



Invariant of Motion



CHESS & LEPP

$$x(s) = \sqrt{2J\beta(s)} \sin(\psi(s) + \phi_0)$$

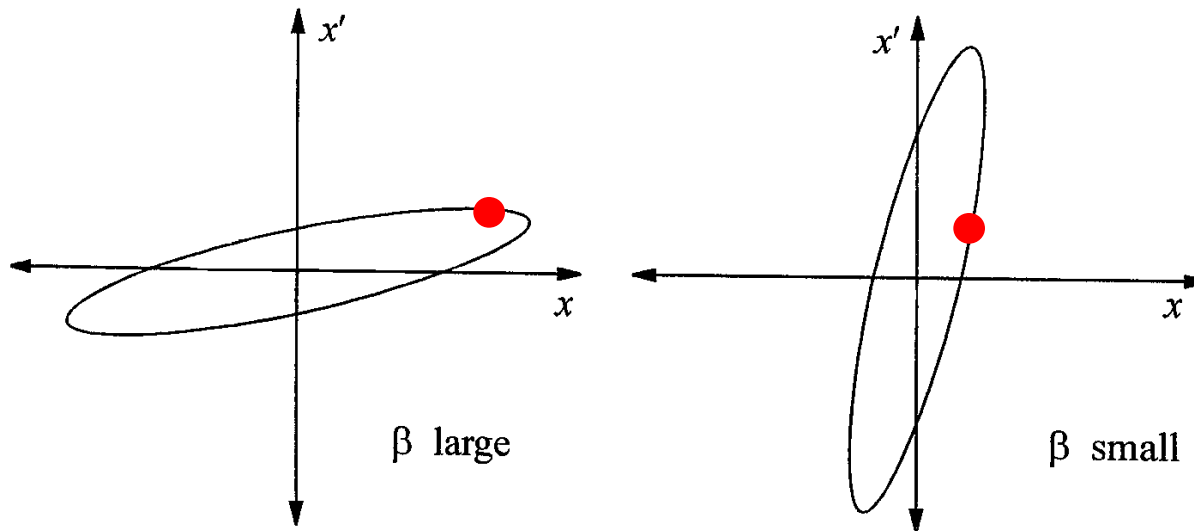
Where J and ϕ are given by the starting conditions x_0 and x'_0 .

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = 2J$$

Leads to the invariant of motion:

$$f(x, x', s) = \gamma(s)x^2 + 2\alpha(s)xx' + \beta(s)x'^2 \Rightarrow \frac{d}{ds} f = 0$$

It is called the **Courant-Snyder invariant**.





Propagation of Twiss Parameters



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$$(x_0, x_0') \begin{pmatrix} \gamma_0 & \alpha_0 \\ \alpha_0 & \beta_0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = 2J$$

$$(x, x') \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} = 2J = (x_0, x_0') \underline{M}^T \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} \underline{M} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

$$\begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} = \underline{M}^{-T} \begin{pmatrix} \gamma_0 & \alpha_0 \\ \alpha_0 & \beta_0 \end{pmatrix} \underline{M}^{-1}$$

$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = \underline{M} \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} \underline{M}^T$$



