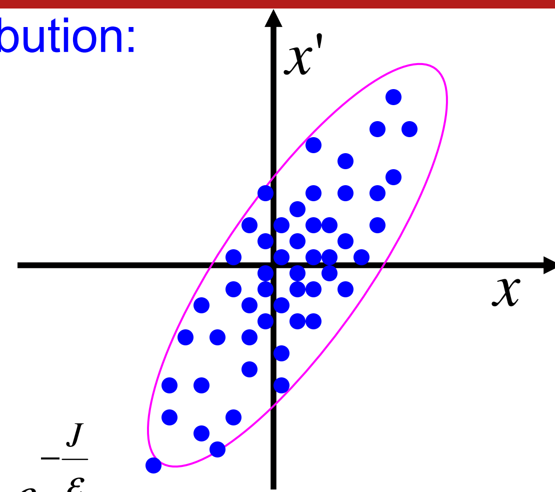
Phase Space Distribution

Often one can fit a Gauss distribution to the particle distribution:

$$\rho(x,x') = \tfrac{1}{2\pi\varepsilon} e^{-\frac{\gamma x^2 + 2\alpha x x' + \beta x'^2}{2\varepsilon}}$$



The equi-density lines are then ellipses. And one chooses the starting conditions for β and α according to these ellipses!

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin\phi_0 \\ \cos\phi_0 \end{pmatrix} \qquad \rho(J,\phi_0) = \tfrac{1}{2\pi\varepsilon} e^{-\frac{J}{\varepsilon}}$$

$$\langle 1 \rangle = \tfrac{1}{2\pi\varepsilon} \int_0^{2\pi}\int_0^{\infty} e^{-J/\varepsilon} dJ d\phi_0 = 1$$

Initial beam distribution ⟶ initial α, β, γ

$$\langle x^2 \rangle = \tfrac{1}{2\pi\varepsilon} \iint 2J\beta \sin\phi_0^2 e^{-J/\varepsilon} dJ d\phi_0 = \varepsilon\beta \quad \longrightarrow \quad \langle x'^2 \rangle = \varepsilon\gamma$$

$$\langle x x' \rangle = -\tfrac{1}{2\pi\varepsilon} \iint 2J\alpha \sin\phi_0^2 e^{-J/\varepsilon} dJ d\phi_0 = \varepsilon\alpha$$

$$\varepsilon = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2}$$ is called the emittance.

Invariant of Motion

$$x(s) = \sqrt{2J\beta(s)}\,\sin(\psi(s) + \phi_0)$$

Where J and ϕ are given by the starting conditions x_0 and x'_0.

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = 2J$$

Leads to the invariant of motion:

$$f(x, x', s) = \gamma(s)x^2 + 2\alpha(s)xx' + \beta(s)x'^2 \quad \Rightarrow \quad \tfrac{d}{ds} f = 0$$

It is called the Courant-Snyder invariant.