Sextupoles cause nonlinear dynamics, which can be chaotic and unstable.

\[
\begin{pmatrix}
  x_{n+1} \\
  x'_{n+1}
\end{pmatrix} = M_0 \begin{pmatrix}
  x_n \\
  x'_n
\end{pmatrix} - \frac{k_2 l_s}{2} \begin{pmatrix}
  0 \\
  x_n^2
\end{pmatrix}
\]

\[
\begin{pmatrix}
  x_{n+1} \\
  x'_{n+1}
\end{pmatrix} = \begin{pmatrix}
  \sqrt{\beta} & 0 \\
  -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}}
\end{pmatrix} \begin{pmatrix}
  \hat{x}_n \\
  \hat{x}'_n
\end{pmatrix}
\]

\[
\begin{pmatrix}
  \hat{x}_{n+1} \\
  \hat{x}'_{n+1}
\end{pmatrix} = \begin{pmatrix}
  \cos \mu & \sin \mu \\
  -\sin \mu & \cos \mu
\end{pmatrix} \begin{pmatrix}
  \hat{x}_n \\
  \hat{x}'_n
\end{pmatrix} - \frac{k_2 l_s}{2} \sqrt{\beta} \begin{pmatrix}
  0 \\
  \beta \hat{x}^2_n
\end{pmatrix}
\]

\[
\begin{pmatrix}
  \hat{x}_f \\
  \hat{x}'_f
\end{pmatrix} = \frac{k_2 l_s}{2} \beta^{\frac{3}{2}} \begin{pmatrix}
  1 - \cos \mu & \sin \mu \\
  -\sin \mu & 1 - \cos \mu
\end{pmatrix}^{-1} \begin{pmatrix}
  0 \\
  \hat{x}^2_f
\end{pmatrix} = \frac{k_2 l_s}{2} \beta^{\frac{3}{2}} \frac{1}{2 \sin \frac{\mu}{2}} \begin{pmatrix}
  -\cos \frac{\mu}{2} \\
  \sin \frac{\mu}{2}
\end{pmatrix} \hat{x}^2_f
\]

\[
\hat{x}_f = -\frac{4}{k_2 l_s} \beta^{-\frac{3}{2}} \tan \frac{\mu}{2}
\]

\[
\hat{x}'_f = \frac{4}{k_2 l_s} \beta^{-\frac{3}{2}} \tan^2 \frac{\mu}{2}
\]

\[
\begin{pmatrix}
  \Delta \hat{x}_{n+1} \\
  \Delta \hat{x}'_{n+1}
\end{pmatrix} = \begin{pmatrix}
  \cos \mu & \sin \mu \\
  -\sin \mu & \cos \mu
\end{pmatrix} \begin{pmatrix}
  \Delta \hat{x}_n \\
  \Delta \hat{x}'_n
\end{pmatrix} - \frac{k_2 l_s}{2} \beta^{\frac{3}{2}} \Delta \hat{x}^2_n - 4 \tan \frac{\mu}{2} \Delta \hat{x}_n
\]
The Dynamic Aperture

\[
\begin{pmatrix}
\Delta \hat{x}_{n+1} \\
\Delta \hat{x}'_{n+1}
\end{pmatrix} =
\begin{pmatrix}
\cos \mu & \sin \mu \\
-\sin \mu & \cos \mu
\end{pmatrix}
\begin{pmatrix}
\Delta \hat{x}_n \\
\Delta \hat{x}'_n
\end{pmatrix} -
\begin{pmatrix}
0 \\
\frac{k_2 l_s}{2} \beta^3 \Delta \hat{x}_n^2 - 4 \tan \frac{\mu}{2} \Delta \hat{x}_n
\end{pmatrix}
\]

\[
\begin{pmatrix}
\Delta \hat{x}_{n+1} \\
\Delta \hat{x}'_{n+1}
\end{pmatrix} =
\begin{pmatrix}
\cos \mu + 4 \sin \mu \tan \frac{\mu}{2} & \sin \mu \\
-\sin \mu + 4 \cos \mu \tan \frac{\mu}{2} & \cos \mu
\end{pmatrix}
\begin{pmatrix}
\Delta \hat{x}_n \\
\Delta \hat{x}'_n
\end{pmatrix} -
\begin{pmatrix}
0 \\
\frac{k_2 l_s}{2} \beta^3 \Delta \hat{x}_n^2
\end{pmatrix}
\]

\[
\text{Tr}[M] = 2 \frac{\cos \frac{\mu}{2} (1 + 2 \sin^2 \frac{\mu}{2})}{\cos \frac{\mu}{2}} \geq 2
\]

