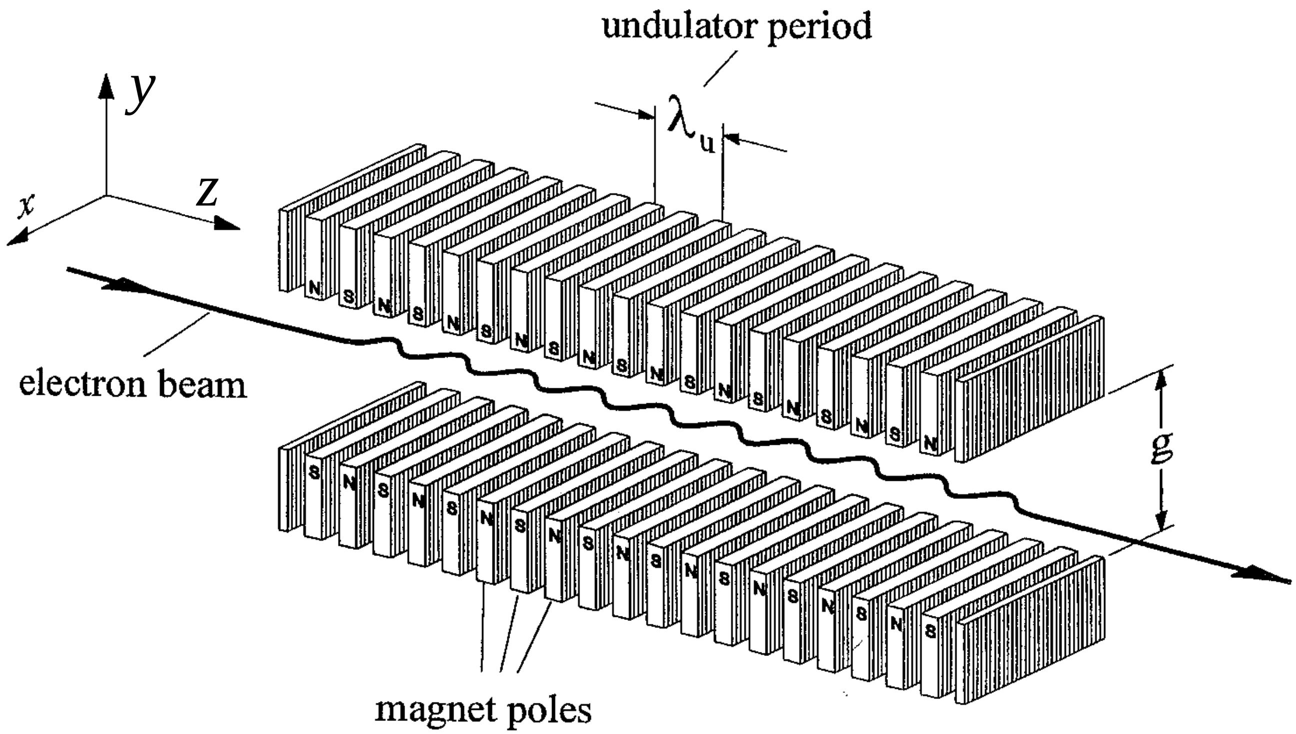
Particle Motion in Undulators

CHESS & LEPP



Conditions to make the beam location and angle independent of the undulator gap.

$$\Delta x' \propto \int_0^L B_y(0,z)dz\ , \quad \Delta x \propto \int_0^L \int_0^{\tilde{z}} B_y(0,z)dz d\tilde{z}$$

Each pole should enhance the same wavelength $\lambda = \frac{1}{n}\lambda_u(\frac{1}{\beta} - \cos\phi)$

Make sure the poles have the same length, and the time spend in each pole is the same:

