E has a similar effect as \( v \times B \).

For relativistic particles \( B = 1 \text{T} \) has a similar effect as

\[
E = cB = 3 \times 10^8 \text{ V/m}
\]

such an Electric field is beyond technical limits.

- Electric fields are only used for very low energies or
- For separating two counter rotating beams with different charge.

\[
\frac{d}{dt} \vec{p} = q(\vec{E} + \vec{v} \times \vec{B})
\]
Static magnetic fields: \( \partial_t \vec{B} = 0 \); \( \vec{E} = 0 \)

\[
\vec{\nabla} \times \vec{B} = \mu_0 (\vec{j} + \varepsilon_0 \partial_t \vec{E}) = 0 \quad \Rightarrow \quad \vec{B} = -\vec{\nabla} \psi(\vec{r})
\]

\[
\vec{\nabla} \cdot \vec{B} = 0 \quad \Rightarrow \quad \vec{\nabla}^2 \psi(\vec{r}) = 0
\]

\( (x=0,y=0) \) is the beam’s design curve

For finite fields on the design curve, \( \Psi \) can be power expanded in \( x \) and \( y \):

\[
\psi(x, y, z) = \sum_{n,m=0}^{\infty} b_{nm}(z)x^n y^m
\]
Surfaces of Equal Potential

\[ \vec{B}_\perp(\text{out}) = \vec{B}_\perp(\text{in}) \]
\[ \vec{H}_{\parallel}(\text{out}) = \vec{H}_{\parallel}(\text{in}) \]
\[ \vec{B}_{\parallel}(\text{out}) = \frac{1}{\mu_r} \vec{B}_{\parallel}(\text{in}) \]
\[ \vec{B}(\vec{r}) = -\nabla \Psi(\vec{r}) \]

\[ 0 = \oint \vec{B} \cdot d\vec{s} = \int_X A \vec{B}_0 \cdot d\vec{s} + \int_A B \vec{B}_0 \cdot d\vec{s} + \int_B X \vec{B}_0 \cdot d\vec{s} \]
\[ = \int_X A \vec{B}_0 \cdot d\vec{s} + \frac{1}{\mu_r} \int_A B \vec{B}_0 \cdot d\vec{s} + \int_B X \vec{B}_0 \cdot d\vec{s} \]
\[ \approx \int_X A \vec{B}_0 \cdot d\vec{s} + \int_B X \vec{B}_0 \cdot d\vec{s} = \Psi(A) - \Psi(B) \]

For large permeability, \( H(\text{out}) \) is perpendicular to the surface.

For highly permeable materials (like iron) surfaces have a constant potential.
Knowledge of the field and the scalar magnetic potential on a closed surface inside a magnet determines the magnetic field for the complete volume which is enclosed.

Green's Theorem

\[ \nabla^2 \psi = 0 \]

Green function:

\[ \nabla_0^2 G(\vec{r}, \vec{r}_0) = \delta(\vec{r} - \vec{r}_0) \]

\[ \psi(\vec{r}) = \int_V \psi(\vec{r}_0) \delta(\vec{r} - \vec{r}_0) \, d^3 \vec{r}_0 \]

\[ = \int_V \left[ \psi(\vec{r}_0) \nabla_0^2 G - G \nabla_0^2 \psi(\vec{r}_0) \right] d^3 \vec{r}_0 \]

\[ = \int_V \nabla_0 \left[ \psi(\vec{r}_0) \nabla_0 G - G \nabla_0 \psi(\vec{r}_0) \right] d^3 \vec{r}_0 \]

\[ = \int_{\partial V} \left[ \psi(\vec{r}_0) \nabla_0 G - G \nabla_0 \psi(\vec{r}_0) \right] \cdot d^2 \vec{r}_0 \]

\[ = \int_{\partial V} \left[ \psi(\vec{r}_0) \nabla_0 G + \vec{B}(\vec{r}_0) G \right] \cdot d^2 \vec{r}_0 \]
If field data in a plane (for example the midplane of a cyclotron or of a beam line magnet) is known, the complete field is determined:

\[ \psi(x, y, z) = \sum_{n=0}^{\infty} b_n(x, z) y^n \quad \Rightarrow \quad \vec{B}(x, 0, z) = - \begin{pmatrix} \partial_x b_0(x, z) \\ b_1(x, z) \\ \partial_z b_0(x, z) \end{pmatrix} \]

\[ 0 = \nabla^2 \psi = \sum_{n=0}^{\infty} \left( \partial_x^2 + \partial_z^2 \right) b_n \, y^n + \sum_{n=2}^{\infty} n(n-1) b_n \, y^{n-2} \]

\[ = \sum_{n=0}^{\infty} \left[ \left( \partial_x^2 + \partial_z^2 \right) b_n + (n+2)(n+1) b_{n+2} \right] y^n \]

\[ b_{n+2}(x, z) = -\frac{1}{(n+2)(n+1)} \left( \partial_x^2 + \partial_y^2 \right) b_n(x, z) \]

