Higher Harmonics from Undulators

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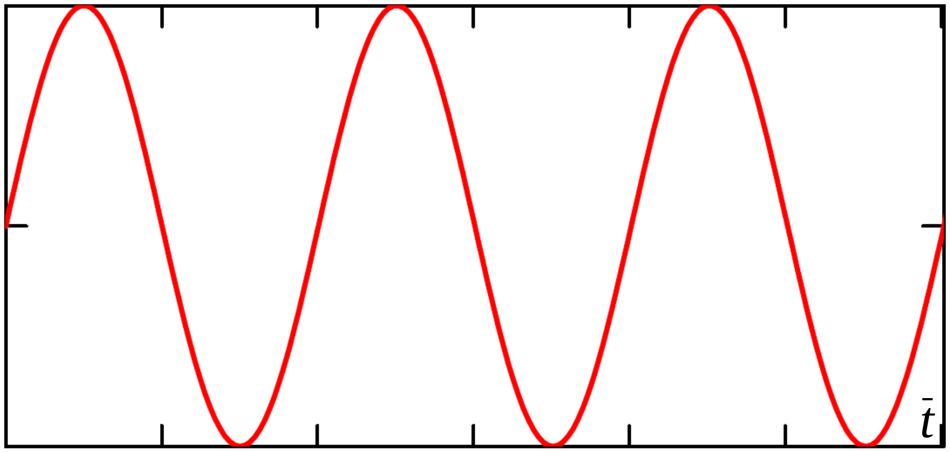
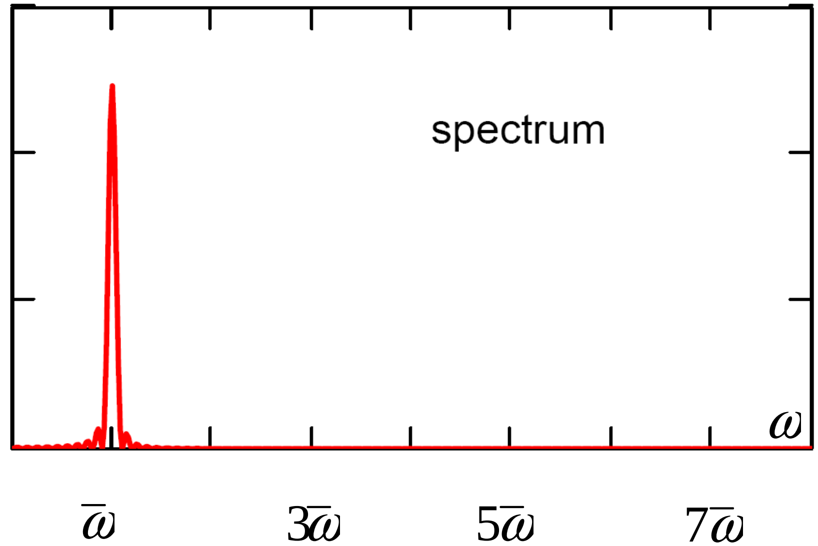
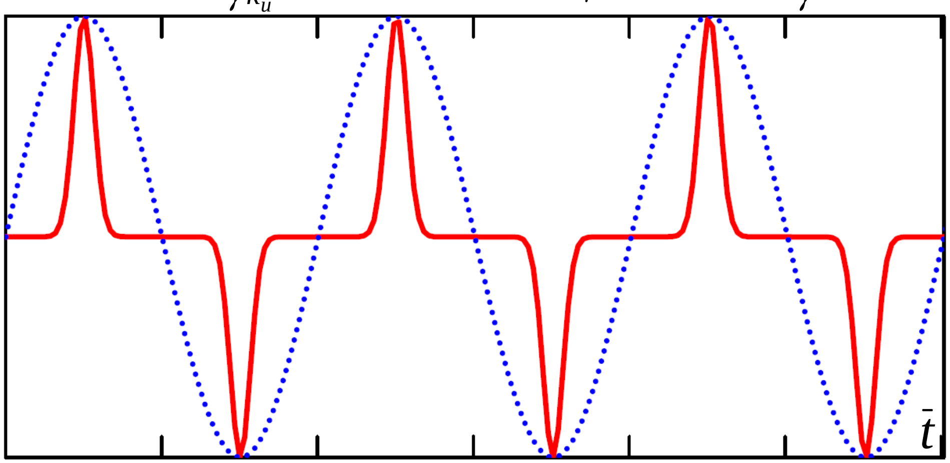
$$\bar{x} = -\frac{K}{\gamma k_u}\cos[\bar{\omega}\bar{t}] + O^3(\tfrac{1}{\gamma}) + \tfrac{1}{\gamma}O^3(K)$$