Undulators Fields

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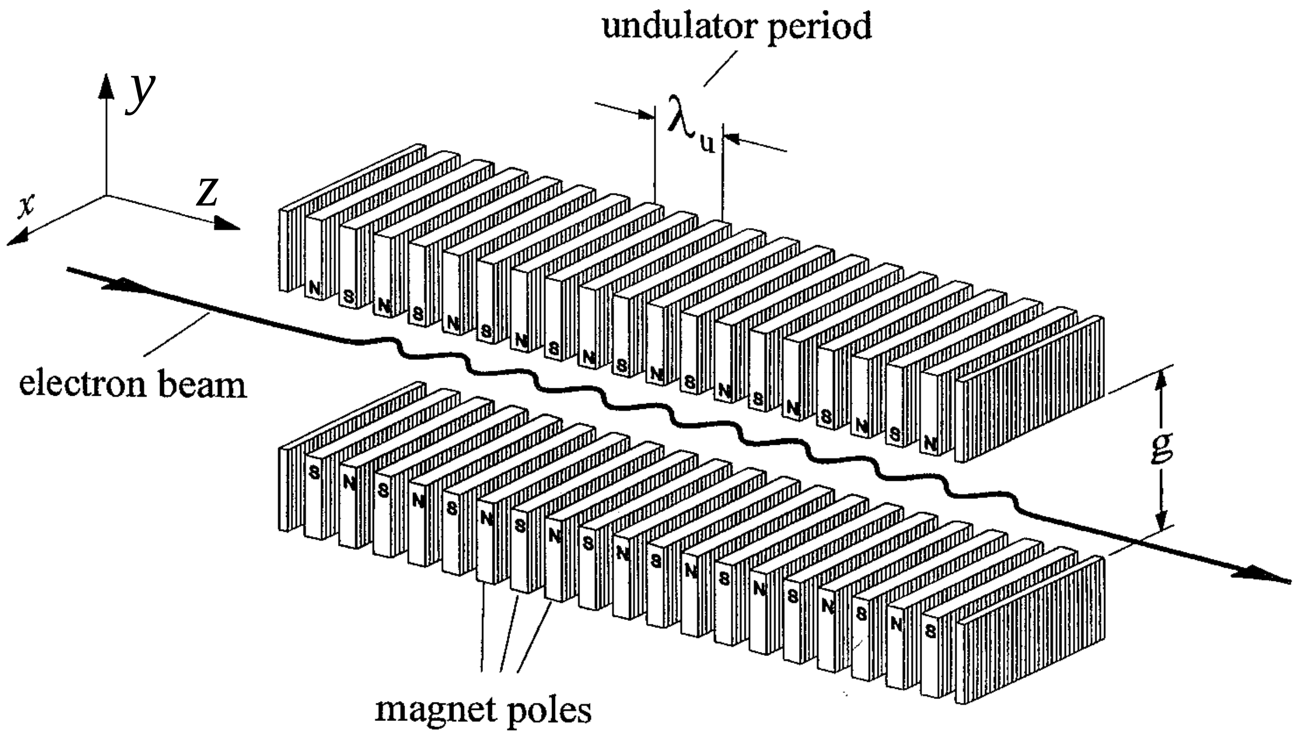

Electrostatics uses magnetic potential: $\vec{B} = -\vec{\nabla}\Psi$

Approximation of infinite width: $\frac{\partial}{\partial x}\Psi = 0$

Periodicity: $\Psi(y,z) = -\Psi(y, z + \frac{\lambda_u}{2})$

Mirror symmetry in z: $\Psi(y,z) = \Psi(y,-z)$

$$\Psi = \sum_{n,\,odd} f_n(y)\cos(nk_u z)\,, \quad k_u = \frac{2\pi}{\lambda_n}$$

Undulator Field Expansion

$$\Psi = \sum_{n,odd} f_n(y)\cos(nk_u z)$$

Magnetic potential equation: $\vec{\nabla}^2\Psi = 0 \quad \Rightarrow \quad f_n''(y) - (nk_u)^2 f_n(y) = 0$

Mid-plane symmetry: $\Psi(y,z) = -\Psi(-y,z)$

$$\vec{\nabla}^2\Psi = 0 \quad \Rightarrow \quad f_n''(y) - (nk_u)^2 f_n(y) = 0$$

$$\Psi = \sum_{n,odd} A_n \sinh(nk_u y)\cos(nk_u z)\,, \quad B_y = -\sum_{n,odd} A_n nk_u \cosh(nk_u y)\cos(nk_u z)$$

Field harmonics at the pole face: $B_y(\frac{g}{2},z) = \sum_{n,odd} a_n \cos(nk_u z) \quad \Rightarrow \quad A_n = \frac{-a_n}{nk_u \cosh(nk_u g/2)}$

The higher field harmonics are exponentially suppressed in the mid plane.

