Synchrotron Radiation Potentials

The instantaneous rest frame

\[ \Delta = \vec{x} - \vec{x}_0(t_r) \]
\[ c(t - t_r) = |\vec{x} - \vec{x}_0(t_r)| \implies t_r(t, \vec{x}) \]

The retarded potentials \( (\frac{1}{c} \Phi, \vec{A}) \)

\[ \tilde{\Phi}(\tilde{t}, \tilde{\vec{x}}) = \frac{q}{4\pi\varepsilon_0} \frac{1}{|\tilde{\vec{x}}|} \]
\[ \frac{1}{c} \Phi(t, \vec{x}) = \gamma(\frac{1}{c} \tilde{\Phi} + \vec{\beta} \cdot \tilde{\vec{A}}) = \gamma(t_r) \frac{1}{c} \tilde{\Phi}(\tilde{\vec{x}}) \]
\[ \tilde{\vec{A}}(\tilde{t}, \tilde{\vec{x}}) = 0 \]
\[ \vec{A}(t, \vec{x}) = \gamma(\tilde{\vec{A}}_{\parallel} + \vec{\beta} \frac{1}{c} \tilde{\Phi}) + \tilde{\vec{A}}_{\perp} = \vec{\beta}(t_r) \frac{1}{c} \Phi(t, \vec{x}) \]

Transformation to the laboratory frame

\[ |\tilde{\vec{x}}| = c\tilde{t} - c\tilde{t}_r = \gamma_r (ct - \vec{\beta}_r \cdot \vec{x}) - \gamma_r (ct_r - \vec{\beta}_r \cdot \vec{x}_0) \]
\[ = \gamma_r (\Delta - \vec{\beta}_r \cdot \Delta) \]

with \( \vec{x}_0 = \vec{x}_0(t_r) \), etc.

and \( \Delta(t, \vec{x}) \)
\[
\vec{B}(t, \vec{x}) = \vec{\nabla} \times \vec{A} \\
\vec{E}(t, \vec{x}) = -\frac{\partial}{\partial t} \vec{A} - \vec{\nabla} \Phi \\
\Rightarrow \vec{\nabla} \times \vec{\Phi} = \vec{\nabla} \times \vec{A} \quad \Rightarrow \quad \vec{\nabla} \times \vec{\Phi} = \vec{\nabla} \times \vec{A}
\]

\[
t_r = t - \frac{1}{c} | \vec{x} - \vec{x}_0(t_r) | \\
\vec{\beta}_r = [\vec{x}_0(t_r)]_{\vec{\beta}_r=0} = 0 \\
\Rightarrow \quad [\vec{\nabla} t_r]_{\vec{\beta}_r=0} = \left[ -\frac{1}{c} (\vec{\nabla} | \vec{x} - \vec{x}_0(t_r) |)_{t_r} \right]_{\vec{\beta}_r=0} = -\frac{1}{c} \vec{\Delta} = -\frac{1}{c} \vec{n}
\]

\[
\vec{\Phi} = \gamma_r \frac{q}{4\pi e_0} \frac{1}{|\vec{x}|}, \\
\vec{\nabla} \times \vec{\Phi} = \gamma_r (\Delta - \vec{\beta}_r \cdot \Delta)
\]

\[
\vec{E}(\vec{\tau}, \vec{\Phi}) = \left[ -\frac{1}{c} \vec{\Phi} \frac{\partial}{\partial t} \vec{\beta}_r - \vec{\nabla} \Phi \right]_{\vec{\beta}_r=0} = \left[ -\frac{1}{c} \vec{\Phi} \frac{\partial}{\partial t} t_r \vec{\beta}_r - \vec{\nabla} \Phi \right]_{\vec{\beta}_r=0} \\
= -\frac{1}{c} \vec{\Phi} \vec{\beta}_r - \left[ \vec{\nabla} t_r \vec{\beta}_r \cdot \left( \frac{\partial}{\partial \beta} \Phi \right)_{\Delta} + \vec{\nabla} \Delta (\frac{\partial}{\partial \Delta} \Phi)_{\beta} \right]_{\vec{\beta}_r=0} \\
= -\frac{1}{c} \vec{\Phi} \vec{\beta}_r + \frac{1}{c} \vec{n} (\vec{\beta}_r \cdot \vec{n}) \vec{\Phi} - \frac{\partial}{\partial \Delta} \vec{\Phi} = -\frac{1}{c} \vec{\Phi} \vec{\beta}_r - \frac{\partial}{\partial \Delta} \vec{\Phi} = -\frac{1}{c} \vec{\Phi} \vec{\beta}_r + \frac{1}{\Delta} \vec{\Phi} \vec{n}
\]
Rest-Frame-Radiation Power