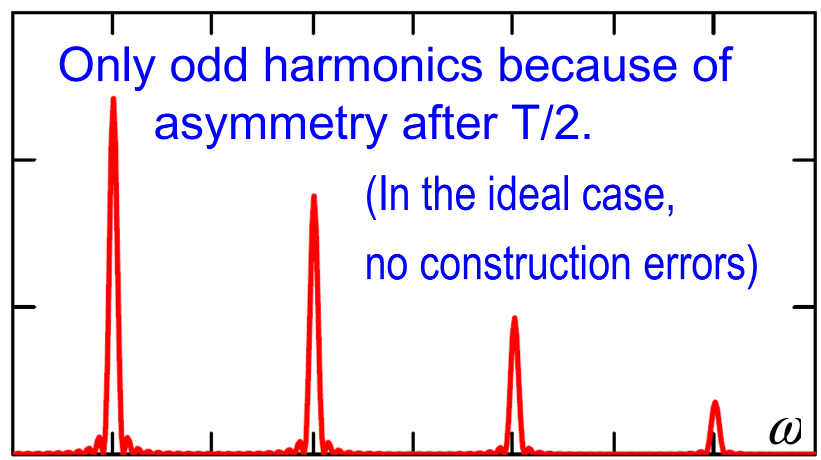


Radiation spectrum



$$\bar{x} = -\frac{K}{\gamma k_u}\cos[\bar{\omega}(\bar{t} + \bar{z}(\bar{t})/c)] + O^3(\tfrac{1}{\gamma})$$





