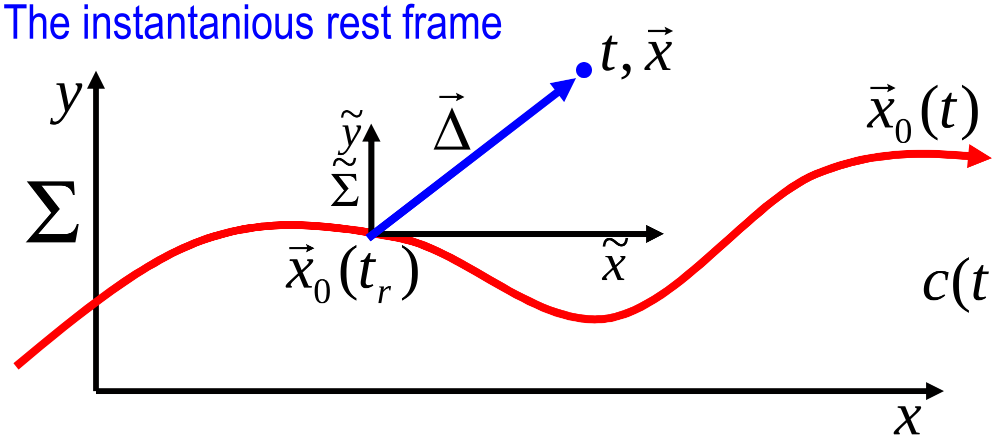
Synchrotron Radiation Potentials

CHESS & LEPP

The instantanious rest frame



$$\vec{\Delta} = \vec{x} - \vec{x}_0(t_r)$$

$$c(t - t_r) = |\vec{x} - \vec{x}_0(t_r)| \quad \Rightarrow \quad t_r(t, \vec{x})$$

The retarded potentials $(\frac{1}{c}\Phi, \vec{A})$

$$\tilde{\Phi}(\tilde{t}, \tilde{\vec{x}}) = \frac{q}{4\pi\varepsilon_0} \frac{1}{|\tilde{\vec{x}}|}$$

$$\tilde{\vec{A}}(\tilde{t}, \tilde{\vec{x}}) = 0$$

$$\tfrac{1}{c}\Phi(t, \vec{x}) = \gamma(\tfrac{1}{c}\tilde{\Phi} + \vec{\beta} \cdot \tilde{\vec{A}}) \qquad = \gamma(t_r)\tfrac{1}{c}\tilde{\Phi}(\tilde{\vec{x}})$$

$$\vec{A}(t, \vec{x}) \quad = \gamma(\tilde{\vec{A}}_\parallel + \vec{\beta}\tfrac{1}{c}\tilde{\Phi}) + \tilde{\vec{A}}_\perp = \vec{\beta}(t_r)\tfrac{1}{c}\Phi(t, \vec{x})$$

Transformation to the laboratory frame

$$|\tilde{\vec{x}}| = c\tilde{t} - c\tilde{t}_r = \gamma_r(ct - \vec{\beta}_r \cdot \vec{x}) - \gamma_r(ct_r - \vec{\beta}_r \cdot \vec{x}_{0r})$$

$$= \gamma_r(\Delta - \vec{\beta}_r \cdot \vec{\Delta})$$

with $\vec{x}_{0r} = \vec{x}_0(t_r)$, etc.

and $\vec{\Delta}(t, \vec{x})$

Synchrotron Radiation Fields

$$\vec{B}(t,\vec{x}) = \vec{\nabla}\times\vec{A} \qquad \tilde{\vec{B}}(\tilde{t},\tilde{\vec{x}}) = [\vec{\nabla}\times\vec{A}]_{\vec{\beta}_r=0,(t,\vec{x})\to(\tilde{t},\tilde{\vec{x}})}$$
$$\vec{E}(t,\vec{x}) = -\tfrac{\partial}{\partial t}\vec{A} - \vec{\nabla}\Phi \quad \Rightarrow \quad \tilde{\vec{E}}(\tilde{t},\tilde{\vec{x}}) = [-\tfrac{\partial}{\partial t}\vec{A} - \vec{\nabla}\Phi]_{\vec{\beta}_r=0,(t,\vec{x})\to(\tilde{t},\tilde{\vec{x}})}$$

