The Single Resonance Model

\[ \frac{d}{d\vartheta} J = \sum_{n,m=-\infty}^{\infty} mH_{nm}(J) \sin(n \vartheta + m \phi + \Psi_{nm}(J)) \]

\[ \frac{d}{d\vartheta} \phi = \nu + \partial J \sum_{n,m=-\infty}^{\infty} H_{nm}(J) \cos(n \vartheta + m \phi + \Psi_{nm}(J)) \]

Strong deviation from: \( J = J_0 \), \( \phi = \nu \vartheta + \phi_0 \)

Occur when there is coherence between the perturbation and the phase space rotation: \( n + m \frac{d}{ds} \phi \approx 0 \)

Resonance condition: tune is rational \( n + m \nu = 0 \)

On resonance the integral would increases indefinitely!

Neglecting all but the most important term

\[ H(\phi, J, \vartheta) \approx \nu J + H_{00}(J) + H_{nm}(J) \cos(n \vartheta + m \phi + \Psi_{nm}(J)) \]
for \( \nu \approx \frac{n}{3} \)

\[ \Phi_f = \frac{1}{3} \pi, \pi, \frac{5}{3} \pi \]

for \( \delta > 0 \)

for \( \nu \approx \frac{n}{4} \)

\[ \Phi_f = \frac{1}{3} \pi, \pi, \frac{5}{3} \pi \]

for \( \delta < 0 \)

or none for:

\( \delta > 0 \)

Either 8 fixed points: \( \delta < 0 \)

How can the motion inside the fixed points be simplified for a real accelerator?

\( \rightarrow \) Normal From Theory
Coupling Resonances

\[ \frac{d}{d\vartheta} J_x = \cos(\tilde{\psi}_x + \varphi_x) \sqrt{2J_x \beta_x \Delta f_x \frac{L}{2\pi}} \quad , \quad \frac{d}{d\vartheta} \varphi_x = v_x - \sin(\tilde{\psi}_x + \varphi_x) \sqrt{\beta_x \frac{L}{2J_x \Delta f_x}} \]

\[ \frac{d}{d\vartheta} J_y = \cos(\tilde{\psi}_y + \varphi_y) \sqrt{2J_y \beta_y \Delta f_y \frac{L}{2\pi}} \quad , \quad \frac{d}{d\vartheta} \varphi_y = v_y - \sin(\tilde{\psi}_y + \varphi_y) \sqrt{\beta_y \frac{L}{2J_y \Delta f_y}} \]

\[ \frac{d}{d\vartheta} \tilde{\varphi} = \tilde{\partial}_J H \quad , \quad \frac{d}{d\vartheta} \tilde{J} = -\tilde{\partial}_\varphi H \quad , \quad H(\tilde{\varphi}, \tilde{J}, \vartheta) = \vec{v} \cdot \vec{J} - \frac{L}{2\pi} \int_0^{\tilde{x}} \Delta \tilde{f}(\tilde{x}, s) d\tilde{x} \]

The integral form can be chosen since it is path independent. This is due to the Hamiltonian nature of the force:

\[ \Delta f_{x,y}(x, y, s) = -\partial_{x,y} \Delta H(x, y, s) \]

Single Resonance model for two dimensions means retaining only the amplitude dependent tune shift and one term in the two dimensional Fourier expansion:

\[ H(\tilde{\varphi}, \tilde{J}, \vartheta) = \vec{v} \cdot \tilde{J} + H_{00}(\tilde{J}) + H_{n\bar{n}}(\tilde{J}) \cos(n \vartheta + m_x \varphi_x + m_y \varphi_y + \Psi_{n\bar{n}}(\tilde{J})) \]

For \( n + m_x \nu_x + m_y \nu_y \approx 0 \)

\[ m_x \varphi_x + m_y \varphi_y = \tilde{m} \cdot \tilde{\varphi} \]
$n + m_x \nu_x + m_y \nu_y \approx 0$ means that oscillations in $y$ can drive oscillations in $x$ in

$$x'' = -Kx + \Delta f_x(x, y, s)$$

The resonance term in the Hamiltonian then changes only slowly:

$$H(\tilde{\phi}, \tilde{J}, \vartheta) = \tilde{v} \cdot \tilde{J} + H_{00}(\tilde{J}) + H_{\tilde{m}}(\tilde{J}) \cos(n \vartheta + \tilde{m} \cdot \tilde{\phi} + \Psi_{n\tilde{m}}(\tilde{J}))$$

$$\frac{d}{d\vartheta} \tilde{\phi} = \tilde{\partial}_J H, \quad \frac{d}{d\vartheta} \tilde{J} = -\tilde{\partial}_\vartheta H$$

$$J = \tilde{m} \cdot \tilde{J}$$

$$J_\perp = m_x J_x - m_y J_y = \tilde{m} \times \tilde{J} \implies \frac{d}{d\vartheta} J_\perp = 0$$

Difference resonances lead to stable motion since:

$$n + |m_x| \nu_x - |m_y| \nu_y \approx 0 \implies |m_x| J_x + |m_y| J_y = \text{const.}$$

Sum resonances lead to unstable motion since:

$$n + |m_x| \nu_x + |m_y| \nu_y \approx 0 \implies |m_x| J_x - |m_y| J_y = \text{const.}$$
\[ n + m_x v_x + m_y v_y \approx 0 \]  means that oscillations in \( y \) can drive oscillations in \( x \) in

\[ x'' = -K x + \Delta f_x (x, y, s) \]

All these resonances have to be avoided by their respective resonance width.

The position of an accelerator in the tune plane \( s \) called its Working Point.