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Electron Sources: Single Particle Dynamics, Space Charge Limited Emission
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So far we have discussed current density available from a cathode. Child-Langmuir law specifies maximum current density for a space-charge limited, nonrelativistic, 1-D beam *regardless* of available current density from the cathode. The law has a limited applicability to photoguns (applies to continuous flow, few 100s kV DC guns), but provides an interesting insight.

1D problem.

\[ \frac{d^2V}{dz^2} = -\frac{\rho}{\varepsilon_0} \]

\[ \frac{1}{2} m v_z^2 = eV \]
Eliminating $\rho$ and $v_z$ from the Poisson equation, we arrive at

$$\frac{d^2V}{dz^2} = -\frac{J_z}{\epsilon_0} \sqrt{\frac{m}{2eV}}$$

Solving for $V$:

$$V = \left(-\frac{9J_z}{4\epsilon_0} \sqrt{\frac{m}{2e}}\right)^{2/3} z^{4/3}$$

$V$ and $E_z$ are shown in the diagrams:

$V \propto z^{4/3}$

$E_z \propto z^{1/3}$

$E_z(0) = 0$

$$J = \frac{4\epsilon_0}{9} \sqrt{\frac{2e}{m}} \frac{V^{3/2}}{d^2}$$

$J [A/cm^2] = 2.33E^{3/2} [MV/m] / \sqrt{d[cm]}$
Let’s estimate bunch charge limit of a short pulse in a gun.

Assume ‘beer-can’ with rms $\sigma_{x,y}$ $\sigma_t$

also that $E_{cath}$ does not change much over the bunch duration (usually true for photoguns)

If

$$\frac{eE_{cath} \times (c\sigma_t)}{mc^2} \ll 1 \quad \text{or} \quad \frac{E_{cath}[\text{MV/m}] \times (c\sigma_t)[\text{mm}]}{511} \ll 1$$

motion during emission stays nonrelativistic.

Aspect ratio of emitted electrons near the cathode after the laser pulse has expired:

$$A = \frac{\perp}{\parallel} = \frac{2\sigma_x}{3(c\sigma_t) eE_{cath}(c\sigma_t)} = \frac{341}{E_{cath}[\text{MV/m}] (c\sigma_t[\text{mm}])^2}$$
More often than not $A >> 1$ in photoinjectors, i.e. the bunch looks like a pancake near the cathode (!).

For short bunch (note a factor of 2 due to image charge)

$$E_{s.c.} = \frac{\sigma}{\epsilon_0} \rightarrow q = 4\pi\epsilon_0 E_{cath} \sigma_x^2 = 0.11 \times E_{cath} \text{[MV/m]} \sigma_x \text{[mm]}^2 \text{ nC}$$

If emittance is dominated by thermal energy of emitted electrons, the following scaling applies (min possible emit.)

$$\epsilon_{n,x} = \sqrt{\frac{q}{4\pi\epsilon_0 E_{cath}}} \frac{k_B T_{\perp}}{m_e c^2} \quad \epsilon_n \text{[mm - mrad]} \geq 4 \sqrt{\frac{q \text{nC} E_{th} \text{[eV]}}{E_{cath} \text{[MV/m]}}}$$

Typically, the best values achieved in the photoguns are $\times 3$ larger
Cathode Field $\leftarrow E_{\text{th cathode}}$

- **Cathode Field**
  - $E_{\text{cath}} = 120$ MV/m
  - $\tau_{\text{laser}} = 2.7$ ps rms
  - $\sigma_{\text{laser}} = 0.5$ mm rms
  - $\tau_{\text{laser}} \rightarrow z = 0.08$ mm

- **Cathode Field**
  - $E_{\text{cath}} = 43$ MV/m
  - $\tau_{\text{laser}} = 5.8$ ps rms
  - $\sigma_{\text{laser}} = 0.85$ mm rms
  - $\tau_{\text{laser}} \rightarrow z = 0.12$ mm

- **Cathode Field**
  - $E_{\text{cath}} = 8$ MV/m
  - $\tau_{\text{laser}} = 13$ ps rms
  - $\sigma_{\text{laser}} = 2$ mm rms
  - $\tau_{\text{laser}} \rightarrow z = 0.12$ mm