$$t_r = t - \tfrac{1}{c}|\vec{x}-\vec{x}_0(t_r)| \qquad [\vec{\nabla}t_r]_{\vec{\beta}_r=0} = [-\tfrac{1}{c}(\vec{\nabla}|\vec{x}-\vec{x}_0(t_r)|)_{t_r}]_{\vec{\beta}_r=0} = -\tfrac{1}{c}\frac{\tilde{\vec{\Delta}}}{\tilde{\Delta}} = -\tfrac{1}{c}\tilde{\vec{n}}$$
$$\tilde{\vec{\beta}}_r = [\dot{\vec{x}}_0(t_r)]_{\vec{\beta}_r=0} = 0 \quad \Rightarrow \quad [\tfrac{\partial}{\partial t}t_r]_{\vec{\beta}_r=0} = [1-\tfrac{1}{c}(\tfrac{\partial}{\partial t}|\vec{x}-\vec{x}_0(t_r)|)_{t_r}]_{\vec{\beta}_r=0} = 1$$

$$\vec{A} = \tfrac{1}{c}\Phi\vec{\beta}_r \quad \Rightarrow \quad \tilde{\vec{B}}(\tilde{t},\tilde{\vec{x}}) = [\tfrac{1}{c}\Phi\vec{\nabla}\times\vec{\beta}_r]_{\vec{\beta}_r=0} = \tfrac{1}{c}\tilde{\Phi}[\vec{\nabla}t_r\times\tfrac{\partial}{\partial t_r}\vec{\beta}_r]_{\vec{\beta}_r=0} = -\tfrac{1}{c^2}\tilde{\Phi}\,\tilde{\vec{n}}\times\tilde{\dot{\vec{\beta}}}_{r\perp}$$

$$\tilde{\Phi} = \gamma_r\frac{q}{4\pi\varepsilon_0}\frac{1}{|\tilde{\vec{x}}|}, \qquad |\tilde{\vec{x}}| = \gamma_r(\Delta - \vec{\beta}_r\cdot\vec{\Delta})$$

$$\tilde{\vec{E}}(\tilde{t},\tilde{\vec{x}}) = [-\tfrac{1}{c}\Phi\tfrac{\partial}{\partial t}\vec{\beta}_r - \vec{\nabla}\Phi]_{\vec{\beta}_r=0} = [-\tfrac{1}{c}\Phi\tfrac{\partial}{\partial t}t_r\dot{\vec{\beta}}_r - \vec{\nabla}\Phi]_{\vec{\beta}_r=0}$$
$$= -\tfrac{1}{c}\tilde{\Phi}\tilde{\dot{\vec{\beta}}}_r - [\vec{\nabla}t_r\dot{\vec{\beta}}_r\cdot(\tfrac{\partial}{\partial\vec{\beta}}\Phi)_{\vec{\Delta}} + \vec{\nabla}\vec{\Delta}^{\mathrm{T}}(\tfrac{\partial}{\partial\vec{\Delta}}\Phi)_{\vec{\beta}}]_{\vec{\beta}_r=0}$$
$$= -\tfrac{1}{c}\tilde{\Phi}\tilde{\dot{\vec{\beta}}}_r + \tfrac{1}{c}\tilde{\vec{n}}(\tilde{\dot{\vec{\beta}}}_r\cdot\tilde{\vec{n}})\tilde{\Phi} - \tfrac{\partial}{\partial\tilde{\vec{\Delta}}}\tilde{\Phi} = -\tfrac{1}{c}\tilde{\Phi}\tilde{\dot{\vec{\beta}}}_{r\perp} - \tfrac{\partial}{\partial\tilde{\vec{\Delta}}}\tilde{\Phi} = -\tfrac{1}{c}\tilde{\Phi}\tilde{\dot{\vec{\beta}}}_{r\perp} + \tfrac{1}{\Delta}\tilde{\Phi}\,\tilde{\vec{n}}$$

Rest-Frame-Radiation Power

$$\vec{\tilde{B}}(\tilde{t},\vec{\tilde{x}}) = -\tfrac{1}{c^2}\tilde{\Phi}\,\vec{\tilde{n}}\times\dot{\vec{\tilde{\beta}}}_{r\perp} \qquad \vec{\tilde{E}}(\tilde{t},\vec{\tilde{x}}) = -\tfrac{1}{c}\tilde{\Phi}\,\dot{\vec{\tilde{\beta}}}_{r\perp} + \tfrac{1}{\tilde{\Delta}}\tilde{\Phi}\,\vec{\tilde{n}}$$