Wave Amplification in a Free Electron Laser

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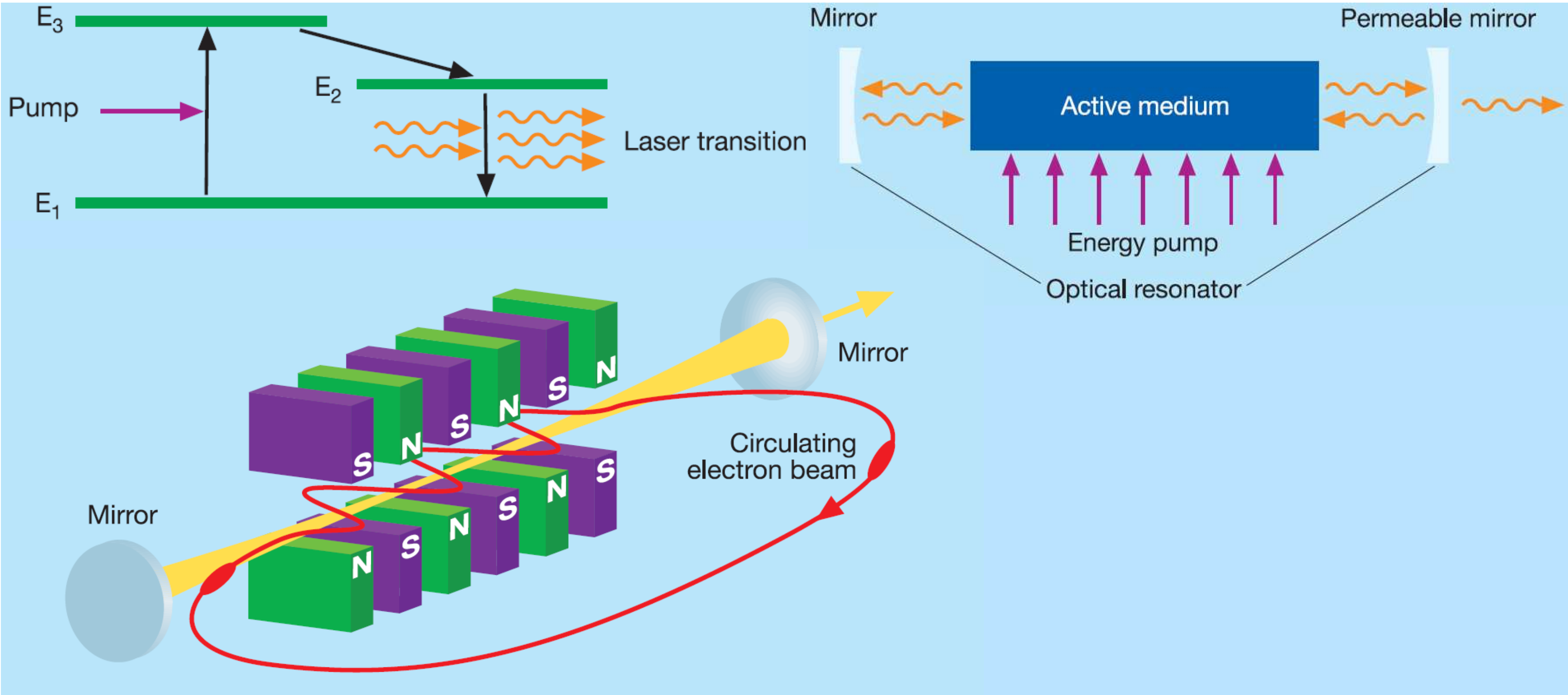
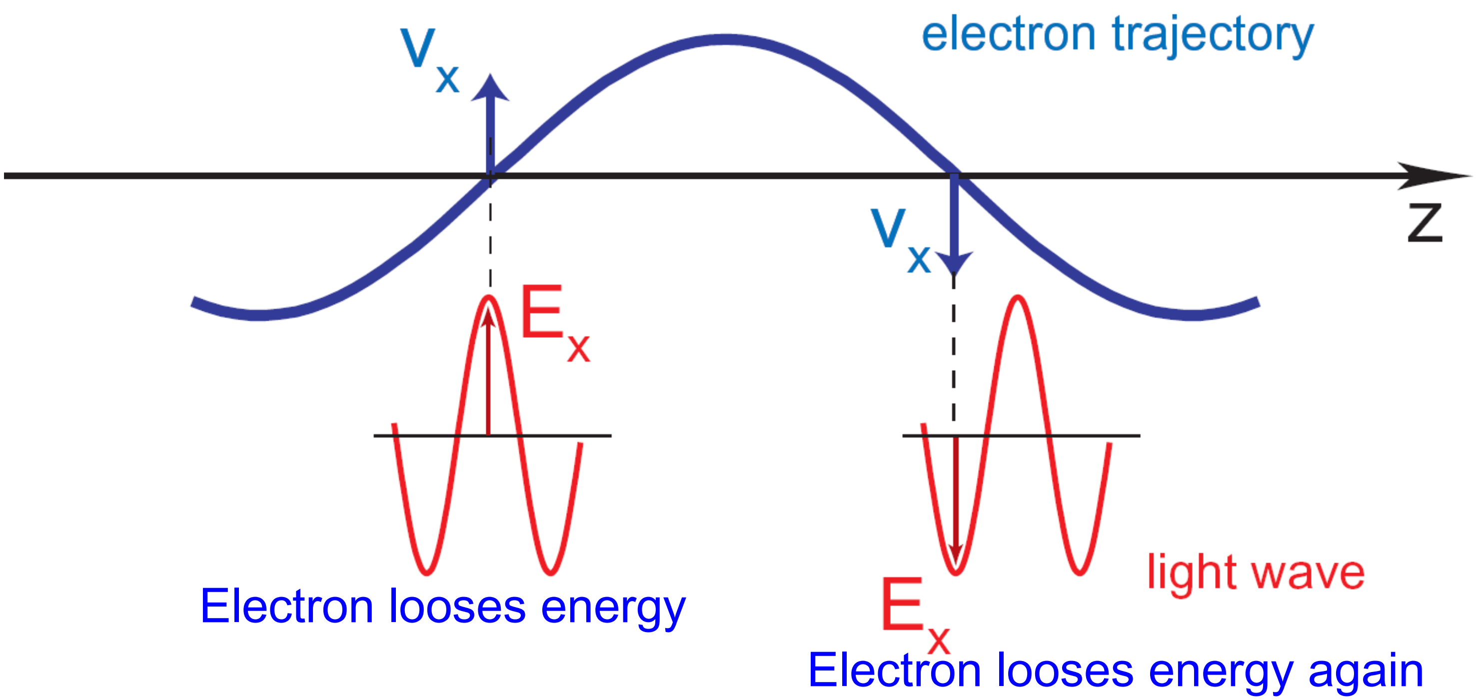