Single-Harmonic Undulator Approximation

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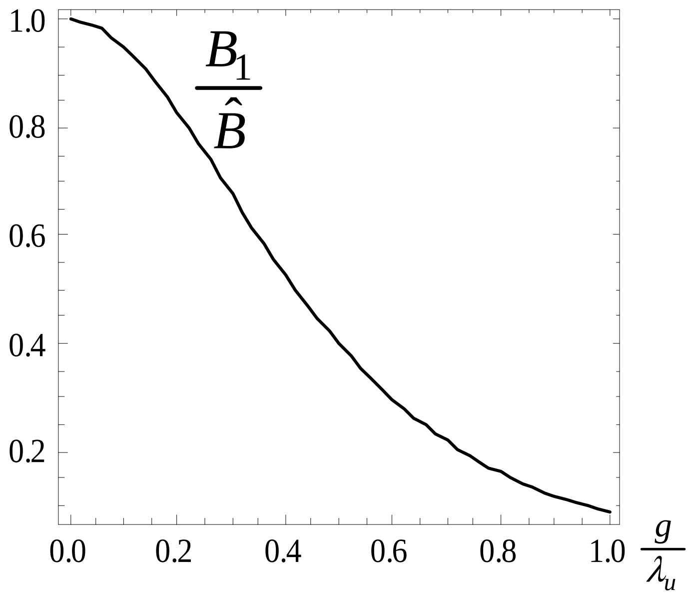
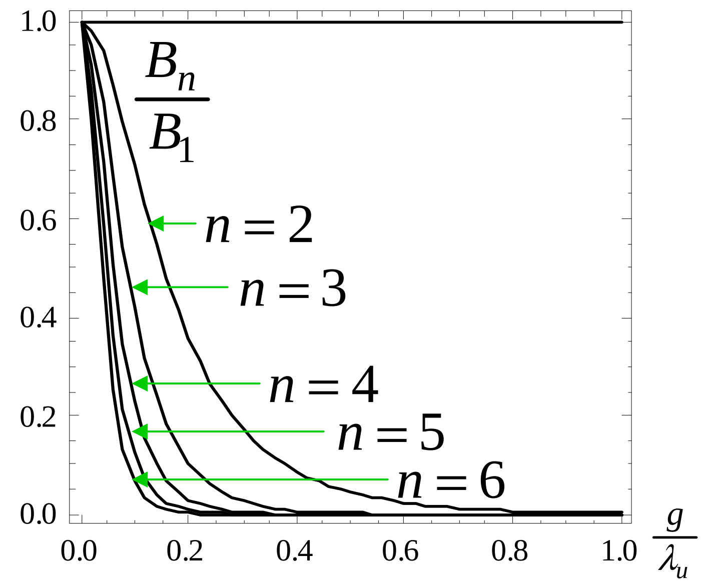
$$\Psi = A\sinh(nk_u y)\cos(nk_u z)$$

$$\begin{pmatrix} B_y \\ B_z \end{pmatrix} \approx B_n \begin{pmatrix} \cosh(nk_u y)\cos(nk_u z) \\ -\sinh(nk_u y)\sin(nk_u z) \end{pmatrix}, \quad B_n = \frac{\hat{B}}{\cosh(nk_u g/2)}$$ with the pole-face field $\hat{B}$



Reduction of mid-plane field with undulator gap



Reduction of higher Fourier component fields

Undulator Strength

$$\begin{pmatrix} B_y \\ B_z \end{pmatrix} \approx B_1 \begin{pmatrix} \cosh(k_u y)\cos(k_u z) \\ -\sinh(k_u y)\sin(k_u z) \end{pmatrix}, \quad B_1 = \frac{\hat{B}}{\cosh(k_u g/2)}$$