The additional fixed point is unstable!
If the chromaticity is corrected by a single sextupole:

\[ \xi_x = \xi_{0x} + \frac{1}{4\pi} \beta_x \eta_x k_2 l \approx 0 \]

\[ J_f = \frac{1}{2\beta^3} \left( \frac{4}{k_2 l_s} \frac{\tan \frac{\mu}{2}}{\cos \frac{\mu}{2}} \right)^2 \approx \frac{1}{2\beta} \left( \frac{\eta}{\xi_0 \pi} \frac{\sin \frac{\mu}{2}}{\cos^2 \frac{\mu}{2}} \right)^2 \]

Often the dynamic aperture is much smaller than the fixed point indicates!

When many sextupoles are used:

\[ \xi_{0x} + \frac{N}{4\pi} \beta_x \eta_x k_2 l \approx 0 \]

The sum of all \(k_2^2\) is then reduced to about

\[ \sum (k_2 l \beta)^2 \approx N (k_2 l \beta)^2 \approx \frac{1}{N} \left( \frac{4\pi}{\eta} \xi_0 \right)^2 \]

The dynamic aperture is therefore greatly increased when distributed sextupoles are used.
Due to the narrow region of unstable trajectories, sextupoles are used for slow particle extraction at a tune of 1/3.

The intersection of stable and unstable manifolds is a certain indication of chaos.
At instable fixed points, there is a stable and an instabile invariant curve.

Intersections of these curves (homoclinic points) lead to chaos.
On the path from Rotation to Libration around the Spin-Orbit-Coupling is a strong chaotic region.

\[ \frac{d^2(\vartheta + \varphi(t))}{dt^2} = -\alpha \left( \frac{a}{r(t)} \right)^3 \sin 2\vartheta \]
Tidal forces from moon and sun cause a stabilization of the rotation axis.
\[
\begin{pmatrix}
  x' \\
  a'
\end{pmatrix} = \begin{pmatrix}
  0 & 1 \\
  -K & 0
\end{pmatrix} \begin{pmatrix}
  x \\
  a
\end{pmatrix} + \begin{pmatrix}
  0 \\
  \Delta f
\end{pmatrix}
\]

\[
\begin{pmatrix}
  x \\
  a
\end{pmatrix} = \sqrt{2J} \begin{pmatrix}
  \sqrt{\beta} & 0 \\
  -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}}
\end{pmatrix} \begin{pmatrix}
  \sin(\psi + \phi_0) \\
  \cos(\psi + \phi_0)
\end{pmatrix} = \sqrt{2J} \beta \ddot{S}
\]

This would be a solution with constant \( J \) and \( \phi \) when \( \Delta f = 0 \).

Variation of constants:

\[
\frac{J'}{\sqrt{2J}} \beta \ddot{S} + \sqrt{2J} \phi_0' \begin{pmatrix}
  0 & \sqrt{\beta} \\
  -\frac{1}{\sqrt{\beta}} & -\frac{\alpha}{\sqrt{\beta}}
\end{pmatrix} \dot{S} = \begin{pmatrix}
  0 \\
  \Delta f
\end{pmatrix}
\]

\[
\frac{J'}{\sqrt{2J}} \ddot{S} + \sqrt{2J} \phi_0' \begin{pmatrix}
  0 & 1 \\
  -1 & 0
\end{pmatrix} \ddot{S} = \beta^{-1} \begin{pmatrix}
  0 \\
  \Delta f
\end{pmatrix}
\]

with

\[
\beta^{-1} = \begin{pmatrix}
  \frac{1}{\sqrt{\beta}} & 0 \\
  \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta}
\end{pmatrix}
\]

\[
\frac{J'}{\sqrt{2J}} = \cos(\psi + \phi_0) \sqrt{\beta} \Delta f \quad \text{and} \quad \sqrt{2J} \phi_0' = -\sin(\psi + \phi_0) \sqrt{\beta} \Delta f
\]
Simplification of linear motion

\[
\begin{pmatrix}
  x \\
a
\end{pmatrix} = \sqrt{2J} \begin{pmatrix}
  \sqrt{\beta} & 0 \\
  -\frac{a}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}}
\end{pmatrix} \begin{pmatrix}
  \sin(\psi + \phi_0) \\
  \cos(\psi + \phi_0)
\end{pmatrix} \Rightarrow \begin{cases}
  J' = 0 \\
  \phi_0' = 0
\end{cases}
\]