Energy flux instantanious rest frame

$$\vec{\tilde{S}} = \tfrac{1}{\mu_0}\vec{\tilde{E}}\times\vec{\tilde{B}} = \tfrac{1}{\mu_0}\tfrac{1}{c^2}\tilde{\Phi}^2[\tfrac{1}{c}\dot{\vec{\tilde{\beta}}}_{r\perp}\times(\vec{\tilde{n}}\times\dot{\vec{\tilde{\beta}}}_{r\perp}) - \tfrac{1}{\Delta}\vec{\tilde{n}}\times(\vec{\tilde{n}}\times\dot{\vec{\tilde{\beta}}}_{r\perp})]$$

$$= \tfrac{1}{\mu_0}\tfrac{1}{c^2}\tilde{\Phi}^2[\tfrac{1}{c}\dot{\vec{\tilde{\beta}}}{}^2_{r\perp}\vec{\tilde{n}} + \tfrac{1}{\tilde{\Delta}}\dot{\vec{\tilde{\beta}}}_{r\perp}]$$

Radiation power in the instantanious rest frame

$$\tilde{P} = \oint_{4\pi}\vec{\tilde{S}}\cdot\vec{\tilde{n}}\Delta^2\sin\vartheta d\vartheta d\varphi = \tfrac{1}{\mu_0}\tfrac{\tilde{\Delta}^2}{c^3}\tilde{\Phi}^2\dot{\vec{\tilde{\beta}}}{}^2_r\oint_{4\pi}\sin^3\vartheta d\vartheta d\varphi = \tfrac{2}{3c}\tfrac{q^2}{4\pi\varepsilon_0}\dot{\vec{\tilde{\beta}}}{}^2_r$$

Now find the radiation power P in the lab frame Σ

Option 1: Derive the much more complicated form of $\vec{E},\vec{B},\vec{S}$ and integrate over $\vec{S}$.

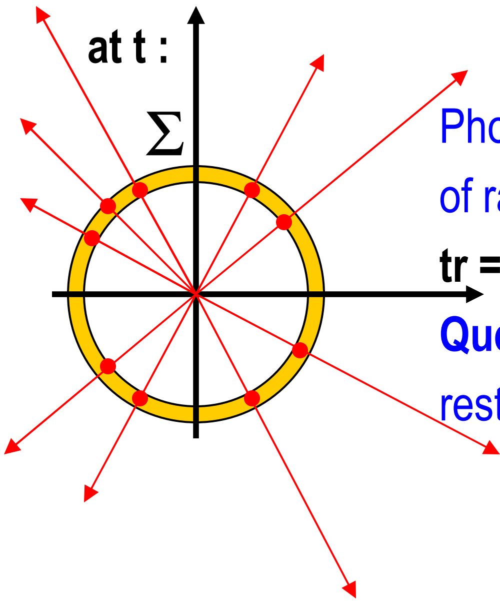
Option 2: Express P in terms of 4-vectors and tensors and see how it transforms.

Option 3: Determine how photons change their frequency by the Lawrence transformation (via Doppler shift) and sum over all photons.

Lab-Frame-Radiation Power

Spectral photon flux in the lab frame

Strategy: The sum over the energy of all photons that are within the shell of a sphere of radius **r** and thickness **dr = cdt** is the energy that was radiated between **tr = t - r/c** and **tr + dt**.



Photons radiate in all directions, those in the shell of radius r have emanated from the source between **tr = t – r/c** and **tr + dt**.

Question: from where in the instantanious rest frame did these photons come?

Lab-Frame-Radiation Power

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Spectral photon flux in the lab frame

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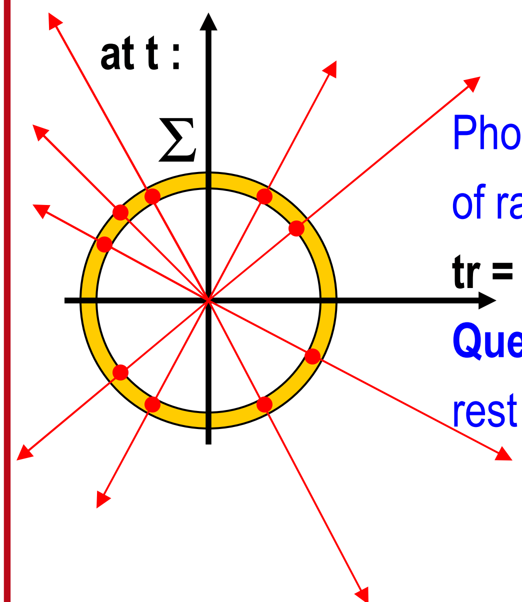

