Wave equation for all components
\[
\nabla^2 \vec{E} = \frac{1}{c^2} \partial_t^2 \vec{E} \\
\nabla^2 \vec{B} = \frac{1}{c^2} \partial_t^2 \vec{B}
\]

Search for simple modes:
Transverse electric and magnetic (TEM) waves cannot exist, since:

\[
E_z = 0 \text{ and } B_z = 0 \implies \vec{E}_\perp = 0 \text{ and } \vec{B}_\perp = 0
\]
Fourier expansion of the z-dependence: 

\[ \vec{E}(x, y, z, t) = \int \vec{E}_{k_z \omega}(x, y) e^{i k_z z - i \omega t} \, dk_z \, d\omega \]

\[ \vec{\nabla}^2 \vec{E} = \frac{1}{c^2} \partial_t^2 \vec{E} \Rightarrow \vec{\nabla}^2 E_z = -\left[\left(\frac{\omega}{c}\right)^2 - k_z^2\right] E_z \]

\[ \vec{\nabla}^2 \vec{B} = \frac{1}{c^2} \partial_t^2 \vec{B} \Rightarrow \vec{\nabla}^2 B_z = -\left[\left(\frac{\omega}{c}\right)^2 - k_z^2\right] B_z \]

Eigenvalue equation with boundary conditions:

\[ \vec{\nabla} \times \vec{B} = \frac{1}{c^2} \partial_t \vec{E} \]

\[ \vec{\nabla} \times B_z + ik_z \vec{e}_z \times \vec{B}_\perp = -i \omega \frac{1}{c^2} \vec{E}_\perp \]

\[ \vec{\nabla}_r \times B_z + ik_z \vec{e}_z \times \vec{B}_r = -i \omega \frac{1}{c^2} \vec{E}_\varphi \Rightarrow \partial_r B_z = 0 \]

Walls:

\[ \vec{E}_{\parallel} = 0 \quad \vec{B}_r = 0 \]

\[ E_z = 0 \quad \partial_r B_z = 0 \]

Solutions for E or B only exist for a discrete set of eigenvalues:

\[ \left(\frac{\omega}{c}\right)^2 - k_z^2 = k_n^{(E)}^2 \]

\[ \left(\frac{\omega}{c}\right)^2 - k_z^2 = k_n^{(B)}^2 \]

Due to different boundary conditions, \( E_z \) and \( B_z \) cannot simultaneously be nonzero.

TE modes have \( E_z = 0 \)

TM modes have \( B_z = 0 \)
Displacement relation

\[ \omega(k_z) = c \sqrt{A_n^2 + k_z^2} \]

Phase velocity \( v_{ph} = \omega / k_z = c \sqrt{1 + \left(\frac{A_n}{k_z}\right)^2} > c \)

Group velocity \( v_{gr} = \frac{d\omega}{dk_z} = c / \sqrt{1 + \left(\frac{A_n}{k_z}\right)^2} < c \)

For each excitation frequency \( \omega \) one obtains a propagation in the wave guide of

\[ e^{ik_zz}, \quad k_z = \sqrt{(\frac{\omega}{c})^2 - A_n^2} \]

Transport for \( \omega \) above the cutoff frequency \( \omega > \omega_n = cA_n \)

Damping for \( \omega \) below the cutoff frequency \( \omega < \omega_n = cA_n \)
Boundary conditions:

\[ E_z(\bar{x}_0) = 0 \quad \vec{\nabla}^2 E_z = [k_z^2 - (\frac{\omega}{c})^2]E_z \]

\[ E_z(\bar{x}) = E_z \sin\left(\frac{n\pi}{a} x\right) \sin\left(\frac{m\pi}{b} y\right) \]

\[ (\frac{\omega}{c})^2 - k_z^2 = k_{nm}^{(B)^2} = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 \]

\[ \partial_r B_z(\bar{x}_0) = 0 \quad \vec{\nabla}^2 B_z = [k_z^2 - (\frac{\omega}{c})^2]B_z \]

\[ B_z(\bar{x}) = B_z \cos\left(\frac{n\pi}{a} x\right) \cos\left(\frac{m\pi}{b} y\right) \]

\[ (\frac{\omega}{c})^2 - k_z^2 = k_{nm}^{(E)^2} = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 \]

TE and TM modes happen to have the same eigenvalues.