with the pole-face field $\hat{B}$

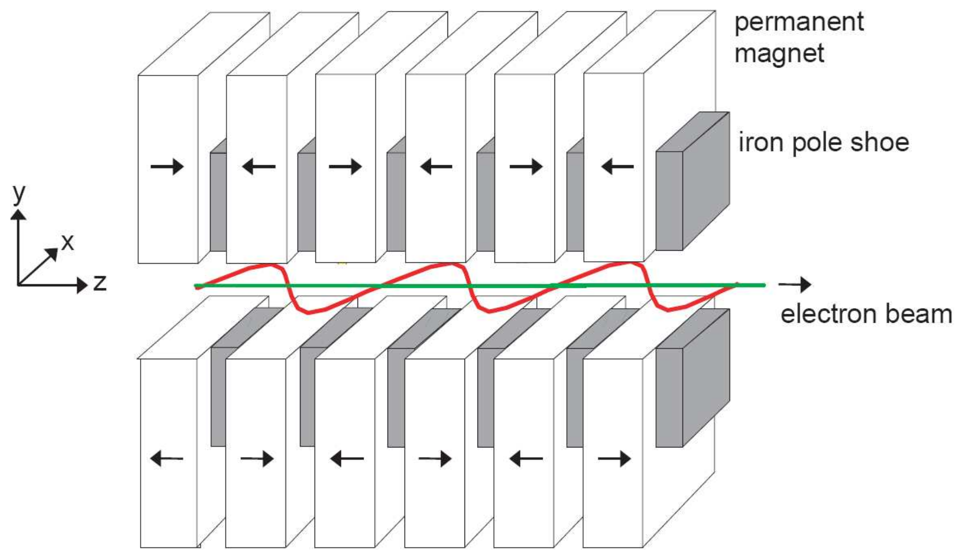
$\hat{B}$ for different magnet types

Electro-magnets with soft iron poles: $\hat{B} \leq 2\mathrm{T}$

Superconducting magnets (for high frequency radiation): $\hat{B} \leq 6\mathrm{T}$

Permanent magnets: $\hat{B} \leq 1\mathrm{T}$

Hybrid magnets with permanent magnets and soft iron yokes: $\hat{B} \leq 2\mathrm{T}$



Equation of Motion in Undulators

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$y(t)=0$ is a solution because of mid-plane symmetry

$$\dot{\vec{p}}=q\vec{v}\times\vec{B}\,,\quad \vec{B}=\begin{pmatrix}B_y\\B_z\end{pmatrix}\quad\Rightarrow\quad\begin{pmatrix}\dot{v}_x\\\ddot{z}\end{pmatrix}=\frac{q}{m\gamma}\begin{pmatrix}-\dot{z}B_y(0,z)\\v_x\,B_y(0,z)\end{pmatrix}$$

Transformation to an integral equation: $v_x(t)=v_x(0)-\int_0^{z(t)}\frac{q}{m\gamma}B_1\cos(k_u\tilde{z})d\tilde{z}$

This implicit integral equation can be iterated:

$$v_x^{(n+1)}(t)=v_x(0)-c\frac{K}{\gamma}\sin[k_u z^{(n)}(t)]\,,\quad K=\frac{qB_1}{mck_u}$$

$$v_z^{(n)}(t)=\sqrt{v^2-v_x^{(n)2}(t)}\,,\quad z^{(n)}(t)=\int_0^t v_z^{(n)}(t)dt$$

Roughly speaking, the particle will wiggle through the device with a mean velocity in the longitudinal direction and with oscillating, small transverse velocity. Therefore, start the iteration with $\beta_x^{(0)}(t)=0\,,\quad z^{(0)}(t)=\beta ct$

Iteration of Motion in Undulators

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$$\beta_x^{(n+1)}(t) = -\tfrac{K}{\gamma}\sin[k_u z^{(n)}(t)]\,,\quad \beta_z^{(n+1)}(t) = c\beta\sqrt{1-\frac{\beta_x^{(n+1)2}(t)}{\beta^2}} = c(1-\tfrac{1}{2\gamma^2}-\tfrac{\beta_x^{(n+1)2}(t)}{2}) + O^4(\tfrac{1}{\gamma})$$

$$z^{(n+1)}(t) = \int_0^t \beta_z^{(n+1)}(t)cdt + O^3(\tfrac{1}{\gamma}) \quad \textit{for} \quad t \propto O^1(\gamma)$$

0th step: $\beta_x^{(0)}(t) = 0\,,\quad z^{(0)}(t) = \beta ct$

1st step: $\beta_x^{(1)}(t) = -\tfrac{K}{\gamma}\sin[k_u\beta ct]\,,\quad \beta_z^{(1)}(t) = c[\underbrace{1-\tfrac{1}{2\gamma^2}-\left(\tfrac{K}{2\gamma}\right)^2}_{\bar\beta_z} + \left(\tfrac{K}{2\gamma}\right)^2\cos(2k_u\beta ct)] + O^4(\tfrac{1}{\gamma})$

2nd step: $\beta_x^{(2)}(t) = -\tfrac{K}{\gamma}\sin[k_u\bar\beta_z ct + \tfrac{1}{2}\left(\tfrac{K}{2\gamma}\right)^2\sin(2k_u\bar\beta_z ct)] + O^4(\tfrac{1}{\gamma}) = -\tfrac{K}{\gamma}\sin[k_u\bar\beta_z ct] + O^3(\tfrac{1}{\gamma})$

$$\beta_z^{(2)}(t) = c[\bar\beta_z + \left(\tfrac{K}{2\gamma}\right)^2\cos(2k_u\bar\beta_z ct)] + O^4(\tfrac{1}{\gamma})$$

3rd step: $\beta_x^{(3)}(t) = -\tfrac{K}{\gamma}\sin[k_u\bar\beta_z ct + \tfrac{1}{2}\left(\tfrac{K}{2\gamma}\right)^2\sin(2k_u\bar\beta_z ct)] + O^4(\tfrac{1}{\gamma}) = -\tfrac{K}{\gamma}\sin[k_u\bar\beta_z ct] + O^3(\tfrac{1}{\gamma})$

$$\beta_z^{(3)}(t) = c[\bar\beta_z + \left(\tfrac{K}{2\gamma}\right)^2\cos(2k_u\bar\beta_z ct)] + O^4(\tfrac{1}{\gamma})$$

Conclusion: The iteration repeats and the solution has thus been found.