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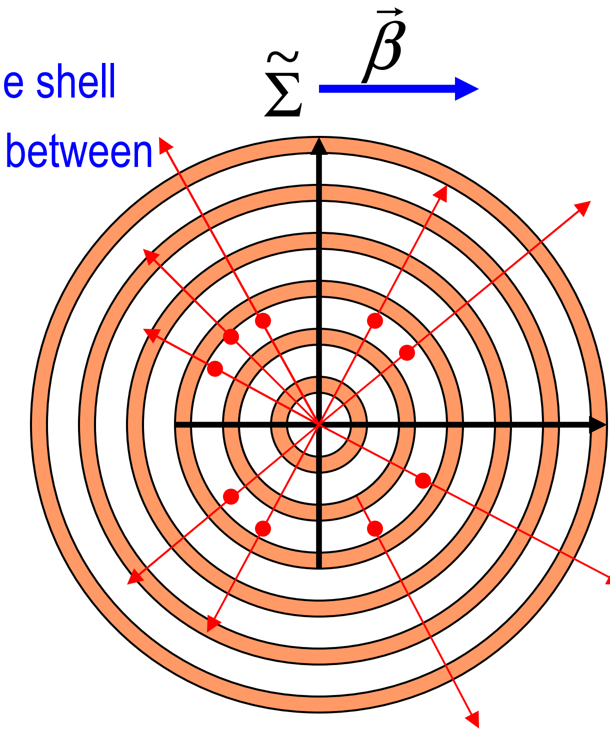

Lab-Frame-Radiation Power

CHESS & LEPP

Spectral photon flux in the lab frame

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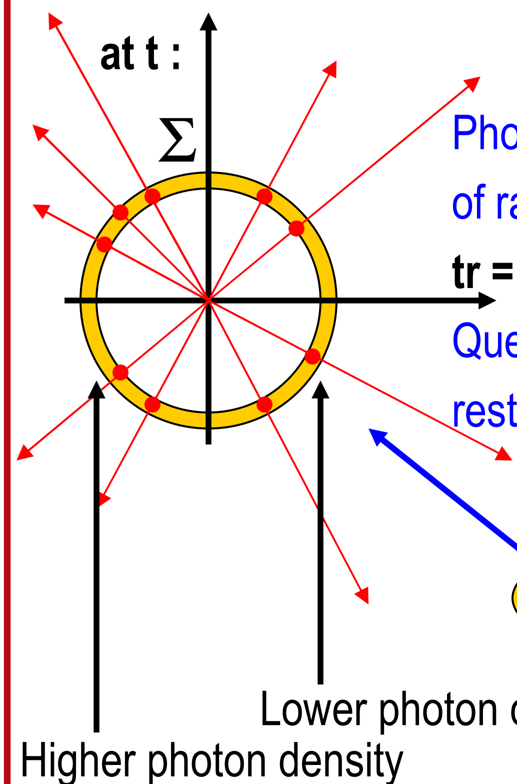

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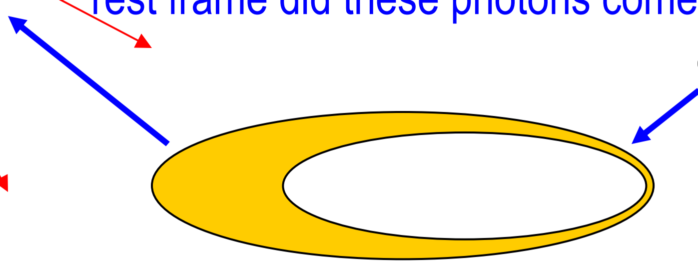