For simplicity one still looks at TE and TM modes separately.
Rectangular TE Modes

Boundary conditions: 
\[ E_z(\vec{x}) = 0 \]
\[ \vec{E}_{\parallel}(\vec{x}_0) = 0 \]
\[ E_x(\vec{x}) = [A \cos\left(\frac{n\pi}{a}x\right) + B \sin\left(\frac{n\pi}{a}x\right)] \sin\frac{m\pi}{b}y \]
\[ E_y(\vec{x}) = \sin\left(\frac{n\pi}{a}x\right) [C \cos\left(\frac{m\pi}{b}y\right) + D \sin\left(\frac{m\pi}{b}y\right)] \]
\[ \vec{\nabla}_\perp \cdot \vec{E}_\perp = 0 \implies D = 0, \quad B = 0, \quad C = -A \frac{n}{a} \frac{b}{m} \]
\[ \vec{\nabla}_\perp \times \vec{E}_\perp = i\omega B_z \cos\left(\frac{n\pi}{a}x\right) \cos\left(\frac{m\pi}{b}y\right) \implies A \left[ \frac{b}{m\pi} \left( \frac{m\pi}{b} \right)^2 + \left( \frac{n\pi}{a} \right)^2 \right] = -i\omega B_z \]

\[ \vec{B}_r(\vec{x}_0) = 0 \quad B_x(\vec{x}) = \sin\left(\frac{n\pi}{a}x\right) [C' \cos\left(\frac{m\pi}{b}y\right) + D' \sin\left(\frac{m\pi}{b}y\right)] \]
\[ B_y(\vec{x}) = [A' \cos\left(\frac{n\pi}{a}x\right) + B' \sin\left(\frac{n\pi}{a}x\right)] \sin\left(\frac{m\pi}{b}y\right) \]
\[ \vec{\nabla}_\perp \times \vec{B}_\perp = 0 \implies D' = 0, \quad B' = 0, \quad C' = A' \frac{n}{a} \frac{b}{m} \]
\[ \vec{\nabla}_\perp \cdot \vec{B}_\perp = -ik_z B_z \cos\left(\frac{n\pi}{a}x\right) \cos\left(\frac{m\pi}{b}y\right) \implies A' \frac{b}{m\pi} k_{nm}^{(E)} = -ik_z B_z \]
Rectangular TE and TM Modes

**TE Modes**

\[ \vec{B}(\vec{x}) = B_z \left( \frac{n\pi}{a} \frac{k_z}{k_{nm}^E} \sin\left(\frac{n\pi}{a} x\right) \cos\left(\frac{m\pi}{b} y\right) \sin(k_z z - \omega t) \right) \]

**TM Modes:**

Exchange of E and B

\[ \vec{E}(\vec{x}) = \frac{\omega}{k_{nm}^E} B_z \left( \cos\left(\frac{n\pi}{a} x\right) \sin\left(\frac{m\pi}{b} y\right) \sin(k_z z - \omega t) \right) \]

Notation: TE_{nm} Mode

- \( n = 1 \)
- \( m = 0 \)
Cylindrical Wave Guides

TM Modes:

\[ E_z(x_0) = 0 \]
\[ \vec{\nabla}_\perp^2 E_z = [k_z^2 - (\omega/c)^2]E_z \]
\[ (\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\phi^2) E_z = [k_z^2 - (\omega/c)^2]E_z \]
\[ (\xi^2 \partial_\xi^2 + \xi\partial_\xi + \xi^2 - n^2) E_z = 0, \quad \xi = k_{nm}^{(E)} r \]

\[ E_z(\vec{x}) = E_z J_n(k_{nm}^{(B)} r)e^{in\phi} \]
\[ k_{nm}^{(B)} \text{ is the } m^{th} \text{ zero of the } n^{th} \text{ Bessel function over } r. \]

TE Modes:

\[ \partial_r B_z(x_0) = 0 \]
\[ \vec{\nabla}_\perp^2 B_z = [k_z^2 - (\omega/c)^2]B_z \]
\[ B_z(\vec{x}) = B_z J_n(k_{nm}^{(E)} r)e^{in\phi} \]
\[ k_{nm}^{(E)} \text{ is the } m^{th} \text{ extremeum of } J_n \text{ over } r. \]