Average Velocity

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$$\beta_x^{(n+1)}(t) = -\tfrac{K}{\gamma}\sin[k_u z^{(n)}(t)], \quad \beta_z^{(n+1)}(t) = c\beta\sqrt{1-\tfrac{\beta_x^{(n+1)2}(t)}{\beta^2}} = c(1-\tfrac{1}{2\gamma^2}-\tfrac{\beta_x^{(n+1)2}(t)}{2}) + O^4(\tfrac{1}{\gamma})$$

$$z^{(n+1)}(t) = \int_0^t \beta_z^{(n+1)}(t)cdt + O^3(\tfrac{1}{\gamma}) \quad for \quad t \propto O^1(\gamma)$$

Note: The time is restricted to order γ, because the average velocity is only know to order γ^3.
This time restriction is avoided when the average velocity is computed accurately, which leads to an elliptic integral of the 1st kind.

$$\dot z = \sqrt{v^2 - \left(c\tfrac{K}{\gamma}\right)^2 \sin[k_u z]}$$

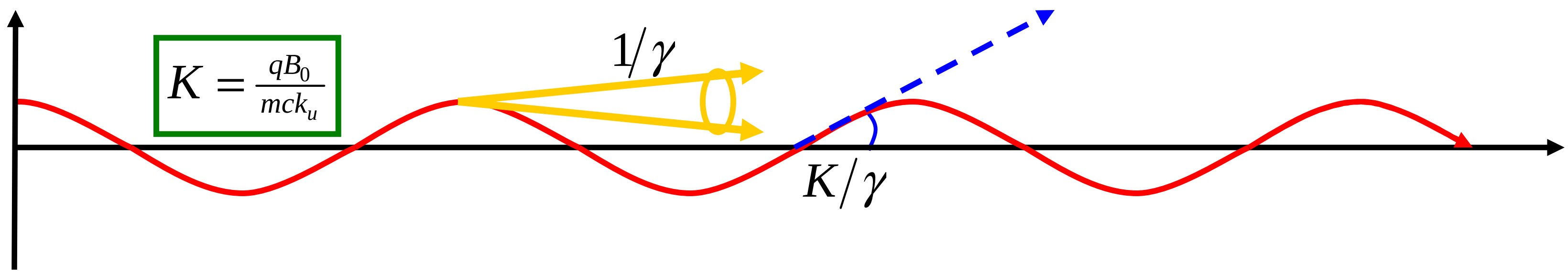
$$\Rightarrow \quad T = \int_0^{\lambda_u} \frac{dz}{\sqrt{v^2-\left(c\tfrac{K}{\gamma}\right)^2\sin[k_u z]}} = \tfrac{1}{\beta c k_u}\int_0^{2\pi}\frac{d\varphi}{\sqrt{1-\left(\tfrac{K}{\beta\gamma}\right)^2\sin[\varphi]}} = \tfrac{4}{\beta c k_u}\mathrm{K}(\tfrac{K}{\beta\gamma})$$

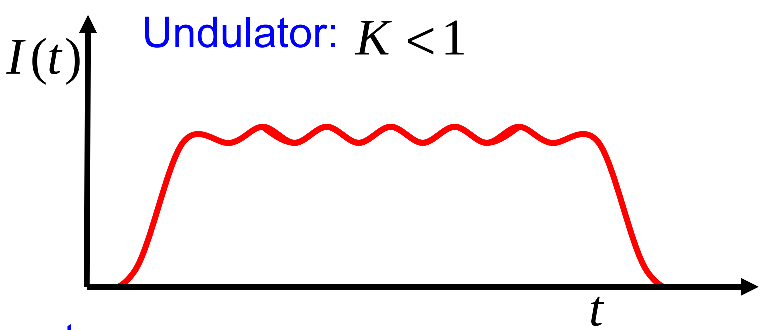
$$\Rightarrow \quad \bar\beta_z = \tfrac{\lambda_u}{cT} = \tfrac{\pi}{2}\frac{\beta}{\mathrm{K}(\tfrac{K}{\beta\gamma})} = 1-\tfrac{1+K^2/2}{2\gamma^2} + O^4(\tfrac{1}{\gamma})$$