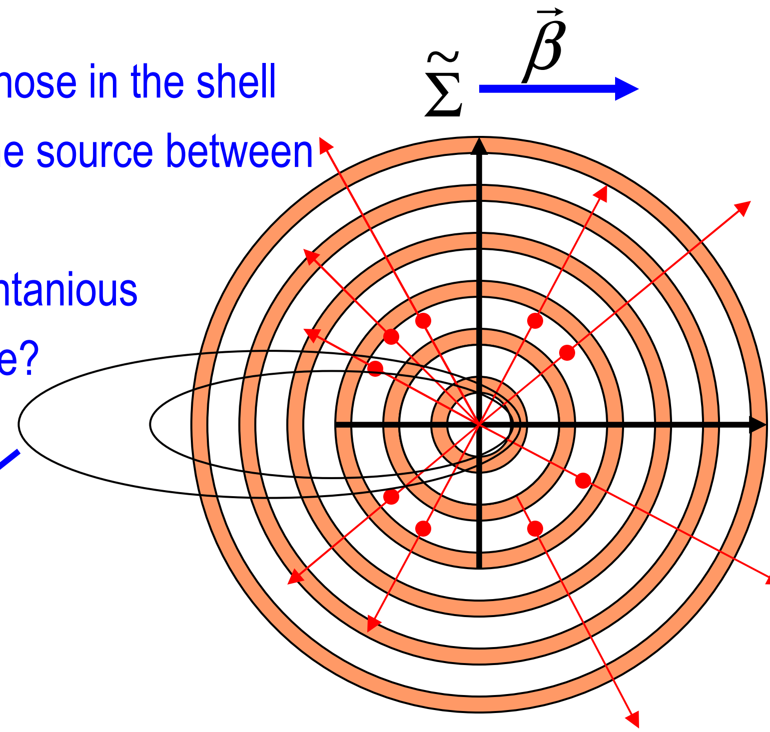

Lab-Frame-Radiation Power

Spectral photon flux in the lab frame

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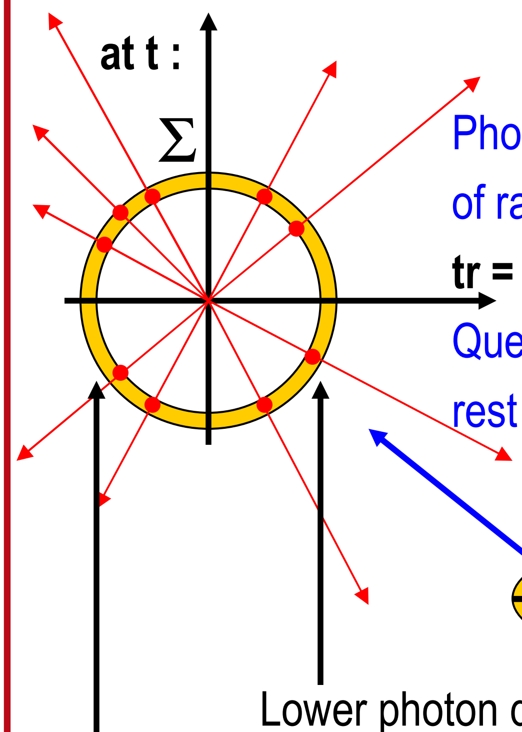

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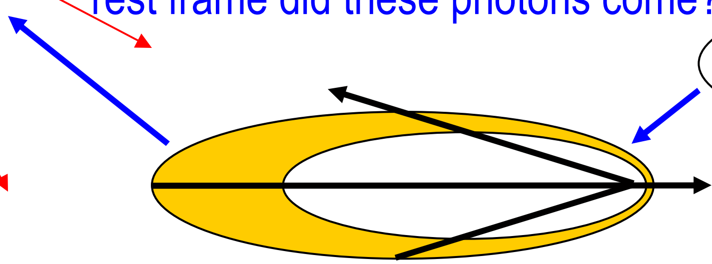
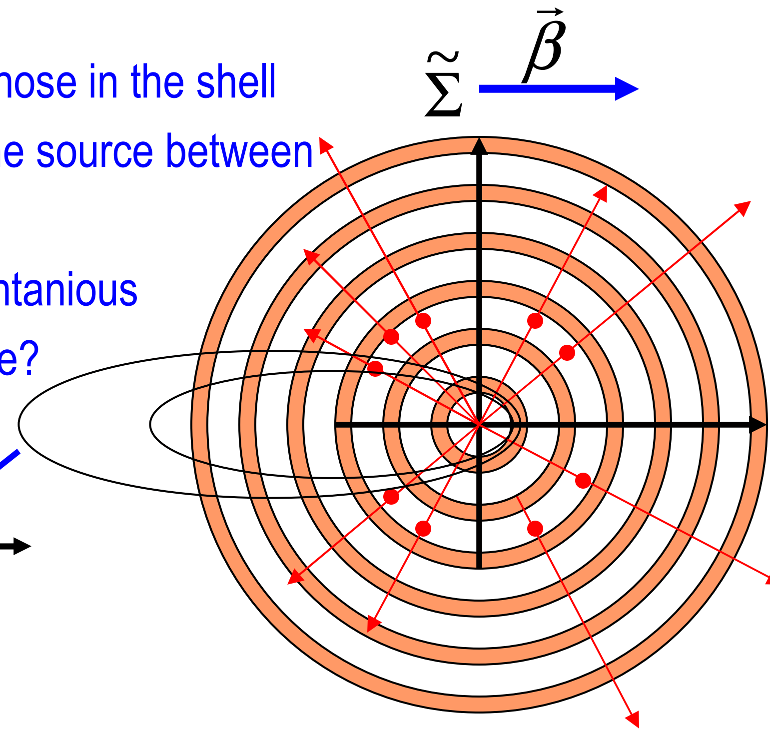