The Free Electron Laser (FEL) is called a laser because some of its components correspond to components of a quantum laser.

The electron beam in an undulator is used to amplify an injected radiation field. The optically pumped medium in quantum lasers also amplifies an injected wave.

Energy Loss in an E&M Wave in a Undulator

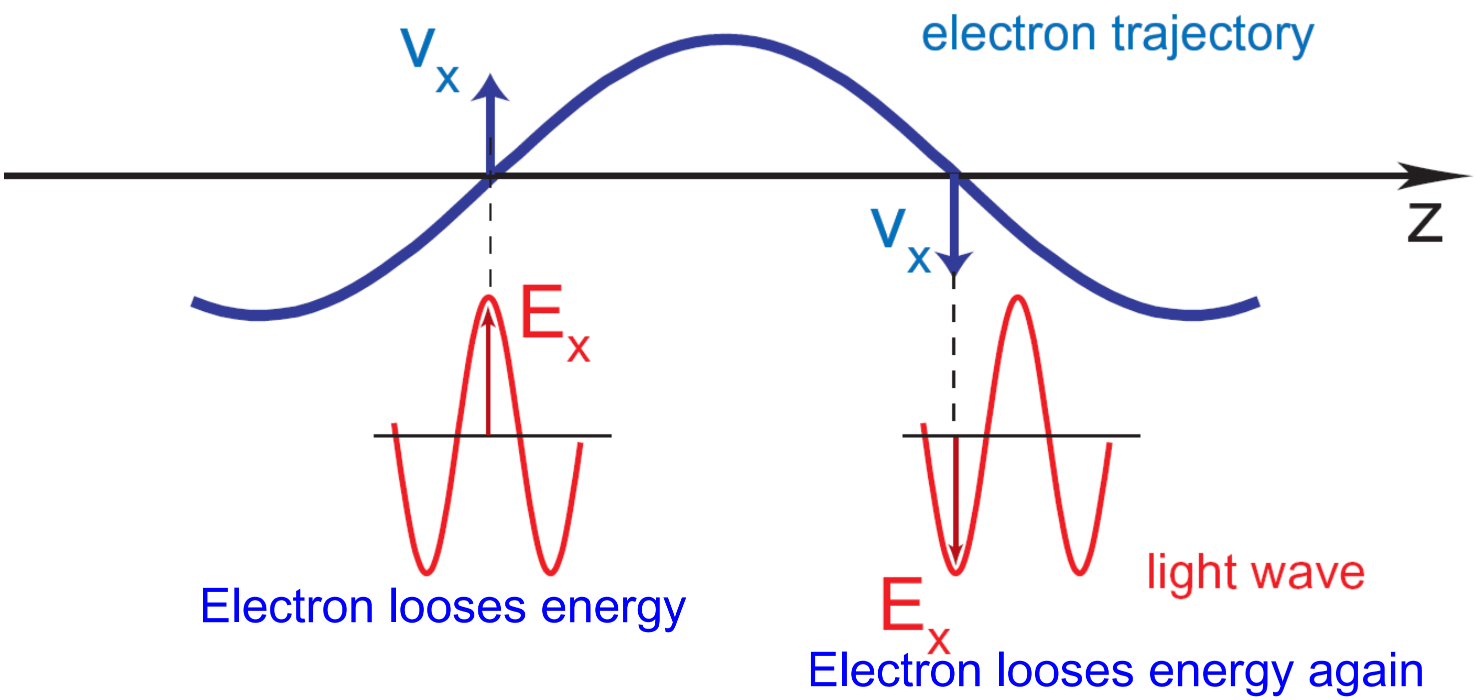


When the light wave passes the electron beam by half a wavelength per half beam oscillation, the electron can continuously loose energy. Only then can the energy transfer between beam and wave be large.