Data of the magnetic field in the plane \( y=0 \) is used to determine \( b_0(x, z) \) and \( b_1(x, z) \).
Complex Potentials

\[ w = x + iy, \quad \bar{w} = x - iy \]

\[ \partial_x = \partial_w + \partial_{\bar{w}}, \quad \partial_y = i\partial_w - i\partial_{\bar{w}} = i(\partial_w - \partial_{\bar{w}}) \]

\[ \nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2 = (\partial_w + \partial_{\bar{w}})^2 - (\partial_w - \partial_{\bar{w}})^2 + \partial_z^2 = 4\partial_w \partial_{\bar{w}} + \partial_z^2 \]

\[ \psi = \text{Im}\left\{ \sum_{\nu, \lambda=0}^{\infty} a_{\nu\lambda}(z) \cdot (w\bar{w})^\lambda \bar{w}^\nu \right\} \]

\[ \nabla^2 \psi = \text{Im}\left\{ \sum_{\nu=0,\lambda=1}^{\infty} 4a_{\nu\lambda}(\lambda + \nu)\lambda(w\bar{w})^{\lambda-1} \bar{w}^\nu + \sum_{\nu=0,\lambda=0}^{\infty} a_{\nu\lambda}''(w\bar{w})^\lambda \bar{w}^\nu \right\} \]

\[ = \text{Im}\left\{ \sum_{\nu, \lambda=0}^{\infty} [4(\lambda + 1 + \nu)(\lambda + 1)a_{\nu\lambda+1} + a_{\nu\lambda}''](w\bar{w})^\lambda \bar{w}^\nu \right\} = 0 \]

Iteration equation: \[ a_{\nu\lambda+1} = -\frac{1}{4(\lambda + 1 + \nu)(\lambda + 1)} a_{\nu\lambda}'' , \quad a_{\nu0} = \Psi_\nu(z) \]

The functions \( \Psi_\nu(z) \) along a line determine the complete field inside a magnet.
\( \Psi_\nu(z) \) are called the \( z \)-dependent multipole coefficients

\[
\psi(x, y, z) = \text{Im}\left\{ \sum_{\nu, \lambda=0}^\infty \frac{(-1)^\lambda \nu!}{(\lambda + \nu)! \lambda!} \left( \frac{ww}{4} \right)^\lambda \bar{w}^{\nu} \Psi_{\nu}^{[2\lambda]}(z) \right\}
\]

\[
\psi(r, \varphi, z) = \sum_{\nu, \lambda=0}^\infty \frac{(-1)^\lambda \nu!}{(\lambda + \nu)! \lambda!} \left( \frac{r}{2} \right)^{2\lambda} r^\nu \text{Im}\{\Psi_{\nu}^{[2\lambda]}(z)e^{-i\nu\varphi}\}
\]

The index \( \nu \) describes \( C_\nu \) Symmetry around the \( z \)-axis \( \hat{e}_z \) due to a sign change after \( \Delta \varphi = \frac{\pi}{\nu} \) for \( \nu = 3 \).
Main fields in accelerator physics: \( \lambda = 0 \), \( \partial_z^2 \psi = 0 \)

\[
\Psi_{\nu} = \begin{cases} 
  e^{i\nu \vartheta} |\Psi_{\nu}| & \text{for } \nu \neq 0 \\
  i |\Psi_{0}| & \text{for } \nu = 0 
\end{cases}
\]

\[
\psi(r, \varphi) = \sum_{\nu=1}^{\infty} r^{\nu} |\Psi_{\nu}| \text{Im} \{e^{-i\nu(\varphi-\vartheta)}\} + |\Psi_{0}|
\]
Main Field Potential

Main field potential: \[ \psi = |\Psi_0| - \sum_{\nu=1}^{\infty} r^\nu |\Psi_\nu| \sin[\nu(\varphi - \vartheta_\nu)] \]

The isolated multipole: \[ \psi = -r^\nu |\Psi_\nu| \sin(\nu\varphi) \]