Notation: TE_{nm} Mode
Fundamental Mode

Mode for particle acceleration: TM$_{01}$

\[ E_r(\vec{x}) = -E_z r_1 k_z J_0 ' \left( \frac{r}{r_1} \right) \sin(k_z z - \omega t) \]

\[ E_\phi(\vec{x}) = 0 \]

\[ B_r(\vec{x}) = 0 \]

\[ B_\phi(\vec{x}) = -E_z r_1 \frac{\omega}{c^2} J_0 ' \left( \frac{r}{r_1} \right) \sin(k_z z - \omega t) \]
Resonant Cavities

TE Modes: Standing waves with nodes

\[ B_z(\vec{x}) \propto \sin(k_z z) \sin(\omega t), \quad k_z = \frac{l\pi}{L} \]

\( l > 0 \)

TM Modes: Standing waves with nodes

\[ E_z(\vec{x}) \propto \cos(k_z z) \cos(\omega t), \quad k_z = \frac{l\pi}{L} \]

\( l \geq 0 \)

For each mode \( \text{TE}_{nm} \) or \( \text{TM}_{nm} \) there is a discrete set of frequencies that can be excited.

\[ \omega_{nm}^{(E/B)} = c \sqrt{k_{nm}^{(E/B)} + \left(\frac{l\pi}{L}\right)^2} \]
Resonant Cavities Examples

Rectangular cavity:

\[ \omega^{(E/B)}_{nm} = c \sqrt{ \left( \frac{n\pi}{L_x} \right)^2 + \left( \frac{m\pi}{L_y} \right)^2 + \left( \frac{l\pi}{L_z} \right)^2 } \]

Fundamental acceleration mode: \( \omega^{(B)}_{110} = c \frac{\pi}{L} \sqrt{2} \)

\( L_x = L_y = 22\text{cm} \quad \Rightarrow \quad f^{(B)}_{110} = 1.0\text{GHz} \)

Pill Box cavity:

\[ \omega^{(E/B)}_{nm} = c \sqrt{ k^{(E/B)}_{nm}^2 + \left( \frac{l\pi}{L} \right)^2 } \]

\( k^{(B)}_{nm} r \) is the m\textsuperscript{th} 0 of the n\textsuperscript{th} Bessel function.

\( k^{(E)}_{nm} r \) is the m\textsuperscript{th} extremeum of \( J_n \)

Fundamental acceleration mode: \( \omega^{(E)}_{010} = c \frac{2.4}{r} \)

\( r = 11\text{cm} \quad \Rightarrow \quad f^{(M)}_{010} = 1.0\text{GHz} \)
500MHz Cavity of DORIS:

\[ r = 23.1 \text{ cm} \quad \Rightarrow \quad f_{010}^{(M)} = 0.4967 \text{GHz} \]

- The frequency is increased and tuned by a tuning plunger.
- An inductive coupling loop excites the magnetic field at the equator of the cavity.
Superconducting Cavities

Q = $10^{10}$

E = 20MV/m

A bell with this Q would ring for a year.

- Very low wall losses.
- Therefore continuous operation is possible.
- Energy recovery becomes possible.

Normal conducting cavities

- Significant wall losses.
- Cannot operate continuously with appreciable fields.
- Energy recovery was therefore not possible.
The field in many cells can be excited by a single power source and a single input coupler in order to have the voltage of several cavities available.

Example: PETRA Cavity

\[ f_{\text{res}} = 500\text{MHz} \]
\[ R_s = 18.0 \times 10^6 \Omega \]
\[ 125\text{kW} \rightarrow 2.12\text{MV} \]

Without the walls: Long single cavity with too large wave velocity.

Thick walls: shield the particles from regions with decelerating phase.
The iris size is chosen to let the phase velocity equal the particle velocity.

- \( \pi \) mode
  - Long initial settling or filling time, not good for pulsed operation.

- \( \frac{\pi}{2} \) mode
  - Small shunt impedance per length.

- \( \frac{2\pi}{3} \) mode
  - Common compromise.
The iris size is chosen to let the phase velocity equal the particle velocity.

Loss free propagation: \( k = \frac{2\pi}{nd} \)

Standing wave cavity.

Traveling wave cavity (wave guide).
The Klystron as Power Source

- DC acceleration to several 10kV, 100kV pulsed
- Energy modulation with a cavity
- Time of flight density modulation
- Excitation of a cavity with output coupler
- Only works for non-relativistic electrons

\[ P = \eta U_0 I_{\text{beam}}, \quad \eta \leq 65\% \]

- Power < 1.5MW
- Power < 40MW pulsed

I up to > 10A