This is the same condition as for constructive interference of radiation: $\lambda = \frac{1}{n_{\text{odd}}}\lambda_{\text{u}}(\frac{1}{\bar{\beta}_z} - 1)$

Similar adding can happen for 3, 5, 7, etc. half beam oscillations. For every strong frequency there are odd order harmonics.

Basis for FEL Radiation Amplification



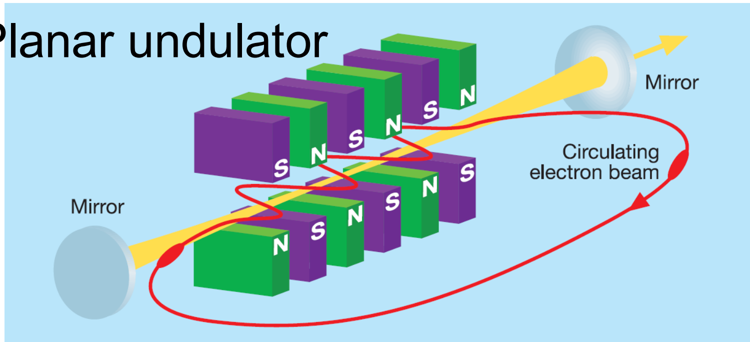
The wave in which the particle looses energy most efficiently has the same wavelength as the emitted radiation in the undulator.

This is the basis for the possibility of radiation amplification:

A wave with one wavelength can enhance the energy loss, and thus the radiation of an electron. And that radiation has the very same wavelength.

A Planar Undulator with Linear Polarization

Planar undulator



Electron motion:

$$v_x = -c\frac{K}{\gamma}\sin(k_u z)$$

Horizontally polarized light:

$$E_{Lx} = E_L \cos[k_L(z - ct) + \phi_0]$$

Change of particle energy due to the E&M wave (stimulated emission):

$$\frac{dE}{dt} = mc^2\frac{d\gamma}{dt} = qv_x E_{Lx} = -qc\frac{E_{L0}K}{\gamma}\sin(k_u z)\cos(k_L[z - ct] + \phi_0)$$

$$= -qc\frac{E_{L0}K}{2\gamma}(\sin\Psi_+ + \sin\Psi_-)\,, \quad \Psi_\pm = k_L[z - ct] + \phi_0 \pm k_u z$$

The energy transfer is large when $\frac{d}{dt}\Psi_+ = 0$ or $\frac{d}{dt}\Psi_- = 0$ with $\dot z(t) = \overline{\beta}_z c + O^2(\frac{1}{\gamma})$

