Phase Space Distribution

Often one can fit a Gauss distribution to the particle distribution:

\[
\rho(x, x') = \frac{1}{2\pi \varepsilon} e^{\frac{-\gamma x^2 + 2\alpha xx' + \beta x'^2}{2\varepsilon}}
\]

The equi-density lines are then ellipses. And one chooses the starting conditions for \( \beta \) and \( \alpha \) according to these ellipses!

\[
\begin{pmatrix}
  x \\
  x'
\end{pmatrix}
= \sqrt{2J}
\begin{pmatrix}
  \sqrt{\beta} & 0 \\
  -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}}
\end{pmatrix}
\begin{pmatrix}
  \sin \phi_0 \\
  \cos \phi_0
\end{pmatrix}
\]

\[
\rho(J, \phi_0) = \frac{1}{2\pi \varepsilon} e^{-\frac{J}{\varepsilon}}
\]

\[
\langle 1 \rangle = \frac{1}{2\pi \varepsilon} \int_0^{2\pi} \int_0^\infty e^{-J/\varepsilon} dJ d\phi_0 = 1
\]

Initial beam distribution

\[
\langle x^2 \rangle = \frac{1}{2\pi \varepsilon} \int_0^{2\pi} \int_0^\infty 2J \beta \sin \phi_0^2 e^{-J/\varepsilon} dJ d\phi_0 = \varepsilon \beta
\]

\[
\langle x^2 \rangle = \frac{1}{2\pi \varepsilon} \int_0^{2\pi} \int_0^\infty 2J \alpha \sin \phi_0^2 e^{-J/\varepsilon} dJ d\phi_0 = \varepsilon \alpha
\]

\[
\varepsilon = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}
\]

is called the emittance.
Invariant of Motion

\[ x(s) = \sqrt{2J\beta(s)} \sin(\psi(s) + \phi_0) \]

Where \( J \) and \( \phi \) are given by the starting conditions \( x_0 \) and \( x'_0 \).

\[ \gamma x^2 + 2\alpha xx' + \beta x'^2 = 2J \]

 Leads to the invariant of motion:

\[ f(x, x', s) = \gamma(s)x^2 + 2\alpha(s)xx' + \beta(s)x'^2 \quad \Rightarrow \quad \frac{d}{ds} f = 0 \]

It is called the Courant-Snyder invariant.