Where the rotation \( \vartheta_\nu \) of the coordinate system is set to 0

The potentials produced by different multipole components have

a) Different rotation symmetry \( C_\nu \)

b) Different radial dependence \( r^\nu \)
Multipoles in Accelerators  \( \nu=0: \text{Solenoids} \)

\[
\begin{align*}
\dot{j} &= j \\
m\gamma \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} &= q \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} \times \begin{pmatrix} -\frac{x}{2} B_z' \\ \frac{y}{2} B_z' \\ B_z \end{pmatrix} \\
&\downarrow \\
\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} &= \frac{q B_z}{m \gamma} \begin{pmatrix} \dot{y} \end{pmatrix} - \frac{q B_z \dot{z}}{2m \gamma} \begin{pmatrix} y \end{pmatrix} \\
\ddot{w} &= -i \frac{q B_z}{m \gamma} \dot{w} - i \frac{q B_z}{2m \gamma} w
\end{align*}
\]

\[\psi = \Psi_0(z) - \frac{w\bar{w}}{4} \Psi_0''(z) \pm \ldots\]

\[\vec{B} = \begin{pmatrix} \frac{x}{2} \Psi_0'' \\ \frac{y}{2} \Psi_0'' \\ -\Psi_0' \end{pmatrix} \quad \Rightarrow \quad \nabla \cdot \vec{B} = 0\]

\[g = \frac{qB_z}{2m\gamma}, \quad w_0 = w e_{0}^{\int_{0}^{t} g \, dt}\]

\[\dot{w}_0 = (\dot{w} + i 2g\dot{w} + igw - g^2 w) e_{0}^{\int_{0}^{t} g \, dt} = -g^2 w_0\]

\[\begin{align*}
\dot{x}_0 &= -g^2 x_0 \\
\dot{y}_0 &= -g^2 y_0
\end{align*}\]

Focusing in a rotating coordinate system
Solenoid vs. Strong Focusing

If the solenoids field was perpendicular to the particle’s motion, its bending radius would be

$$\rho_z = \frac{p}{qB_z}$$

$$\ddot{r} = -\left(\frac{qB_z}{2m\gamma}\right)^2 r = -\frac{qv_z}{m\gamma} B_z \frac{r}{4\rho_z}$$

Solenoid focusing is weak compared to the deflections created by a transverse magnetic field.

Transverse fields: $$\vec{B} = B_x \vec{e}_x + B_y \vec{e}_y$$

$$m\gamma \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = q \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} \times \begin{pmatrix} B_x \\ B_y \\ 0 \end{pmatrix} \implies \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \frac{qv_z}{m\gamma} \begin{pmatrix} -B_y \\ B_x \end{pmatrix}$$

Strong focusing

Weak focusing < Strong focusing by about $$\frac{r}{\rho}$$
Solenoid magnets are used in detectors for particle identification via

\[ \rho = \frac{p}{qB} \]

The solenoid’s rotation \[ \dot{\phi} = -\frac{qB_z}{2m\gamma} \] of the beam is often compensated by a reversed solenoid called compensator.

Solenoid or Weak Focusing:

Solenoids are also used to focus low \( \gamma \) beams:

\[ \ddot{w} = -\left(\frac{qB_z}{2m\gamma}\right)^2 w \]

Weak focusing from natural ring focusing:

\[ \Delta r = r - R \]

\[ [(R + \Delta r) \cos \varphi - \Delta x_0]^2 + [(R + \Delta r) \sin \varphi - \Delta y_0]^2 = R^2 \]

Linearization in \( \Delta \):

\[ \Delta r = (\cos \varphi \Delta x_0 + \sin \varphi \Delta y_0) \]

\[ \partial_{\varphi}^2 \Delta r = -\Delta r \quad \Rightarrow \quad \Delta \ddot{r} = -\dot{\varphi}^2 \Delta r \quad = \quad -\left(\frac{v}{\rho}\right)^2 \Delta r = \left(\frac{qB}{m\gamma}\right)^2 \Delta r \]
\[ \psi = \Psi_1 \text{Im}\{x - iy\} = -\Psi_1 \cdot y \quad \Rightarrow \quad \vec{B} = -\nabla \psi = \Psi_1 \vec{e}_y \]

**Equipotential**

\[ y = \text{const.} \]

**Dipole Magnets**

Dipole magnets are used for steering the beams direction.