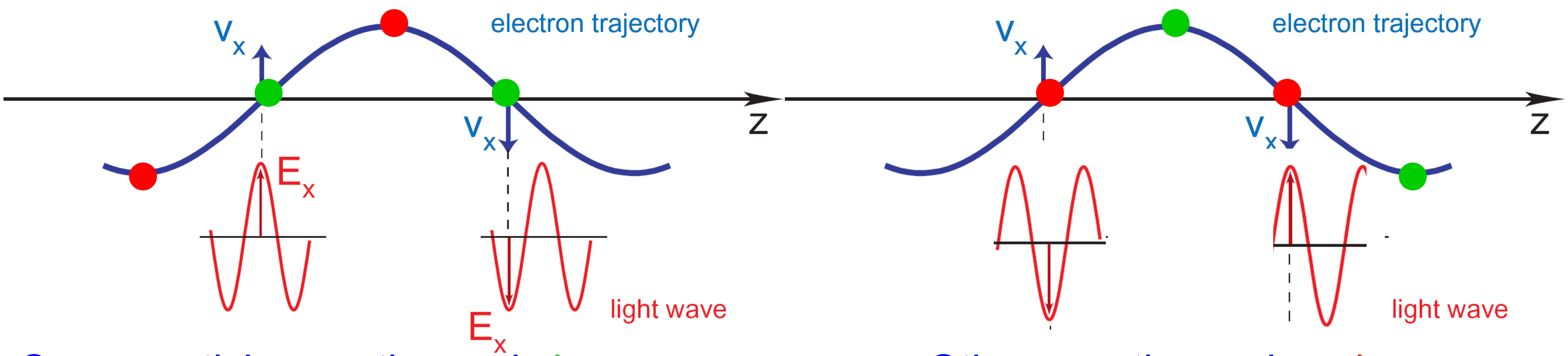
$k_L c(1 - \overline{\beta}_z) = \pm k_u \overline{\beta}_z$ (only + is possible) $\Rightarrow \lambda_L = \lambda_u(\frac{1}{\overline{\beta}_z} - 1)$

The same as for undulator radiation !

The Ponderomotive Phase

The ponderomotive phase $\Psi_+ = k_u z - k_L(ct - z) - \varphi_0$ is the phase of the particle's oscillation relative to the phase of the E&M wave.

$$\frac{d\gamma}{dz} = -\frac{qE_{L0}}{\beta_z}\frac{K}{2\gamma}\sin\psi_+ = -\frac{K_L k_u}{\beta_z}\frac{K}{2\gamma}\sin\psi_+ \,, \quad K_L = \frac{qE_L}{mc^2 k_u} \,, \quad K = \frac{qcB_0}{mc^2 k_u}$$



Some particles continuously loose energy. Others continuously gain energy.

How can such a beam amplify, if the net energy transfer is zero ?

The Pendulum Equation of FELs

The ponderomotive phase $\Psi_+ = k_u z - k_L(ct-z) - \varphi_0$ is the phase of the particle's oscillation relative to the phase of the E&M wave. It changes by 2π appox. after a light wavelength.

$$\frac{d\gamma}{dz} = -\frac{qE_{L0}}{\bar\beta_z}\frac{K}{2\gamma}\sin\psi_+ = -\frac{K_L}{\bar\beta_z}\frac{k_u K}{2\gamma}\sin\psi_+ \, , \quad K_L = \frac{qE_L}{mc^2 k_u} \, , \quad K = \frac{qcB_0}{mc^2 k_u}$$

$$\bar\beta_z = 1 - \frac{1+K^2/2}{2\gamma^2} + O^4(\tfrac{1}{\gamma}) \, , \quad \bar\beta_z^{-1} = 1 + \frac{1+K^2/2}{2\gamma^2} + O^4(\tfrac{1}{\gamma})$$

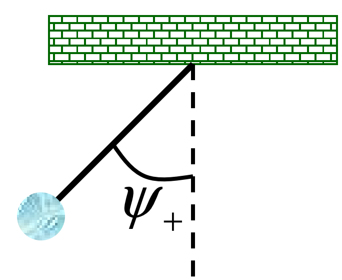
$$\frac{d}{dz}\psi_+ = k_u - k_L\left(\frac{1}{\bar\beta_z(\gamma)} - 1\right) = k_u - k_L\frac{1+K^2/2}{2\gamma^2} + O^4(\tfrac{1}{\gamma})$$

Optimum energy transfer for γ_r with $\frac{d}{dz}\psi_+\big|_{\gamma=\gamma_r} = 0$, i.e. for $k_u = k_L\frac{1+K^2/2}{2\gamma_r^2} + O^4(\tfrac{1}{\gamma_r})$

$$\frac{d}{dz}\psi_+ = \frac{2k_u}{\gamma}\cdot\Delta\gamma + O^5(\tfrac{1}{\gamma})$$

$$\frac{d^2}{dz^2}\psi_+ = -\frac{K_L k_u^2 K}{\gamma^2}\sin\psi_+ + O^4(\tfrac{1}{\gamma})$$

This is the differential equation for a pendulum !