\[ \vec{B}(\vec{t}, \vec{x}) = -\frac{1}{c^2} \vec{\Phi} \vec{n} \times \vec{\beta}_{r\perp} \quad \vec{E}(\vec{t}, \vec{x}) = -\frac{1}{c} \vec{\Phi} \vec{\beta}_{r\perp} + \frac{1}{\Delta} \vec{\Phi} \vec{n} \]

Energy flux instantaneous rest frame

\[ \vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} = \frac{1}{\mu_0} \frac{1}{c^2} \vec{\Phi}^2 \left[ \frac{1}{c} \vec{\beta}_{r\perp} \times (\vec{n} \times \vec{\beta}_{r\perp}) - \frac{1}{\Delta} \vec{n} \times (\vec{n} \times \vec{\beta}_{r\perp}) \right] \]

\[ = \frac{1}{\mu_0} \frac{1}{c^2} \vec{\Phi}^2 \left[ \frac{1}{c} \vec{\beta}_{r\perp}^2 \vec{n} + \frac{1}{\Delta} \vec{\beta}_{r\perp} \right] \]

Radiation power in the instantaneous rest frame

\[ \tilde{P} = \int \vec{S} \cdot \vec{n} \Delta^2 \sin \vartheta d\vartheta d\varphi = \frac{1}{\mu_0} \frac{\tilde{\Phi}^2}{c^3} \vec{\beta}_{r\perp}^2 \int \sin^3 \vartheta d\vartheta d\varphi = \frac{2}{3c} \frac{q^2}{4\pi\varepsilon_0} \tilde{\beta}_{r\perp}^2 \]

Now find the radiation power \( \tilde{P} \) in the lab frame \( \Sigma \)

Option 1: Derive the much more complicated form of \( \vec{E}, \vec{B}, \vec{S} \) and integrate over \( \vec{S} \).

Option 2: Express \( \tilde{P} \) in terms of 4-vectors and tensors and see how it transforms.

Option 3: Determine how photons change their frequency by the Lawrence transformation (via Doppler shift) and sum over all photons.
Spectral photon flux in the lab frame

Strategy: The sum over the energy of all photons that are within the shell of a sphere of radius $r$ and thickness $dr = cdt$ is the energy that was radiated between $tr = t - r/c$ and $tr + dt$.

At $t$:

$\Sigma$

Photons radiate in all directions, those in the shell of radius $r$ have emanated from the source between $tr = t - r/c$ and $tr + dt$.

Question: from where in the instantaneous rest frame did these photons come?
Spectral photon flux in the lab frame

Strategy: The sum over the energy of all photons that are within the shell of a sphere of radius $r$ and thickness $dr = c dt$ is the energy that was radiated between $tr = t - r/c$ and $tr + dt$.

At $t$:

$\sum$

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Photons radiate in all directions, those in the shell of radius \( r \) have emanated from the source between \( tr = t - r/c \) and \( tr + dt \).

Question: from where in the instantaneous rest frame did these photons come?

Higher photon density

Lower photon density
Spectral photon flux in the lab frame

Strategy: The sum over the energy of all photons that are within the shell of a sphere of radius \( r \) and thickness \( dr = c dt \) is the energy that was radiated between \( tr = t - r/c \) and \( tr + dt \).

\[
\sum \text{at } t:
\]

Photons radiate in all directions, those in the shell of radius \( r \) have emanated from the source between \( tr = t - r/c \) and \( tr + dt \).

Question: from where in the instantaneous rest frame did these photons come?