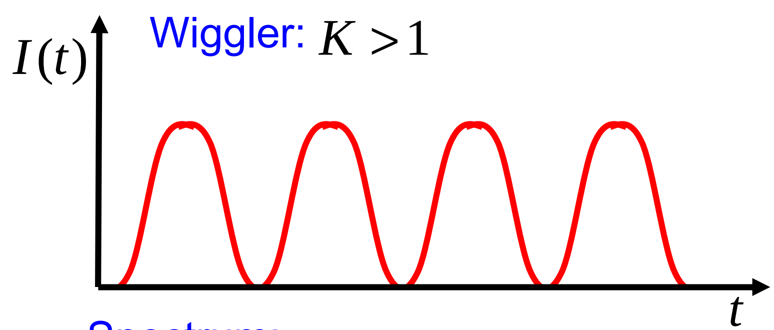
Motion in Undulators and Wigglers

$$\beta_x(t) = -\tfrac{K}{\gamma}\sin[k_u\bar{\beta}_z ct] + O^3(\tfrac{1}{\gamma}) \qquad x(t) = \tfrac{K}{\gamma k_u}\cos[k_u\bar{\beta}_z ct] + O^3(\tfrac{1}{\gamma})$$

$$\beta_z(t) = c[\bar{\beta}_z + \left(\tfrac{K}{2\gamma}\right)^2\cos(2k_u\bar{\beta}_z ct)] + O^4(\tfrac{1}{\gamma}) \qquad z(t) = \bar{\beta}_z ct + \left(\tfrac{K}{2\gamma}\right)^2 \tfrac{1}{2k_u}\sin(2k_u\bar{\beta}_z ct) + O^4(\tfrac{1}{\gamma})$$







Spectrum:
Bulk is a coherent superposition for the appropriate wavelength $\lambda = \frac{1}{n}\lambda_u(\frac{1}{\bar{\beta}_z} - \cos\phi)$.
Fluctuations are from incoherent superposition of short-magnet radiation.

Spectrum:
Incoherent superposition of long-magnet spectra with $\omega_c = \frac{3}{2}\frac{c}{\rho}\gamma^3$
Lower frequencies at larger angles where field is smaller.

Motion in the Co-moving Frame

$$\beta_x(t) = -\frac{K}{\gamma}\sin[k_u\bar{\beta}_z ct] + O^3(\tfrac{1}{\gamma})$$

$$x(t) = \frac{K}{\gamma k_u}\cos[k_u\bar{\beta}_z ct] + O^3(\tfrac{1}{\gamma})$$

$$\beta_z(t) = c[\bar{\beta}_z + \left(\tfrac{K}{2\gamma}\right)^2\cos(2k_u\bar{\beta}_z ct)] + O^4(\tfrac{1}{\gamma})$$

$$z(t) = \bar{\beta}_z ct + \left(\tfrac{K}{2\gamma}\right)^2\frac{1}{2k_u}\sin(2k_u\bar{\beta}_z ct) + O^4(\tfrac{1}{\gamma})$$

Lorentz transformation to the co-moving frame:

$$\bar{\gamma} = \frac{1}{\sqrt{1-\bar{\beta}_z^2}} = \frac{1}{\sqrt{1-[1-\frac{1+K^2/2}{\gamma^2}+O^4(\frac{1}{\gamma})]}} = \frac{1}{\sqrt{1+K^2/2}}\gamma + O^3(\tfrac{1}{\gamma})$$

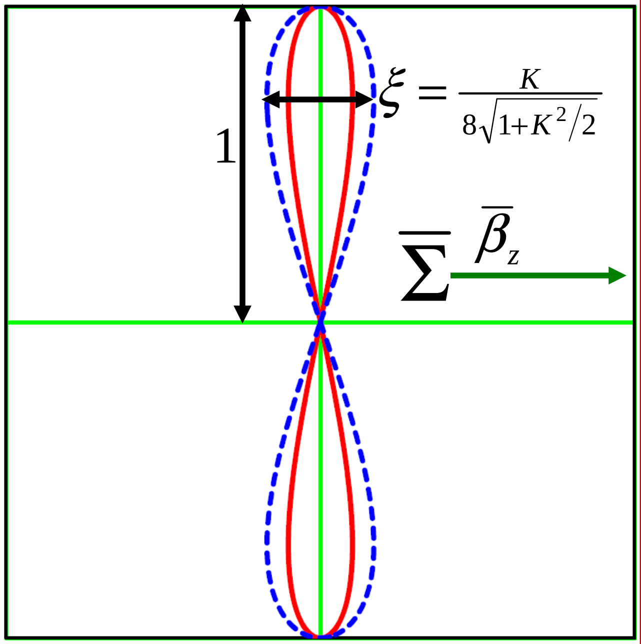
$$\bar{x} = -\frac{K}{\gamma k_u}\cos[k_u\bar{\beta}_z ct] + O^3(\tfrac{1}{\gamma})$$