The Ponderomotive Phase Space

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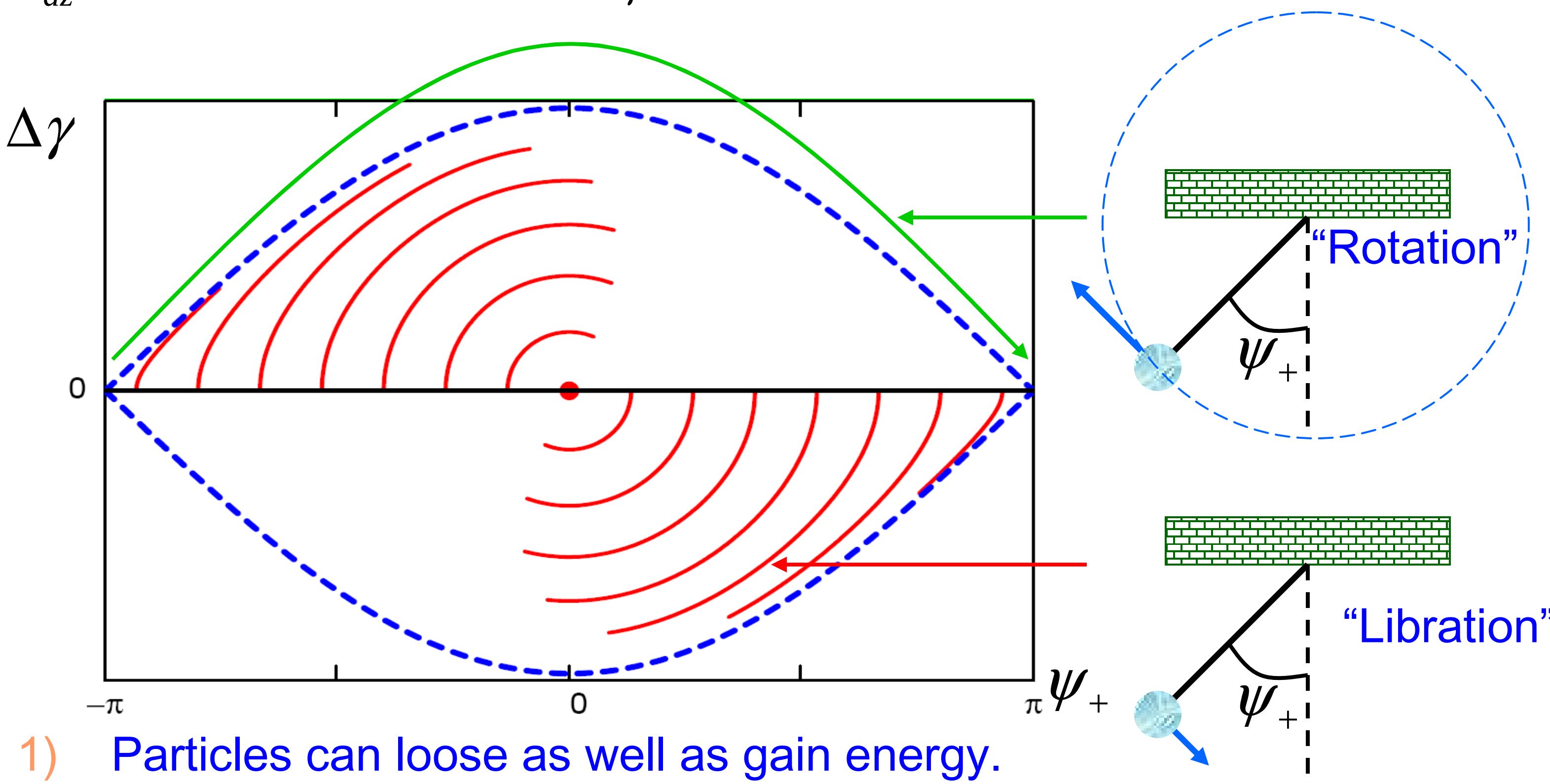
$$\frac{d}{dz}\psi_+ = \frac{2k_u}{\gamma}\cdot\Delta\gamma + O^5(\tfrac{1}{\gamma}), \quad \frac{d\gamma}{dz} = -\frac{K_L k_u K}{2\gamma}\sin\psi_+ + O^3(\tfrac{1}{\gamma})$$

$$\frac{d^2}{dz^2}\psi_+ = -\Omega^2\sin\psi_+ + O^4(\tfrac{1}{\gamma})$$ Wave number for small amplitudes: $\Omega = \frac{k_u}{\gamma}\sqrt{K_L K}$