**Bending radius**:

\[ \rho = \frac{p}{qB} \]

\[ \frac{d\rho}{d\phi} = q\vec{v} \times \vec{B} \quad \Rightarrow \quad \frac{dp}{dt} = qvB_{\perp} \quad \Rightarrow \quad \rho = \frac{dl}{d\phi} = \frac{vdt}{dp / p} = \frac{p}{qB_{\perp}} \]
Different Dipoles

C-shape magnet:

H-shape magnet:

Window frame magnet:

\[ \mathbf{B}_{\perp} \text{(out)} = \mathbf{B}_{\perp} \text{(in)} \]

\[ \mathbf{H}_{\perp} \text{(out)} = \mu_r \mathbf{H}_{\perp} \text{(in)} \]

\[ 2nI = \iint \mathbf{H} \cdot d\mathbf{s} = H_{Fe} l_{Fe} + H_0 2a \]

\[ = \frac{1}{\mu_r} H_0 l_{Fe} + H_0 2a \approx H_0 2a \]

Dipole strength:

\[ \frac{1}{\rho} = \frac{q \mu_0}{p} \frac{nI}{a} \]
B = 2 T: Typical limit, since the field becomes dominated by the coils, not the iron. Limiting j for Cu is about 100A/mm²

B < 1.5 T: Typically used region

B < 1 T: Region in which \( B_0 = \mu_0 \frac{nI}{a} \)

Shims reduce the space that is open to the beam, but they also reduce the fringe field region.
Where is the vertical Dipole?
\[ \psi = \Psi_2 \text{Im}\{(x - iy)^2\} = -\Psi_2 \cdot 2xy \quad \Rightarrow \quad \vec{B} = -\nabla \psi = \Psi_2 2\begin{pmatrix} y \\ x \end{pmatrix} \]

In a quadrupole particles are focused in one plane and defocused in the other plane. Other modes of strong focusing are not possible.
\[
\psi = -\Psi_2 \cdot 2xy \quad \Rightarrow \quad \text{Equipotential: } x = \frac{\text{const.}}{y}
\]

\[
\vec{B} = 2\Psi_2 \begin{pmatrix} y \\ x \end{pmatrix} \quad \Rightarrow \quad \vec{B}(0 \mapsto 1) = 2\Psi_2 r \hat{e}_r
\]

Quadrupole strength:

\[
k_1 = \frac{q}{p} \left. \frac{\partial}{\partial x} B_y \right|_0 = \frac{q\mu_0}{p} \frac{2nI}{a^2}
\]
Real Quadrupoles

The coils show that this is an upright quadrupole not a rotated or skew quadrupole.
\[ \psi = \Psi_3 \text{Im}\{(x - iy)^3\} = \Psi_3 \cdot (y^3 - 3x^2y) \implies \vec{B} = -\vec{\nabla} \psi = \Psi_3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix} \]

**C\text{\textsubscript{3}} Symmetry**

i) Sextupole fields hardly influence the particles close to the center, where one can linearize in \(x\) and \(y\).

ii) In linear approximation a by \(\Delta x\) shifted sextupole has a quadrupole field.

iii) When \(\Delta x\) depends on the energy, one can build an energy dependent quadrupole.

\[ \vec{B} \approx \Psi_3 \left( \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix} + 6\Psi_3 \Delta x \begin{pmatrix} y \\ x \end{pmatrix} + O(\Delta x^2) \right) \]
$\psi = \Psi_2 \cdot (y^3 - 3x^2y) \implies \text{Equipotential: } x = \sqrt{\frac{\text{const.}}{y}} + \frac{1}{3}y^2$

$B_y \bigg|_{x=0} = -\Psi_3 3y^2$

$nI = \oint \vec{H} \cdot d\vec{s} \approx \int_0^a H_r dr = \Psi_3 \frac{a^3}{\mu_0}$

$\Phi = +\Phi_0$

$\Phi = -\Phi_0$

$\Phi = +\Phi_0$

$\Phi = -\Phi_0$

Quadrupole strength:

$k_2 = \frac{q}{p} \frac{\partial^2 B_y}{\partial x^2} \bigg|_{0} = \frac{q\mu_0}{p} \frac{6nI}{a^3}$
Real Sextupoles
Higher order Multipoles

\[
\psi = \Psi_n \text{Im}\{(x - iy)^n\} = \Psi_n \cdot (\ldots - i \ n \ x^{n-1} \ y) \quad \Rightarrow \quad \vec{B}(y = 0) = \Psi_n \ n \begin{pmatrix} 0 \\ x^{n-1} \end{pmatrix}
\]