$$\bar{z} = \frac{K}{8\sqrt{1+K^2/2}}\frac{K}{\gamma k_u}\sin(2k_u\bar{\beta}_z ct) + O^3(\tfrac{1}{\gamma})$$

$$ct = \bar{\gamma}\{c\bar{t} + \bar{\beta}_z\bar{z}\} = \bar{\gamma}(c\bar{t} + \bar{z}) + O^2(\tfrac{1}{\gamma})$$

$$\bar{x} = -\frac{K}{\gamma k_u}\cos(\bar{\omega}\bar{t}) + O^3(\tfrac{1}{\gamma}) + \tfrac{1}{\gamma}O^3(K)\,, \quad \bar{\omega} = k_u\bar{\beta}_z\bar{\gamma}c$$

$$\bar{z} = \xi\frac{K}{\gamma k_u}\sin(2\bar{\omega}\bar{t}) + O^3(\tfrac{1}{\gamma}) + \tfrac{1}{\gamma}O^4(K)$$



Radiation in the Co-moving Frame

Radiation code at http://www-xfel.spring8.or.jp/cband/e

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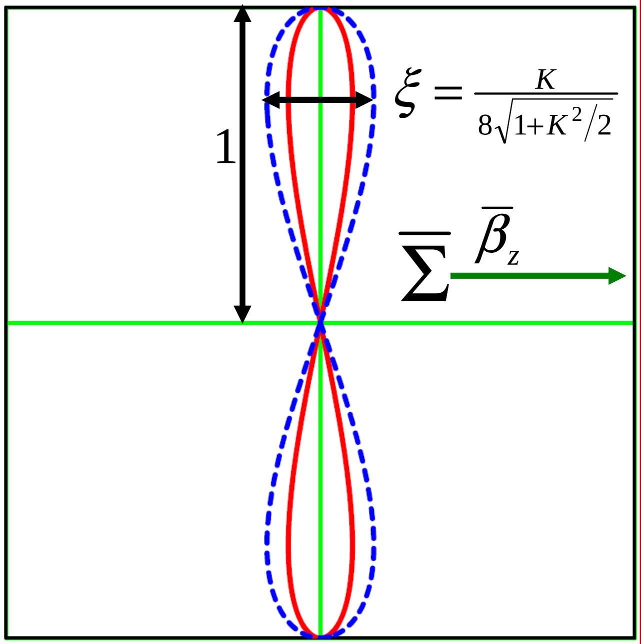
Hertz-dipole radiation with $\bar{\omega} = k_u \bar{\beta}_z \bar{\gamma} c$ for

$$\bar{x} = -\frac{K}{\gamma k_u}\cos(\bar{\omega}\bar{t}) + O^3(\tfrac{1}{\gamma}) + \tfrac{1}{\gamma} O^3(K)$$

$$\bar{z} = \xi \frac{K}{\gamma k_u}\sin(2\bar{\omega}\bar{t}) + O^3(\tfrac{1}{\gamma}) + \tfrac{1}{\gamma} O^4(K)$$

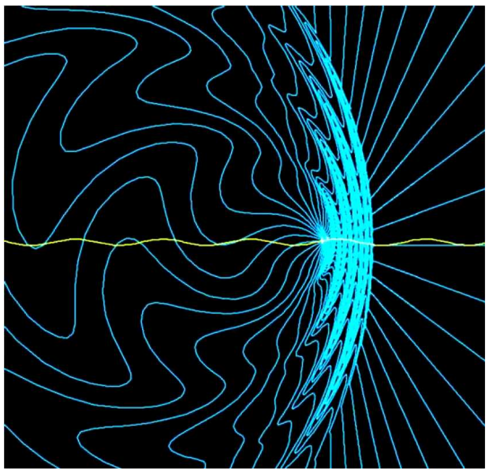


Lorentz transformation to the lab frame

$$\begin{pmatrix} \frac{1}{c}\omega \\ \vec{k} \end{pmatrix} = \begin{pmatrix} \frac{1}{c}\omega \\ \omega\vec{n} \end{pmatrix} \Rightarrow \bar{\omega} = \omega\bar{\gamma}(1-\bar{\beta}_z \cos\vartheta)$$