\[
\begin{pmatrix}
  x \\
a
\end{pmatrix} = \sqrt{2J} \begin{pmatrix}
  \sqrt{\beta} & 0 \\
  -\frac{a}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}}
\end{pmatrix} \begin{pmatrix}
  \sin \phi \\
  \cos \phi
\end{pmatrix} \Rightarrow \begin{cases}
  J' = 0 \\
  \phi' = \frac{1}{\beta}
\end{cases}
\]

\[
\begin{pmatrix}
  x \\
a
\end{pmatrix} = \sqrt{2J} \begin{pmatrix}
  \sqrt{\beta} & 0 \\
  -\frac{a}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}}
\end{pmatrix} \begin{pmatrix}
  \sin(\psi - \mu \frac{s}{L} + \varphi) \\
  \cos(\psi - \mu \frac{s}{L} + \varphi)
\end{pmatrix} \Rightarrow \begin{cases}
  J' = 0 \\
  \varphi' = \mu \frac{1}{L}
\end{cases}
\]

\[
\tilde{\psi} = \psi - \mu \frac{s}{L} \Rightarrow \tilde{\psi}(s + L) = \tilde{\psi}(s)
\]

Corresponds to Floquet's Theorem
Quasi-periodic Perturbation

\[ J' = \cos(\psi + \phi)\sqrt{2J\beta} \Delta f \quad , \quad \phi' = -\sin(\psi + \phi)\sqrt{\frac{\beta}{2J}} \Delta f \]

\[ J' = \cos(\tilde{\psi} + \phi)\sqrt{2J\beta} \Delta f \quad , \quad \phi' = \mu \frac{1}{L} - \sin(\tilde{\psi} + \phi)\sqrt{\frac{\beta}{2J}} \Delta f \]

New independent variable \[ \mathcal{G} = 2\pi \frac{S}{L} \]

\[ \frac{d}{d\mathcal{G}} J = \cos(\tilde{\psi} + \phi)\sqrt{2J\beta} \Delta f \frac{L}{2\pi} \quad , \quad \frac{d}{d\mathcal{G}} \phi = \nu - \sin(\tilde{\psi} + \phi)\sqrt{\frac{\beta}{2J}} \Delta f \frac{L}{2\pi} \]

\[ \Delta f(x) = \Delta f(\sqrt{2J\beta} \sin(\tilde{\psi} + \phi)) \]

The perturbations are \(2\pi\) periodic in \(\mathcal{G}\) and in \(\phi\)

\(\phi\) is approximately \(\phi \approx \nu \cdot \mathcal{G}\)

For irrational \(\nu\), the perturbations are quasi-periodic.
The motion remains Hamiltonian in the perturbed coordinates!

If there is a part in $\partial_\varphi H$ that does not depend on $\varphi, s \quad \Rightarrow$ Tune shift

The effect of other terms tends to average out.

$$\varphi(\mathcal{G}) - \varphi_0 \approx \mathcal{G} \cdot \partial_\varphi \langle H \rangle_{\varphi, \mathcal{G}}(J)$$

$$\nu(J) = \nu + \partial_\varphi \langle \Delta H \rangle_{\varphi, \mathcal{G}}(J)$$
Tune Shift Examples

\[ H(\phi, J) = \nu \cdot J - \frac{L}{2\pi} \int_0^x \Delta f(\hat{x}, s) \, d\hat{x} \quad , \quad \Delta \nu(J) = \partial_J \langle \Delta H \rangle_{\phi, \vartheta} \]

**Quadrupole:**
\[ \Delta f = -\Delta k \cdot x \]
\[ \Delta H = \frac{L}{2\pi} \Delta k \frac{1}{2} x^2 = \frac{L}{2\pi} \Delta k J \beta \sin^2(\tilde{\psi} + \phi) \]
\[ \langle \Delta H \rangle_{\phi, \vartheta} = \frac{1}{2\pi} \int_0^{2\pi} \Delta k \beta \, d\vartheta \, L \frac{J}{4\pi} = \int_0^L \Delta k \beta \, ds \frac{J}{4\pi} \Rightarrow \Delta \nu = \frac{1}{4\pi} \oint \Delta k \beta \, ds \]

