Normalization and probability density

Probability to find a particle in the volume element \( d^3 \vec{x} \) is given by \( |\Psi(\vec{x}, t)|^2 \ d^3 \vec{x} \)

Probability to find the particle somewhere is one:
\[
\int |\Psi(\vec{x}, t)|^2 \ d^3 \vec{x} = 1
\]

Examples:
\[
\int \int \int |\Psi(\vec{x}, t)|^2 \ dx \ dy \ dz = 1
\]
\[
\int \int \int r^2 \sin \vartheta \ d \vartheta \ d \varphi \ dr = 1
\]

Spherical symmetric wave functions: \( 4\pi \int_0^{\infty} |\Psi(r, t)|^2 \ r^2 \ dr = 1 \)

Probability to find a particle with a radius between \( r \) and \( r + dr \):
\[
n \int |\Psi(r, t)|^2 \ r^2 \ dr = 1
\]

For a stationary state this is
\[
n \int r \Phi(r)^2 \ dr = \int u(r)^2 \ dr
\]

\( u(r) \) is not normalized to 1 but to \( 1/4\pi \):
\[
\int_0^{\infty} |u(r)|^2 \ dr = \frac{1}{4\pi}
\]
Expectation values

The most probable value of \( r \) is given by \( \frac{\partial}{\partial r} u(r) = 0 \)

The average radius after many measurements in identically prepared states:
After \( N \) measurements one might have measured a radius \( r_1, n_1 \) time \( r_2, n_2 \) times, etc.
The average radius is then

\[
\langle r \rangle = \frac{1}{N} (n_1 r_1 + n_2 r_2 + n_3 r_3 + \ldots), \quad N = n_1 + n_2 + n_3 + \ldots \\
= p_1 r_1 + p_2 r_2 + p_3 r_3 + \ldots, \quad \text{probability } p_j \text{ to measure } r_j
\]

\[
= 4\pi \int_0^\infty r |u(r)|^2 \, dr
\]

\[
= \int_0^\infty \int_0^\frac{\pi}{2} \int_0^{2\pi} r |\Psi(r, \vartheta, \phi, t)|^2 \, r^2 \sin \vartheta \, d\phi \, d\vartheta \, dr
\]

Similarly in one dimension:

\[
\langle x \rangle = \int_{-\infty}^\infty x |\Psi(x, t)|^2 \, dx
\]
A limit of the Bohr model

\[ E = E_1 \quad E = \frac{1}{4} E_1 \]
\[ r_{\text{max}} = a_0 \quad r_{\text{max}} = (3 + \sqrt{5})a_0 > 4a_0 \]

**The Bohr radius is not the most likely radius.**

\[ \langle r \rangle_{E_1} = \frac{3}{2} a_0 \quad \langle r \rangle_{E_2} = 6a_0 \]

**The Bohr radius is also not the expectation value.**