Lower photon density than by angular boost

Higher photon density than by angular boost
Spectral photon flux in the lab frame

Strategy: The sum over the energy of all photons that are within the shell of a sphere of radius $r$ and thickness $dr = c dt$ is the energy that was radiated between $tr = t - r/c$ and $tr + dt$.

$$\sum \text{at } t :$$

Photons radiate in all directions, those in the shell of radius $r$ have emanated from the source between $tr = t - r/c$ and $tr + dt$.

Question: from where in the instantaneous rest frame did these photons come?

Result: $P = \tilde{P}$

Higher photon density

Higher photon energy due to the Doppler shift

Lower photon density than by angular boost

Lower photon energy
Spectral photon flux in the lab frame

Strategy: The sum over the energy of all photons that are within the shell of a sphere of radius $r$ and thickness $dr = c dt$ is the energy that was radiated between $t_r = t + r/c$ and $t_r + dt$.

\[
P = \frac{dE}{dt} = c \frac{dE}{dr} = c \frac{1}{dr} \int_0^\infty \omega \rho(r) d\omega r^2 d\Omega dr = c \frac{1}{dr} \int_0^\infty \omega dN(r)
\]

\[
= c \frac{1}{dr} \int_0^\infty \omega d\tilde{N}(\tilde{r} (\tilde{\varphi})) = c \frac{1}{dr} \int_0^\infty \omega d\tilde{N}(\tilde{r})
\]

\[
= c \frac{1}{d\tilde{r}} \int_0^\infty \tilde{\omega} d\tilde{N}(\tilde{r}) \frac{\omega}{\tilde{\omega}} \frac{d\tilde{r}}{dr}
\]

\[
= \frac{\omega / \tilde{\omega}}{dr / d\tilde{r}} \tilde{P} = \tilde{P}
\]
Spectral photon flux in the lab frame

Strategy: The sum over the energy of all photons that are within the shell of a sphere of radius \( r \) and thickness \( dr = cdt \) is the energy that was radiated between \( tr = t + r/c \) and \( tr + dt \).

\[
P = \frac{dE}{dt} = c \frac{dE}{dr} = c \frac{1}{dr} \int \int_0^\infty \omega \rho(r) d\omega r^2 d\Omega dr = c \frac{1}{dr} \int \int_0^\infty \omega dN(r)
\]

\[
= c \frac{1}{dr} \int \int_0^\infty \omega d\tilde{N}(\tilde{r}(\tilde{\vartheta})) = c \frac{1}{dr} \int \int_0^\infty \omega d\tilde{N}(\tilde{r}) = c \frac{1}{d\tilde{r}} \int \int_0^\infty \tilde{\omega} d\tilde{N}(\tilde{r}) \frac{\omega}{\tilde{\omega}} \frac{d\tilde{r}}{dr}
\]

\[
= \frac{\omega}{\tilde{\omega}} \frac{dr}{d\tilde{r}} \tilde{P} = \tilde{P}
\]

At the surface of the sphere with radius \( r = c(t - t_r) \):

\[
(\begin{pmatrix} ct \\ \tilde{r} \end{pmatrix}) = (\begin{pmatrix} r \\ r \tilde{n} \end{pmatrix}) = \frac{1}{c} \begin{pmatrix} \omega \\ \tilde{k} \end{pmatrix} = \frac{1}{c} \begin{pmatrix} \omega \\ \omega \tilde{n} \end{pmatrix}
\]

The Lawrence transformation thus leads to the same linear transformation for \( r \) and \( \omega \), and therefore

\[
\frac{\tilde{\omega}}{\omega} = \frac{d\tilde{r}}{dr}
\]
Synchrotron Radiation Power

Power in the instantaneous rest frame: \( \tilde{P} = \frac{2r_e}{3c} mc^2 \tilde{\beta}_r^2 \)

Because it equals the power in the lab frame, there must be a Lorentz-invariant quantity that has the value \( \tilde{\beta}_r^2 \) in the rest frame.