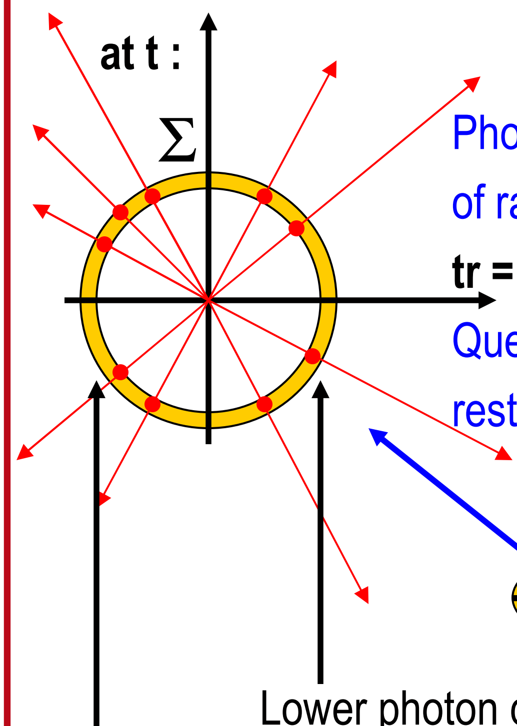
Lower photon density than by angular boost

Higher photon density than by angular boost

Lab-Frame-Radiation Power

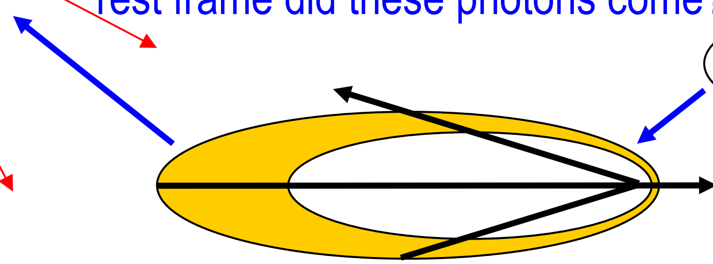
Spectral photon flux in the lab frame

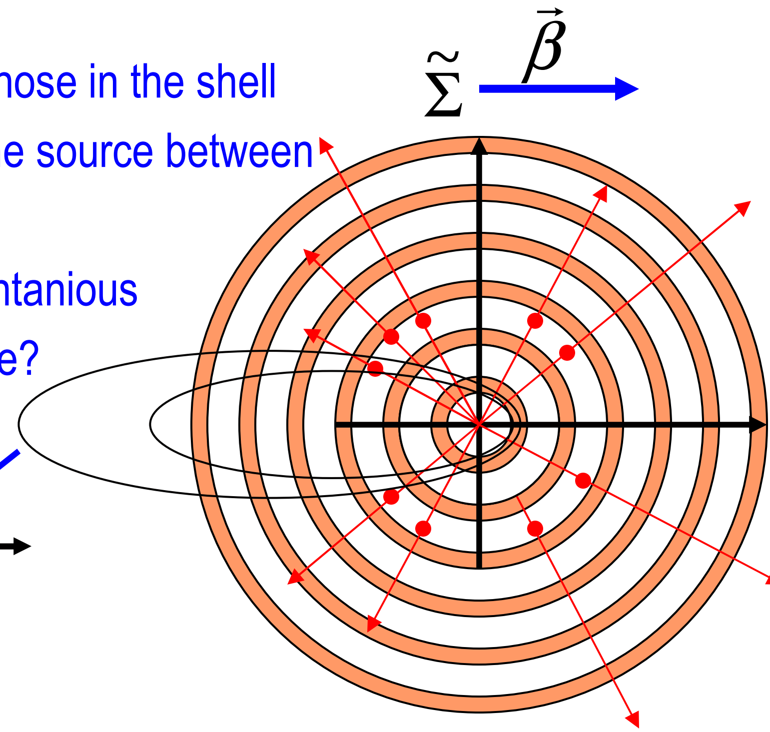
Strategy: The sum over the energy of all photons that are within the shell of a sphere of radius **r** and thickness **dr = cdt** is the energy that was radiated between **tr = t - r/c** and **tr + dt**.