**Notes:**
- $E_{\text{cath}} / E_{\text{s.charge}} = \text{same simulated emittance}$
- (but the measured is best for the higher gradient)
Fixed laser power, varied laser spot size at $V = 250$ kV

Graph: Limit predicted from the simple formula on previous slide.
Beam dynamics

Beam dynamics without collective forces is simple.

\[ \nabla \cdot \vec{B} = 0 \quad \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J} \quad \nabla \cdot \vec{D} = \rho \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]

\[ \vec{B} = \mu \vec{H} \]

\[ \vec{D} = \varepsilon \vec{E} \]

\[ \frac{d}{dt} (\gamma m \vec{\beta} c) = e (\vec{E} + \vec{v} \times \vec{B}) \]

\[ \nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \]

Calculating orbits in known fields is a single particle problem.
Effects on the phase space

- **Time varying fields:**
  - RF focusing
  - Coupler kicks

- **Aberrations:**
  - Geometric
  - Chromatic

- **Collective space charge forces**

- **Bunch phase space**

- Single particle solution integrated over finite bunch dimensions / energy (this lecture)

- Trickier space charge forces (next lecture)
Since emittance is such a central concept / parameter in the accelerator physics, it warrants few comments. For Hamiltonian systems, the phase space density is conserved (a.k.a. Liouville’s theorem). Rms (normalized) emittance most often quoted in accelerators’ field is based on the same concept and defined as following [and similarly for (y, py) or (E, t)]

\[
\varepsilon_{n,x} = \frac{1}{mc} \sqrt{\langle x^2 \rangle \langle p_x^2 \rangle - \langle xp_x \rangle^2} = \beta \gamma \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2} = \beta \gamma \varepsilon_x
\]

Strictly speaking, this quantity is not what Liouville’s theorem refers to, i.e. it does not have to be conserved in Hamiltonian systems (e.g. geometric aberrations ‘twist’ phase space, increasing effective area, while actual phase space area remains constant). Rms emittance is conserved for linear optics (and no coupling) only.

example of beam matched into periodic focusing with spherical aberrations
Emittance measurement

Usefulness of the rms emittance: it enters the envelope equations & can be readily measured, but provides *limited* info about the beam.

The combination of two slits give position and divergence → direct emittance measurement. Applicable for space charge dominated beams (if slits are small enough).

We’ll see later that the envelope equation in drift is

$$\sigma'' \approx \frac{1}{2I_0} \frac{1}{\gamma^3} \frac{1}{\sigma} + \frac{\varepsilon^2}{\sigma^3}$$

Vary lens strength and measure size to fit in eqn with 3 unknowns to find $\varepsilon$. OK to use if

$$\sigma << \varepsilon_n \sqrt{2\gamma_0 / I}$$
Emittance measurement system

- Main design challenge: for heavily space charge dominated beams, even beamlets passing through the slits are affected by space charge
- E.g. in ERL injector slits are 10 µm
- “Armor” slit intercepts most of the beam; kW beam power handling

measured phase space

70 pC/bunch

weak space charge
Busch’s theorem

Consider axially symmetric magnetic field, azimuthal force

\[ F_\theta = -e(rB_z - \dot{z}B_r) = \frac{1}{r} \frac{d}{dt}(\gamma mr^2 \dot{\theta}) \]

Flux through a circle centered on the axis and passing through \( e \)

\[ \Psi = \int_0^r 2\pi rB_z dr \]

When particle moves from \((r,z)\) to \((r+dr, z+dz)\) from Busch's theorem simply states that canonical angular momentum is conserved

\[ \frac{d\Psi}{dt} = 2\pi r(rB_z - \dot{z}B_r) \Rightarrow \dot{\theta} = \frac{-e}{2\pi \gamma mr^2} (\Psi - \Psi_0) \]