$$\lambda = \frac{2\pi c}{\omega} = \frac{2\pi c}{\bar{\omega}}\bar{\gamma}(1-\bar{\beta}_z\cos\vartheta) = \lambda_u(\tfrac{1}{\bar{\beta}_z} - \cos\vartheta)$$

$$\lambda = \tfrac{1}{n}\lambda_u(\tfrac{1}{\bar{\beta}_z} - \cos\phi)$$



The Doppler shifted frequencies are those with constructive interference !

x-dipole radiates in forward direction with the first harmonic wavelength λ_1,
the z-dipole in transverse direction with the second harmonic wavelength.

But where are the higher harmonics ?

Motion in the Co-moving Frame

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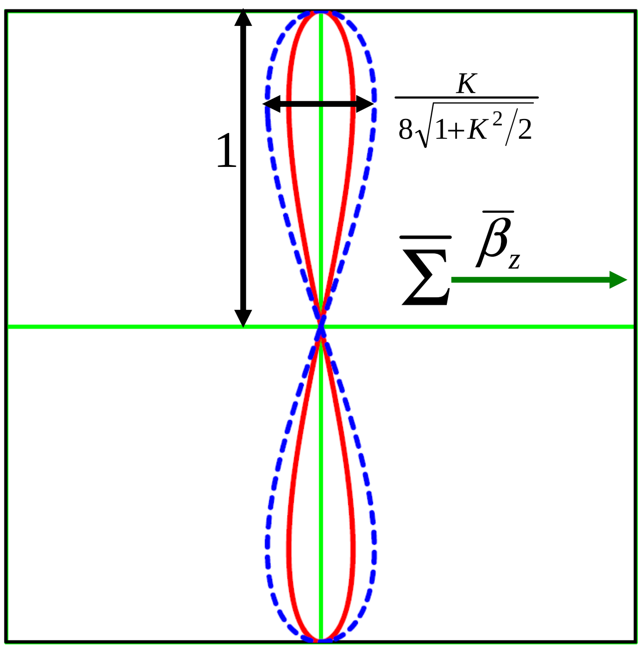
Lorentz transformation to the co-moving frame

$$\bar{x} = -\tfrac{K}{\gamma k_u}\cos[k_u \bar{\beta}_z ct] + O^3(\tfrac{1}{\gamma})$$

$$\bar{z} = \xi \tfrac{K}{\gamma k_u}\sin(2k_u \bar{\beta}_z ct) + O^3(\tfrac{1}{\gamma})$$

$$ct = \bar{\gamma}\{c\bar{t} + \bar{\beta}_z \bar{z}\} = \bar{\gamma}(c\bar{t} + \bar{z}) + O^2(\tfrac{1}{\gamma})$$



$$\bar{z} = \xi \tfrac{K}{\gamma k_u}\sin(2\bar{\varpi}[\bar{t} + \tfrac{\bar{z}}{c}]) + O^3(\tfrac{1}{\gamma})$$
$$= \xi \tfrac{K}{\gamma k_u}[\sin(2\bar{\varpi}\bar{t}) + 2\bar{\varpi}\tfrac{\bar{z}}{c}\cos(2\bar{\varpi}\bar{t})] + O^3(\tfrac{1}{\gamma})$$
$$= \frac{\xi \tfrac{K}{\gamma k_u}\sin(2\bar{\varpi}\bar{t})}{1 - \xi\tfrac{K}{\gamma}\cos(2\bar{\varpi}\bar{t})} + O^3(\tfrac{1}{\gamma})$$

$$\bar{x} = -\tfrac{K}{\gamma k_u}\cos(\bar{\varpi}[\bar{t} + \tfrac{\bar{z}}{c}]) + O^3(\tfrac{1}{\gamma})$$
$$= -\tfrac{K}{\gamma k_u}[\cos(\bar{\varpi}\bar{t}) + \bar{z}\tfrac{\bar{\varpi}}{c}\sin(\bar{\varpi}\bar{t})] + O^3(\tfrac{1}{\gamma})$$

The Hertz-dipole in **z** produces all even harmonics, that in **x** produces all odd harmonics.

Ideal helical undulators:

Electrons have a helical orbit with constant z-velocity.
There is thus no z-dipole and no even harmonics.

There is also no oscillation in the co-moving time, and there are thus no odd harmonics.