Photons radiate in all directions, those in the shell of radius r have emanated from the source between **tr = t – r/c** and **tr + dt**.

Question: from where in the instantanious rest frame did these photons come?





Lower photon density than by angular boost

Higher photon density

Higher photon energy due to the Doppler shift

Lower photon energy

Result : $P = \tilde{P}$

Lab-Frame-Radiation Power

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Spectral photon flux in the lab frame

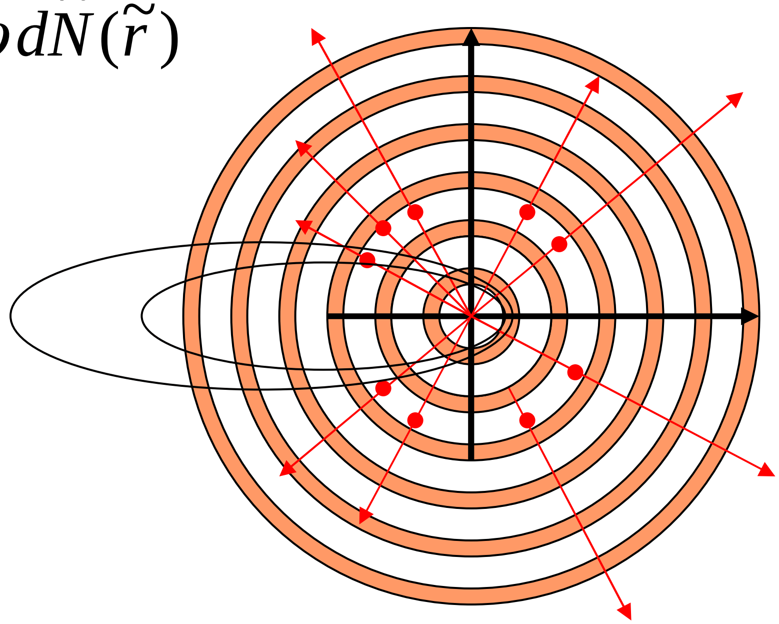
Strategy: The sum over the energy of all photons that are within the shell of a sphere of radius **r** and thickness **dr = cdt** is the energy that was radiated between **tr = t + r/c** and **tr + dt**.

$$P = \frac{dE}{dt} = c\frac{dE}{dr} = c\frac{1}{dr}\int_{4\pi}\int_0^\infty \omega\rho(r)d\omega r^2 d\Omega dr = c\frac{1}{dr}\int_{4\pi}\int_0^\infty \omega dN(r)$$

$$= c\frac{1}{dr}\int_{4\pi}\int_0^\infty \omega d\tilde{N}(\tilde{r}(\tilde{\vartheta})) = c\frac{1}{dr}\int_{4\pi}\int_0^\infty \omega d\tilde{N}(\tilde{r})$$

$$= c\frac{1}{d\tilde{r}}\int_{4\pi}\int_0^\infty \tilde{\omega} d\tilde{N}(\tilde{r})\frac{\omega}{\tilde{\omega}}\frac{d\tilde{r}}{dr}$$

$$= \frac{\omega/\tilde{\omega}}{dr/d\tilde{r}}\tilde{P} = \tilde{P}$$



Lab-Frame-Radiation Power

Spectral photon flux in the lab frame

Strategy: The sum over the energy of all photons that are within the shell of a sphere of radius **r** and thickness **dr = cdt** is the energy that was radiated between **tr = t + r/c** and **tr + dt**.

$$P = \frac{dE}{dt} = c\frac{dE}{dr} = c\tfrac{1}{dr}\int_{4\pi}\int_0^\infty \omega\rho(r)d\omega r^2 d\Omega dr = c\tfrac{1}{dr}\int_{4\pi}\int_0^\infty \omega dN(r)$$

$$= c\tfrac{1}{dr}\int_{4\pi}\int_0^\infty \omega d\tilde{N}(\tilde{r}(\tilde{\vartheta})) = c\tfrac{1}{dr}\int_{4\pi}\int_0^\infty \omega d\tilde{N}(\tilde{r}) = c\tfrac{1}{d\tilde{r}}\int_{4\pi}\int_0^\infty \tilde{\omega} d\tilde{N}(\tilde{r})\tfrac{\omega}{\tilde{\omega}}\tfrac{d\tilde{r}}{dr}$$

$$= \frac{\omega/\tilde{\omega}}{dr/d\tilde{r}}\tilde{P} = \tilde{P}$$

At the surface of the sphere with radius **r = c(t – tr):** $\begin{pmatrix} ct \\ \vec{r} \end{pmatrix} = \begin{pmatrix} r \\ r\vec{n} \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{c}\omega \\ \vec{k} \end{pmatrix} = \frac{1}{c}\begin{pmatrix} \omega \\ \omega\vec{n} \end{pmatrix}$