The 4-vector of acceleration: \( b^\mu = \frac{d}{d\tau} P^\mu = \left( \frac{1}{c} \frac{d}{d\tau} E, \frac{d}{d\tau} \tilde{p} \right) \)

\[
b^\mu = \frac{d}{d\tau} P^\mu = \gamma \frac{d}{dt} P^\mu = \gamma \left( \frac{1}{c} \dot{E}, \dot{\tilde{p}} \right) \quad \Rightarrow \quad \gamma^2 \left( \dot{\tilde{p}}^2 - \frac{1}{c^2} \dot{E}^2 \right) = \text{const.}
\]

Evaluation in the instantaneous rest frame:

\[
\left[ \frac{d}{dt} \gamma \right] \tilde{\beta} = 0 \quad \Rightarrow \quad \left[ \gamma^2 \left( \dot{\tilde{p}}^2 - \frac{1}{c^2} \dot{E}^2 \right) \right] \tilde{\beta} = (mc)^2 \tilde{\beta}_r^2
\]

Radiation power in the laboratory frame

\[
P = \frac{2r_e}{3mc} \gamma^2 \left( \dot{\tilde{p}}^2 - \frac{1}{c^2} \dot{E}^2 \right)
\]
Transverse and Longitudinal Acceleration

\[ P = \frac{2r_e q^2}{3mc} \gamma^2 \left( \ddot{p}^2 - \frac{1}{c^2} \dot{E}^2 \right) \]

Transverse acceleration:

Magnetic:
\[ P = \frac{2r_e q^2}{3mc} \gamma^2 \left( \vec{v} \times \vec{B} \right)^2 = \frac{2r_e q^2}{3mc} \gamma^2 \cdot B^2 = \frac{2r_e}{3m^3c} \cdot \frac{p^4}{\rho^2} = \frac{2r_e c}{3} \cdot mc^2 \cdot \frac{\left(\beta \gamma\right)^4}{\rho^2} \]

Electric:
\[ \vec{E} \perp \vec{v} \quad \Rightarrow \quad P = \frac{2r_e q^2}{3mc} \gamma^2 \vec{E}^2 = \frac{2r_e c}{3} \cdot mc^2 \cdot \frac{\left(\beta \gamma\right)^4}{\rho^2} \]

Relative loss per turn:
\[ \eta = \frac{2\pi \rho}{v} \cdot \frac{1}{m\gamma c^2} = \frac{4\pi \cdot r_e}{\rho} \cdot \left(\beta \gamma\right)^3 \]

Example:
\[ \rho = 60m, \quad E = 5\text{GeV} \rightarrow \eta = 2 \cdot 10^{-4} \]

Longitudinal acceleration:

\[ P = \frac{2r_e q^2}{3mc} \gamma^2 \left[ \vec{E}^2 - \frac{1}{c^2} \left( \vec{E} \cdot \vec{v} \right)^2 \right] = \frac{2r_e q^2}{3mc} \cdot \vec{E}^2 \]

Efficiency:
\[ \eta = \frac{P}{q\vec{v} \cdot \vec{E}} = \frac{2r_e}{3\beta} \cdot \frac{q\vec{E}}{mc^2} \]

Example:
\[ \left| \vec{E} \right| = 30 \text{ MeV} \quad \rightarrow \quad \eta = 10^{-13} \]
### Examples

<table>
<thead>
<tr>
<th>Accelerator</th>
<th>Energy (GeV)</th>
<th>Radius (m)</th>
<th>Energy Loss (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>CESR @ CORNELL</td>
<td>5.3</td>
<td>60</td>
<td>0.02</td>
</tr>
<tr>
<td>PETRA @ DESY</td>
<td>23.5</td>
<td>195</td>
<td>0.6</td>
</tr>
<tr>
<td>LEP @ CERN</td>
<td>105</td>
<td>3000</td>
<td>3.4</td>
</tr>
<tr>
<td>HERA-e @ DESY</td>
<td>27.5</td>
<td>608</td>
<td>0.3</td>
</tr>
<tr>
<td>HERA-p @ DESY</td>
<td>920</td>
<td>582</td>
<td>2.0 x 10^-8</td>
</tr>
</tbody>
</table>

For the same momentum, the energy loss scales with the third power of the mass, i.e. Electrons radiate 6.0 x 10^9 times more than protons.