**Sextupole:**
\[ \Delta f = -k_2 \frac{1}{2} x^2 \]
\[ \Delta H = \frac{L}{2\pi} k_2 \frac{1}{3!} x^3 = \frac{L}{2\pi} k_2 \frac{1}{3!} \sqrt{2J} \beta^3 \sin^3(\tilde{\psi} + \phi) \]
\[ \langle \Delta H \rangle_{\phi, \vartheta} = 0 \Rightarrow \Delta \nu = 0 \]

**Octupole:**
\[ \Delta f = -k_3 \frac{1}{3!} x^3 \]
\[ \Delta H = \frac{L}{2\pi} k_3 \frac{1}{4!} x^4 = \frac{L}{2\pi} k_3 \frac{1}{3!} (J\beta)^2 \sin^4(\tilde{\psi} + \phi) \]
\[ \langle \Delta H \rangle_{\phi, \vartheta} = \frac{J^2}{3!2\pi} \oint k_3 \beta^2 \, ds \left\langle \frac{1}{2^4} (e^{i\phi} - e^{-i\phi})^4 \right\rangle_{\phi} \Rightarrow \Delta \nu = J \frac{1}{16\pi} \oint k_3 \beta^2 \, ds \]
Nonlinear Resonances

\[
\frac{d}{d\vartheta} J = \cos(\widetilde{\psi} + \varphi) \sqrt{2J \beta \Delta f} \frac{L}{2\pi}, \quad \frac{d}{d\vartheta} \varphi = \nu - \sin(\widetilde{\psi} + \varphi) \sqrt{\frac{\beta}{2J} \Delta f} \frac{L}{2\pi}
\]

\[
\frac{d}{d\vartheta} \varphi = \partial J H, \quad \frac{d}{d\vartheta} J = -\partial_\varphi H, \quad H(\varphi, J, \vartheta) = \nu \cdot J - \frac{L}{2\pi} \int_0^x \Delta f(\hat{x}, s) d\hat{x}
\]

The effect of the perturbation is especially strong when

\[
\cos(\widetilde{\psi} + \varphi) \sqrt{\beta \Delta f} \quad \text{or} \quad \sin(\widetilde{\psi} + \varphi) \sqrt{\beta \Delta f}
\]

has contributions that hardly change, i.e. the change of

\[
\sqrt{\beta(\vartheta) \Delta f(x(\vartheta), \vartheta)}
\]

is in resonance with the rotation angle \( \varphi(\vartheta) \).

Periodicity allows Fourier expansion:

\[
H(\varphi, J, \vartheta) = \sum_{n,m=-\infty}^{\infty} \widehat{H}_{nm}(J) e^{i[n\vartheta + m\varphi]} = \sum_{n,m=-\infty}^{\infty} H_{nm}(J) \cos(n\vartheta + m\varphi + \Psi_{nm}(J))
\]

\[
H_{00}(J) = \left\langle H(\varphi, J, s) \right\rangle_{\varphi,s} \Rightarrow \text{Tune shift}
\]
The Single Resonance Model

\[
\frac{d}{d\vartheta} J = \sum_{n,m=-\infty}^{\infty} mH_{nm}(J) \sin(n\vartheta + m\varphi + \Psi_{nm}(J))
\]

\[
\frac{d}{d\vartheta} \varphi = \nu + \partial_J \sum_{n,m=-\infty}^{\infty} H_{nm}(J) \cos(n\vartheta + m\varphi + \Psi_{nm}(J))
\]

Strong deviation from: \( J = J_0, \quad \varphi = \nu \vartheta + \varphi_0 \)

Occur when there is coherence between the perturbation and the phase space rotation:

\[ n + m \frac{d}{ds} \varphi \approx 0 \]

Resonance condition: tune is rational \( n + m \nu = 0 \)

On resonance the integral would increases indefinitely!

Neglecting all but the most important term

\[
H(\varphi, J, \vartheta) \approx \nu J + H_{00}(J) + H_{nm}(J) \cos(n\vartheta + m\varphi + \Psi_{nm}(J))
\]
Fixed points

\[
\frac{d}{d\vartheta} J = mH_{nm}(J) \sin(n\vartheta + m\varphi + \Psi_{nm}(J))
\]

\[
\frac{d}{d\vartheta} \varphi = \nu + \Delta \nu(J) + \partial J [H_{nm}(J) \cos(n\vartheta + m\varphi + \Psi_{nm}(J))]
\]

\[
\Phi = \frac{1}{m} [n\vartheta + m\varphi + \Psi_{nm}(J)] , \quad \delta = \nu + \frac{n}{m}
\]

\[
\frac{d}{d\vartheta} J = mH_{nm}(J) \sin(m\Phi) , \quad \frac{d}{d\vartheta} \Phi = \delta + \Delta \nu(J) + H'_{nm}(J) \cos(m\Phi)
\]

\[
H(\varphi, J, \vartheta) \approx \delta J + H_{00}(J) + H_{nm}(J) \cos(m\Phi)
\]