The Lawrence transformation thus leads to the same linear transformation for **r** and ω, and therefore $\frac{\tilde{\omega}}{\omega} = \frac{d\tilde{r}}{dr}$

Synchrotron Radiation Power

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Power in the instantanious rest frame: $\tilde{P} = \frac{2r_e}{3c} mc^2 \tilde{\dot{\beta}}_r^2$

Because it equals the power in the lab frame,
there must be a Lorentz-invariant quantity that has the value $\tilde{\dot{\beta}}_r^2$ in the rest frame.

The 4-vector of acceleration: $b^\mu = \frac{d}{d\tau} P^\mu = (\frac{1}{c}\frac{d}{d\tau} E, \frac{d}{d\tau}\vec{p})$

$$b^\mu = \frac{d}{d\tau} P^\mu = \gamma \frac{d}{dt} P^\mu = \gamma(\tfrac{1}{c}\dot{E}, \dot{\vec{p}}) \quad \Rightarrow \quad \gamma^2(\dot{\vec{p}}^{\,2} - \tfrac{1}{c^2}\dot{E}^2) = \text{const.}$$

Evaluation in the instantanious rest frame:

$$[\tfrac{d}{dt}\gamma]_{\vec{\beta}=0} = 0 \quad \Rightarrow \quad [\gamma^2(\dot{\vec{p}}^{\,2} - \tfrac{1}{c^2}\dot{E}^2)]_{\vec{\beta}=0} = (mc)^2 \dot{\beta}_r^2$$

Radiation power in the laboratory frame

$$P = \frac{2r_e}{3mc}\gamma^2(\dot{\vec{p}}^{\,2} - \tfrac{1}{c^2}\dot{E}^2)$$

Transverse and Longitudinal Acceleration

$$P = \frac{2r_e}{3mc}\gamma^2(\dot{\vec{p}}^{\,2} - \frac{1}{c^2}\dot{E}^2)$$

Transverse acceleration:

Magnetic: $P = \frac{2r_e q^2}{3mc}\gamma^2(\vec{v}\times\vec{B})^2 = \frac{2r_e q^2}{3mc}v^2\gamma^2 B_\perp^2 = \frac{2r_e}{3m^3 c}\frac{p^4}{\rho^2} = \frac{2r_e c}{3}mc^2\frac{(\beta\gamma)^4}{\rho^2}$

Electric: $\vec{E}\perp\vec{v} \quad \Rightarrow \quad P = \frac{2r_e q^2}{3mc}\gamma^2\vec{E}^2 = \frac{2r_e c}{3}mc^2\frac{(\beta\gamma)^4}{\rho^2}$

Relative loss per turn: $\eta = \frac{2\pi\rho}{v}P\frac{1}{m\gamma c^2} = \frac{4\pi}{3}\frac{r_e}{\rho}(\beta\gamma)^3$

Example: $\rho = 60\text{m}, E = 5\text{GeV} \rightarrow \eta = 2\cdot 10^{-4}$

Longitudinal acceleration:

$$P = \frac{2r_e q^2}{3mc}\gamma^2[\vec{E}^2 - \frac{1}{c^2}(\vec{E}\cdot\vec{v})^2] = \frac{2r_e q^2}{3mc}\vec{E}^2$$

Efficiency: $\eta = \frac{P}{q\vec{v}\cdot\vec{E}} = \frac{2}{3\beta}\frac{r_e|q\vec{E}|}{mc^2}$

Example: $|\vec{E}| = 30\,\frac{\text{MeV}}{\text{m}} \rightarrow \eta = 10^{-13}$

Examples

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CESR @ CORNELL:	5.3GeV	ρ = 60m	η = 0.02%
PETRA @ DESY:	23.5GeV	ρ = 195m	η = 0.6%
LEP @ CERN:	105GeV	ρ = 3000m	η = 3.4%
HERA-e @ DESY:	27.5GeV	ρ = 608m	η = 0.3%
HERA-p @ DESY:	920GeV	ρ = 582m	η = 2.e-8

For the same momentum, the energy loss scales with the third power of the mass, i.e.
Electrons radiate 6.e9 times more than protons.