1) Particles can loose as well as gain energy.
2) For a mono-energetic beam, the net energy change is zero.
3) For a beam with $\gamma=\gamma_r$, i.e. $\Delta\gamma=0$, the net energy change will always remain 0.

Producing Net Energy Loss / Field Gain

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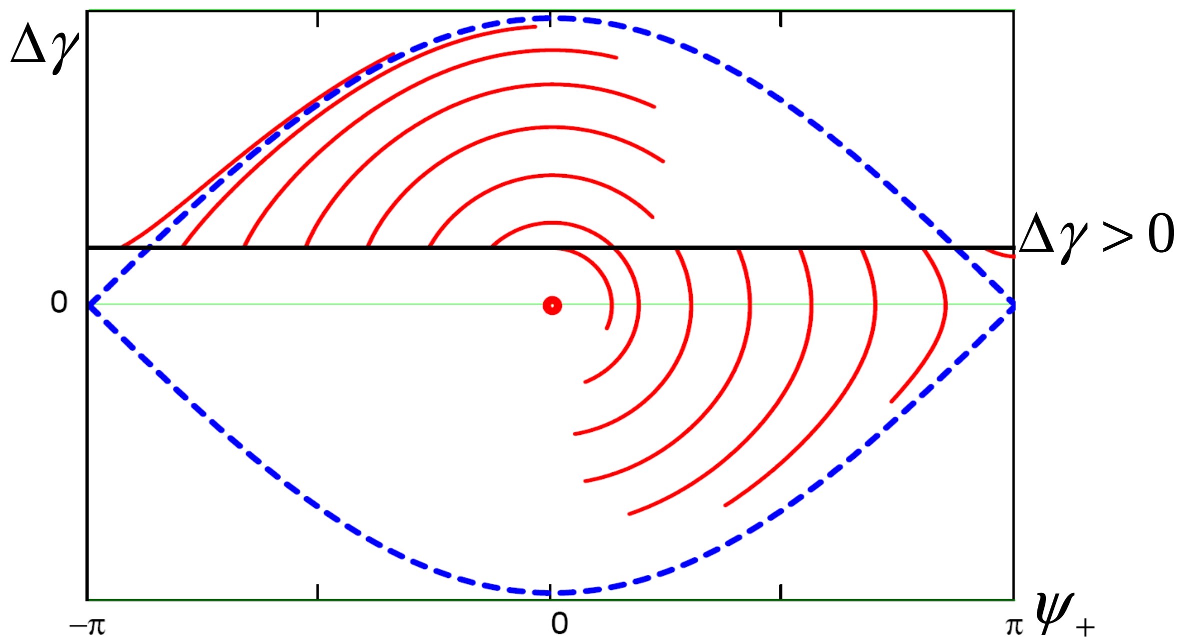
$$\frac{d}{dz}\psi_+ = \frac{2k_u}{\gamma}\cdot\Delta\gamma + O^5(\tfrac{1}{\gamma}), \quad \frac{d\gamma}{dz} = -\frac{K_L k_u K}{2\gamma}\sin\psi_+ + O^3(\tfrac{1}{\gamma})$$

$$\frac{d^2}{dz^2}\psi_+ = -\Omega^2\sin\psi_+ + O^4(\tfrac{1}{\gamma})$$ Wave number for small amplitudes: $\Omega = \frac{k_u}{\gamma}\sqrt{K_L K}$



For a monoenergetic beam with $\Delta\gamma$>0, after some time, more beam energy is lost than gained.

This is the basis process of E&M wave amplification in an FEL.