Busch’s theorem simply states that canonical angular momentum is conserved

\[ P_\theta = erA_\theta + \gamma mr^2 \dot{\theta} \quad (\Psi \rightarrow 2\pi rA_\theta, \Psi_0 \rightarrow 2\pi P_\theta/e \rightarrow \text{get Busch's formula}) \]
If magnetic field $B_z \neq 0$ at the cathode, the bunch acquires angular velocity

$$\dot{\theta} = -\frac{eB_z}{2\gamma m} \rightarrow \sigma_{\rho \perp} = \gamma m \sigma_{x,y} \dot{\theta}$$

$$\varepsilon_{n,mag} \sim \frac{\sigma_{\rho \perp}}{mc} \sigma_x \sim \frac{eB_0}{2mc} \sigma_x^2$$

$$\varepsilon_{n,mag} \text{ [mm - mrad]} \sim 0.03B[G]\sigma_x \text{ [mm]}^2$$

Normally, magnetic field at the cathode is a nuisance. However, it is useful for a) magnetized beams; b) round to flat beam transformation.
Similarly, rms emittance inside a solenoid is increased due to Busch’s theorem. This usually does not pose a problem (it goes down again) except when the beam is used in the sections with non-zero longitudinal magnetic field. In the latter case, producing magnetized beam from the gun becomes important.
Paraxial ray equation

Paraxial ray equation is equation of ‘about’-axis motion (angle with the main axis small & only first terms in off-axis field expansion are included).

\[
\frac{d}{dt} \left( \gamma m \mathbf{r} \right) - \gamma m r \dot{\theta}^2 = e(E_r + r \dot{B}_z)
\]

with \(- \dot{\theta} = \frac{q}{2 \gamma m} \left( B_z - \frac{\Psi_0}{\pi r^2} \right)\) and \(\dot{\gamma} = \beta e E_z / mc\)

\[
\ddot{r} + \frac{\beta e E_z}{\gamma mc} \dot{r} + \frac{e^2 B_z^2}{4 \gamma^2 m^2} r - \frac{e^2 \Psi_0^2}{4 \pi^2 \gamma^2 m^2 r^3} - \frac{e E_r}{\gamma m} = 0
\]

eliminating time and using \(E_r \approx -\frac{1}{2} r E_z' = -\frac{1}{2} r \gamma'' mc^2 / e\)

\[
r'' + \frac{\gamma r'}{\beta^2 \gamma} + \left( \frac{\gamma}{2 \beta^2 \gamma} + \frac{\Omega_L^2}{\beta^2 c^2} \right) r - \left( \frac{P_\theta}{\beta \gamma mc} \right)^2 \frac{1}{r^3} = 0
\]

\(\dot{p}_\theta \equiv e r A_\theta + \gamma m r^2 \dot{\theta}\)

\(\Omega_L \equiv -e B_z / 2 \gamma m\)

\[\theta_L = \int_0^z \frac{\Omega_L}{\beta c} \frac{dz}{\beta c}\]
With paraxial ray equation, the focal length can be determined

\[ f = 4V \frac{1 + \frac{1}{2} eV / mc^2}{1 + eV / mc^2} \frac{1}{E_2 - E_1} \]

\( eV \) is equal to beam K.E., \( E_1 \) and \( E_2 \) are electric fields before and after the aperture

\[ \frac{1}{f} = \int \left( \frac{\Omega_L}{\beta c} \right)^2 dz \\
= \int \left( \frac{eB_z}{2\beta \gamma mc} \right)^2 dz \approx \frac{1}{4} \left( \frac{e}{cp} \right)^2 B_z^2 L \]

\( cp/e[1\text{ MeV}/c] \rightarrow (B\rho)[33.4\text{ G - m}] \)
Real “lenses” are never perfect

R128 data: pincushion effect

Double-solenoid

DC gun

Double solenoid aberrations ($l_{sd}=3.5$ A). GUN CR[H, V]1 scan on VC2.

