Resonant Cavities

TE Modes: Standing waves with nodes

\[ B_z(\vec{x}) \propto \sin(k_z z) \sin(\omega t), \quad k_z = \frac{l\pi}{L} \]
\[ l > 0 \]

TM Modes: Standing waves with nodes

\[ E_z(\vec{x}) \propto \cos(k_z z) \cos(\omega t), \quad k_z = \frac{l\pi}{L} \]
\[ l \geq 0 \]

For each mode TE_{nm} or TM_{nm} there is a discrete set of frequencies that can be excited.

\[ \omega_{nm}^{(E/B)} = c \sqrt{k_{nm}^{(E/B)^2} + \left(\frac{l\pi}{L}\right)^2} \]
Rectangular cavity:

$$\omega_{nm}^{(E/B)} = c \sqrt{\left(\frac{n\pi}{L_x}\right)^2 + \left(\frac{m\pi}{L_y}\right)^2 + \left(\frac{l\pi}{L_z}\right)^2}$$

Fundamental acceleration mode: $$\omega_{110}^{(B)} = c \frac{\pi}{L} \sqrt{2}$$

$$L_x = L_y = 21.2cm \Rightarrow f_{110}^{(B)} = 1.0\text{GHz}$$

Pill Box cavity:

$$\omega_{nm}^{(E/B)} = c \sqrt{k^{(E/B)}_n^2 + \left(\frac{l\pi}{L}\right)^2}$$

- $$k^{(B)}_{nm} r$$ is the $$m^{th}$$ 0 of the $$n^{th}$$ Bessel function.
- $$k^{(E)}_{nm} r$$ is the $$m^{th}$$ extremeum of $$J_n$$

Fundamental acceleration mode: $$\omega_{010}^{(B)} = c \frac{2.40}{r}$$

$$2r = 22.9cm \Rightarrow f_{010}^{(B)} = 1.0\text{GHz}$$
500MHz Cavity of DORIS:

\[ r = 23.1\text{cm} \implies f_{010}^{(M)} = 0.4967\text{GHz} \]

- The frequency is increased and tuned by a tuning plunger.
- An inductive coupling loop excites the magnetic field at the equator of the cavity.
3 dominant features of RF systems

- Bandwidth

\[ I_{in} \Rightarrow V(\omega) \]

(1) The RF system has a resonant frequency \( \omega_0 \)

(2) The resonance curve has a characteristic width \( \Delta \omega = \frac{\omega_0}{2Q} \)

A resonant L/C/R circuit also has such characteristics
RF systems for accelerators

L and C: determined by the cavity geometry

\( R_s \): shunt impedance, related to surface res. R

\[
I_{in} = \left( \frac{1}{R_s} + iC\omega + \frac{1}{iL\omega} \right) U_C
\]

\[
\hat{U}_C = \sqrt{\frac{1}{R_s^2 + \left( \frac{1}{L\omega} - C\omega \right)^2}} \hat{I}_{in} \rightarrow \hat{U}_{Cres} = R_s \hat{I}_{in}
\]

\[
P_{RF} = \left\langle U_C I_{in} \right\rangle_t = \frac{1}{T} \int_0^T \text{Re}\left[\left( \frac{1}{R_s} + iC\omega + \frac{1}{iL\omega} \right) U_C \right] \text{Re}[U_C] dt = \frac{1}{2} \frac{1}{R_s} \hat{U}_C^2
\]

Quality factor:

\[
Q = 2\pi \frac{E}{\Delta E} = 2\pi \frac{\frac{1}{2} C U_C^2}{T P_{RF}} = \omega R_s C = R_s \sqrt{\frac{C}{L}}
\]

Geometry factor:

\[
\frac{R_s}{Q} = \sqrt{\frac{L}{C}}
\]
Geometry factor

\[ \vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho \]

\[ \vec{\nabla} \times \vec{E} = -\partial_t \vec{B} \]

\[ \vec{\nabla} \cdot \vec{B} = 0 \]

\[ \vec{\nabla} \times \vec{B} = \frac{1}{c^2} \left[ \frac{1}{\varepsilon_0} \vec{j} + \partial_t \vec{E} \right] \]

\[ \vec{E}' = \frac{1}{\alpha} \vec{E}(\alpha \vec{r}', \alpha t') \]

\[ \vec{B}' = \frac{1}{\alpha} \vec{B}(\alpha \vec{r}', \alpha t') \]

\[ \vec{j}' = \vec{j}(\alpha \vec{r}', \alpha t') \]

Reducing all sizes by \( \alpha \), letting the time pass \( \alpha \) times faster, reducing all charges by \( \alpha^3 \) and all currents by \( \alpha^2 \) leads to fields that are \( \alpha \) times smaller!

\[ L = \frac{V}{I} = \frac{\alpha^2 V'}{\alpha I'} = \alpha L' \]

\[ C = \frac{Q}{V} = \frac{\alpha^3 Q'}{\alpha^2 V'} = \alpha C \]

For any oscillating circuit \( \sqrt{\frac{L}{C}} \) is a size independent geometry factor!