Multipole strength: \[
k_n = \frac{q}{p} \left. \partial_x^n B_y \right|_{x,y=0} = \frac{q}{p} \Psi_{n+1} (n+1)! \quad \text{units: } \frac{1}{m^{n+1}}
\]

p/q is also called B\(p\) and used to describe the energy of multiply charge ions

Names: dipole, quadrupole, sextupole, octupole, decapole, duodecapole, …

Higher order multipoles come from

- Field errors in magnets
- Magnetized materials
- From multipole magnets that compensate such erroneous fields
- To compensate nonlinear effects of other magnets
- To stabilize the motion of many particle systems
- To stabilize the nonlinear motion of individual particles
The discussed multipoles produce midplane symmetric motion. When the field is rotated by $\pi/2$, i.e. $\vartheta_n = \pi/2n$, one speaks of a skew multipole.
Above 2T the field from the bare coils dominate over the magnetization of the iron. But Cu wires cannot create much filed without iron poles: 5T at 5cm distance from a 3cm wire would require a current density of

\[ j = \frac{I}{d^2} = \frac{1}{d^2} \frac{2\pi r B}{\mu_0} = 1389 \frac{A}{mm^2} \]

Cu can only support about 100A/mm².

- Superconducting cables routinely allow current densities of 1500A/mm² at 4.6 K and 6T. Materials used are usually Nb alloys, e.g. NbTi, Nb₃Ti or Nb₃Sn.
- Superconducting magnets are not only used for strong fields but also when there is no space for iron poles, like inside a particle physics detector.
Problems:

- Superconductivity brakes down for too large fields
- Due to the Meissner-Ochsenfeld effect superconductivity current only flows on a thin surface layer.

Remedy:

- Superconducting cable consists of many very thin filaments (about 10µm).
Complex Potential of a Wire

Straight wire at the origin: \[ \nabla \times \vec{B} = \mu_0 \vec{j} \quad \Rightarrow \quad \vec{B}(r) = \frac{\mu_0 I}{2\pi r} \vec{e}_\phi = \frac{\mu_0 I}{2\pi r} \begin{pmatrix} -y \\ x \end{pmatrix} \]

Wire at \( \vec{a} \): \[ \vec{B}(x, y) = \frac{\mu_0 I}{2\pi (\vec{r} - \vec{a})^2} \begin{pmatrix} -[y - a_y] \\ x - a_x \end{pmatrix} \]

This can be represented by complex multipole coefficients \( \Psi_v \)

\[ \vec{B}(x, y) = -\nabla \Psi \quad \Rightarrow \quad B_x + iB_y = -\left( \partial_x + i\partial_y \right)\psi = -2\partial_w \psi \]

\[ B_x + iB_y = \frac{\mu_0 I}{2\pi} \frac{-i(w_a - w)}{(w_a - w)(\bar{w}_a - \bar{w})} = i \frac{\mu_0 I}{2\pi} \frac{-w_a}{1 - \frac{w_a}{a^2}} \]

\[ = i \frac{\mu_0 I}{2\pi} \partial_w \ln(1 - \frac{w_a}{a^2}) = -2\partial_w \Im\left\{ \frac{\mu_0 I}{2\pi} \ln(1 - \frac{w_a}{a^2}) \right\} \]

\[ \psi = \Im\left\{ \frac{\mu_0 I}{2\pi} \ln(1 - \frac{w_a}{a^2}) \right\} = -\Im\left\{ \frac{\mu_0 I}{2\pi} \sum_{\nu=1}^{\infty} \frac{1}{\nu} \left( \frac{w_a}{a^2} \right)^\nu \bar{w}^\nu \right\} \quad \Rightarrow \quad \Psi_v = \frac{\mu_0 I}{2\pi} \frac{1}{\nu} \frac{1}{a^\nu} e^{i\nu \phi_a} \]
Creating a multipole be created by an arrangement of wires:

\[ \Psi_v = \int_0^{2\pi} \frac{\mu_0}{2\pi} \frac{1}{v} \frac{1}{a'} e^{i\nu \varphi_a} \frac{dl}{d\varphi_a} d\varphi_a \]

\[ \Psi_v = \delta_{vn} \frac{\mu_0}{2} \frac{1}{n} \frac{1}{\hat{a}^n} \hat{I} \quad \text{if} \quad I(\varphi_a) = \hat{I} \cos n \varphi_a \]
Real Air-coil Multipoles
Quadrupole corrector

LHC dipole

RHIC Tunnel
Special SC Air-coil Magnets

LHC double quadrupole

Accuracy

RHIC Siberian Snake dipole