Fixed points: \( \frac{d}{d\vartheta} J = mH_{nm}(J_f) \sin(m\Phi_f) = 0 \quad \Rightarrow \quad \Phi_f = \frac{k}{m} \pi \)

If \( \delta + \Delta \nu(J_f) \pm H'_{nm}(J_f) = 0 \) has a solution.

\[
\frac{d}{d\vartheta} \Delta J = \pm m^2 H_{nm}(J_f) \Delta \Phi , \quad \frac{d}{d\vartheta} \Delta \Phi = [\Delta \nu'(J_f) \pm H''_{nm}(J_f)] \Delta J
\]

Stable fixed point for: \( H_{nm}(J_f)[H''_{nm}(J_f) \pm \Delta \nu'(J_f)] < 0 \)
Sextupole: \[
\Delta f = -k_2 \frac{1}{2} x^2
\]

\[
\Delta H = \frac{L}{2\pi} k_2 \frac{1}{3!} x^3 = \frac{L}{2\pi} k_2 \frac{1}{3!} \sqrt{2J\beta}^3 \sin^3 (\psi + \phi)
\]

\[
= \frac{L}{2\pi} k_2 \frac{1}{3!} \sqrt{2J\beta}^3 \left[ \sin(3[\psi + \phi]) + 3 \sin(\psi + \phi) \right]
\]

Simplification: one sextupole \[
k_2(\mathcal{I}) = k_2 \delta(\mathcal{I}) = k_2 \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \cos(n \mathcal{I})
\]

\[
\Delta H = \frac{L}{2\pi} k_2 \frac{1}{3!} \sqrt{2J\beta}^3 \frac{1}{2\pi} \cos(-n \mathcal{I} + 3\phi + \psi - \frac{\pi}{2}) + \ldots
\]

\[
\Delta H \approx A_2 \sqrt{J}^3 \cos(3\Phi)
\]

\[
\Phi_f = 0, \frac{1}{3} \pi, \frac{2}{3} \pi, \ldots \bigg\} \Phi_f = \frac{1}{3} \pi, \pi, \frac{5}{3} \pi
\]

\[
\delta \pm A_2 \frac{3}{2} \sqrt{J} = 0 \quad \text{for} \quad \delta > 0
\]

All these fixed points are instable since \[
H_{nm}^{'} (J_f) H_{nm}^{''} (J_f) > 0
\]
Octupole:
\[ \Delta f = -k_3 \frac{1}{3!} x^3 \]
\[ \Delta H = \frac{L}{2\pi} k_3 \frac{1}{4!} x^4 = \frac{L}{2\pi} k_3 \frac{1}{3!} J^2 \beta^2 \sin^4(\tilde{\psi} + \varphi) \]
\[ = \frac{L}{2\pi} k_3 \frac{1}{3!8} J^2 \beta^2 [\cos(4[\tilde{\psi} + \varphi]) - 4\cos(\tilde{\psi} + \varphi) + 3] \]

Simplification: one octupole
\[ k_3(\vartheta) = k_3 \delta(\vartheta) = k_3 \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \cos(n\vartheta) \]
\[ \Delta H \approx A_3 J^2 [3 + \cos(4\Phi)] \quad \text{for} \quad \nu \approx \frac{n}{4} \]
\[ \Phi_f = 0, \frac{1}{4} \pi, \frac{2}{4} \pi, \ldots \quad \text{Either 8 fixed points:} \quad \delta < 0 \]
\[ \delta + A_3 2J (3 \pm 1) = 0 \quad \text{or none for:} \quad \delta > 0 \]

\[ H_{nm}(J_f)[H''_{nm}(J_f) \pm \Delta \nu'(J_f)] < 0 \]

Stability for \( (2A_3J)^2 [1 \pm 3] < 0 \), i.e. for the 4 outer fixed points.
Fixed points: \[ \frac{d}{d\vartheta} J = mH_{nm}(J_f) \sin(m\Phi_f) = 0 \quad \Rightarrow \quad \Phi_f = \frac{k}{m} \pi \]

If \( \delta + \Delta \nu(J_f) \pm H'_{nm}(J_f) = 0 \) has a solution.