250 kV gun laser spot scan.
Solenoid: a fascinating lens

Thick lens + Larmor rotation produce interesting results

Misaligned solenoid with space charge beam produces asymmetric tail in the phase space

Beam-based alignment: solenoid current scan
RF focusing effect

SW longitudinal field in RF cavities requires transverse components from Maxwell’s equations → cavity can impart transverse momentum to the beam

Chambers (1965) and Rosenzweig & Serafini (1994) provide a fairly accurate (≥ 5 MeV) matrix for RF cavities (Phys. Rev. E 49 (1994) 1599 – beware, formula (13) has a mistypo)

Edges of the cavities do most of the focusing. For \( \gamma >> 1 \)

\[
\frac{1}{f} = -\frac{\gamma'}{\gamma_2} \left[ \frac{\cos^2 \varphi}{\sqrt{2}} + \frac{1}{\sqrt{8}} \right] \sin \alpha
\]

with \( \alpha = \frac{\ln(\gamma_2 / \gamma_1)}{\sqrt{8} \cos \varphi} \)

\( \gamma_1, \gamma_2, \gamma', \varphi \) Lorentz factor before, after the cavity, cavity gradient and off-crest phase

On crest, and when \( \Delta \gamma = \gamma' L << \gamma \):

\[
\frac{1}{f} \approx -\frac{3}{8} \frac{\gamma'^2 L}{\gamma^2}
\]
Emittance growth from RF focusing and kick

\[ \varepsilon_n = \frac{1}{mc} \sqrt{\langle x^2 \rangle \langle p_x^2 \rangle - \langle xp_x \rangle^2} \]

\[ p_x(x, z) = p_x(0, 0) + \frac{\partial p_x}{\partial x} x + \frac{\partial p_x}{\partial z} z + \frac{\partial^2 p_x}{\partial x \partial z} xz + \ldots \]

- **Kick**
- **Focusing**

\( p_x \) vs. \( x \)

**Kick**

**Focusing**
\[ \varepsilon_n^2 = \varepsilon_0^2 + \varepsilon_{\text{kick}}^2 + \varepsilon_{\text{focus}}^2 \]

- Kick effect on emittance is energy independent (modulo beam size) and can be cancelled downstream
- RF focusing effect scales \( \propto \frac{1}{\gamma} \) (in terms of \( p_x \)) and generally is not cancelled

\[
\varepsilon_{\text{kick}} = \frac{1}{mc} \left| \frac{\partial p_x}{\partial z} \right| \sigma_x \sigma_z
\]

\[
\varepsilon_{\text{focus}} = \frac{1}{mc} \left| \frac{\partial^2 p_x}{\partial z \partial x} \right| \sigma_x^2 \sigma_z
\]
Example: RF focusing in 2-cell SRF injector cavity

on crest

e.g. 500 kV, 3 (1) MV in the 1st cell, 1 mm x 1 mm bunch receives 0.26 (0.13) mm-mrad

Initial kinetic energy (MeV)
\[ \epsilon_{\text{kick}} \approx \sigma_x \sigma_z \left[ \theta_{\text{tilt}} \frac{\Delta E}{mc^2} k_{RF} \sin \varphi + x_{\text{off}} \frac{1}{mc} \frac{\partial}{\partial z} \left( \frac{\partial p_x}{\partial x} \right) \right] \]

- One would prefer on-crest running in the injector (and elsewhere!) from tolerances’ point of view.

For example, for a 3 MeV energy gain:

0.16 \sin \varphi \text{ mm-mrad per mrad of tilt}
For bunch compression, two approaches are used: magnetic compression (with lattice) and drift bunching. Magnetic compression relies on path vs. beam energy dependence, while drift bunching relies on velocity vs. energy dependence (i.e. it works only near the gun when $\gamma \geq 1$).

\[
\frac{\Delta \beta}{\beta} \approx \frac{\Delta \gamma}{\gamma} \frac{1}{\gamma^2 - 1}
\]

\[
L = c \beta \frac{\Delta l}{c \Delta \beta} = \frac{\Delta l}{(\Delta \gamma / \gamma)} (\gamma^2 - 1) = \frac{\lambda_{RF}}{2\pi} \frac{E}{eV_{\text{bun}}} \left( \frac{E^2}{(mc^2)^2} - 1 \right)
\]
• Space charge treatment
  – Debye length
  – Plasma frequency
  – Beam temperature
• Stationary distributions
• Emittance growth due to excess free energy
• Beam envelope equation
• Concept of equivalent beams
• Emittance compensation
• Computational aspects