg}, \quad \Delta x \approx 10^{-14}\,\mathrm{m}$$

$$\mathrm{E}_0 \approx \frac{3}{2\,10^{-27}\,\mathrm{kg}}\left(\frac{10^{-34}\,\mathrm{Js}}{10^{-14}\,\mathrm{m}}\right)^2 = 1.5\,10^{-13}\,\mathrm{J} \approx 1\mathrm{MeV}$$

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