\( \delta \) has to avoid the region \( \delta + \Delta \nu(J) \pm H'_{nm}(J) = 0 \) for all particles.

Assuming that the tune shift and perturbation are monotonous in \( J \):

This tune region has the width \( \Delta_{nm} = 2 |H'_{nm}(J_{\text{max}})| \) for strong resonances.

\( \Delta_{nm} \) is called Resonance Width, Resonance Strength, or Stop-Band Width.
Coupling Resonances

\[
\frac{d}{d\vartheta} J_x = \cos(\tilde{\psi}_x + \varphi_x) \sqrt{2 J_x x \beta_x \Delta f_x} \frac{L}{2\pi}, \quad \frac{d}{d\vartheta} \varphi_x = \nu_x - \sin(\tilde{\psi}_x + \varphi_x) \sqrt{\frac{\beta_x}{2 J_x} \Delta f_x} \frac{L}{2\pi}
\]

\[
\frac{d}{d\vartheta} J_y = \cos(\tilde{\psi}_y + \varphi_y) \sqrt{2 J_y y \beta_y \Delta f_y} \frac{L}{2\pi}, \quad \frac{d}{d\vartheta} \varphi_y = \nu_y - \sin(\tilde{\psi}_y + \varphi_y) \sqrt{\frac{\beta_y}{2 J_y} \Delta f_y} \frac{L}{2\pi}
\]

\[
\frac{d}{d\vartheta} \vec{\varphi} = \vec{\partial} J H, \quad \frac{d}{d\vartheta} \vec{J} = -\vec{\partial} \varphi H, \quad H(\vec{\varphi}, \vec{J}, \vartheta) = \vec{\nu} \cdot \vec{J} - \frac{L}{2\pi} \int_0^\infty \Delta f(\hat{x}, s) d\hat{x}
\]

The integral form can be chosen since it is path independent. This is due to the Hamiltonian nature of the force:

\[
\Delta f_{x,y}(x, y, s) = -\partial_{x,y} \Delta H(x, y, s)
\]

Single Resonance model for two dimensions means retaining only the amplitude dependent tune shift and one term in the two dimensional Fourier expansion:

\[
H(\vec{\varphi}, \vec{J}, \vartheta) = \vec{\nu} \cdot \vec{J} + H_{00}(\vec{J}) + H_{n\bar{m}}(\vec{J}) \cos(n\vartheta + m_x \varphi_x + m_y \varphi_y + \Psi_{n\bar{m}}(\vec{J}))
\]

For \( n + m_x \nu_x + m_y \nu_y \approx 0 \)

\[
m_x \varphi_x + m_y \varphi_y = \hat{m} \cdot \vec{\varphi}
\]
\[ n + m_x \nu_x + m_y \nu_y \approx 0 \]

means that oscillations in \( y \) can drive oscillations in \( x \) in

\[ x'' = -K x + \Delta f_x (x, y, s) \]

The resonance term in the Hamiltonian then changes only slowly:

\[
H(\vec{\phi}, \vec{J}, \mathcal{J}) = \vec{\nu} \cdot \vec{J} + H_{00}(\vec{J}) + H_{nm}(\vec{J}) \cos(n \mathcal{J} + m_x \varphi_x + m_y \varphi_y + \Psi_{nm}(\vec{J}))
\]

\[
\frac{d}{d \mathcal{J}} \vec{\phi} = \vec{\partial} J H , \quad \frac{d}{d \mathcal{J}} \vec{J} = -\vec{\partial} \varphi H
\]

\[ J = \vec{m} \cdot \vec{J} \]

\[ J_\perp = m_x J_x - m_y J_y = \vec{m} \times \vec{J} \quad \Rightarrow \quad \frac{d}{d \mathcal{J}} J_\perp = 0 \]

Difference resonances lead to stable motion since:

\[ n+ | m_x | \nu_x - | m_y | \nu_y \approx 0 \Rightarrow | m_x | J_x + | m_y | J_y = \text{const.} \]

Sum resonances lead to unstable motion since:

\[ n+ | m_x | \nu_x + | m_y | \nu_y \approx 0 \Rightarrow | m_x | J_x - | m_y | J_y = \text{const.} \]
$n + m_x \nu_x + m_y \nu_y \approx 0$ means that oscillations in y can drive oscillations in x in

\[ x'' = -Kx + \Delta f_x(x, y, s) \]

All these resonances have to be avoided by their respective resonance width.

The position of an accelerator in the tune plane $s$ called its Working Point.