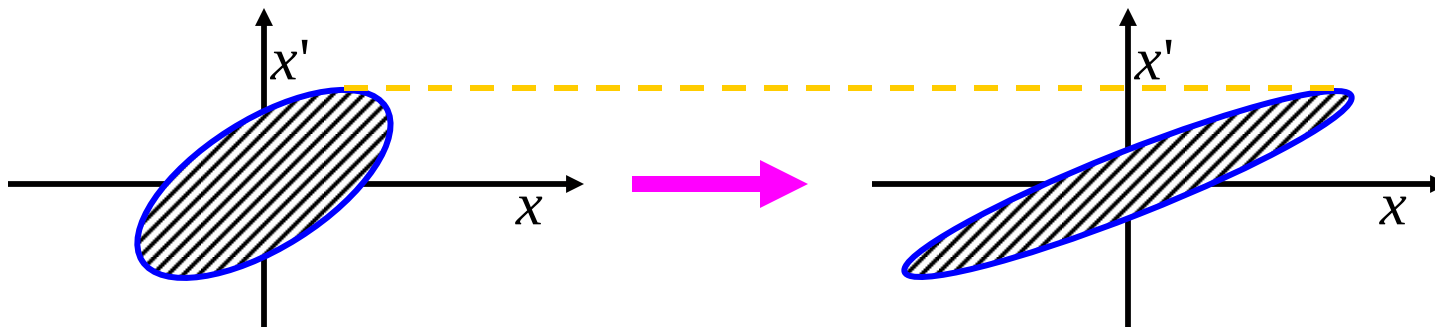
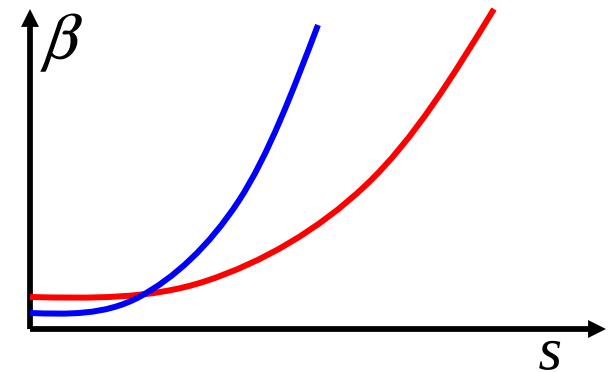
Twiss Parameters in a Drift



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$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} = \begin{pmatrix} \beta_0 - 2\alpha_0 s + \gamma_0 s^2 & \gamma_0 s - \alpha_0 \\ \gamma_0 s - \alpha_0 & \gamma_0 \end{pmatrix}$$

$$\beta = \beta_0^* \left[1 + \left(\frac{s}{\beta_0^*} \right)^2 \right] \quad \text{for } \alpha_0^* = 0$$

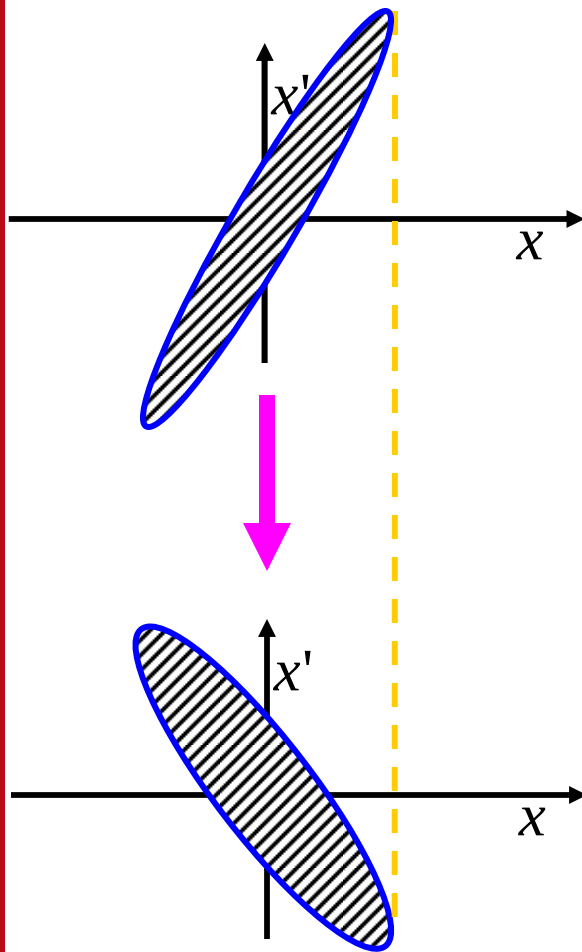




Twiss Parameters after a thin Quadrupole

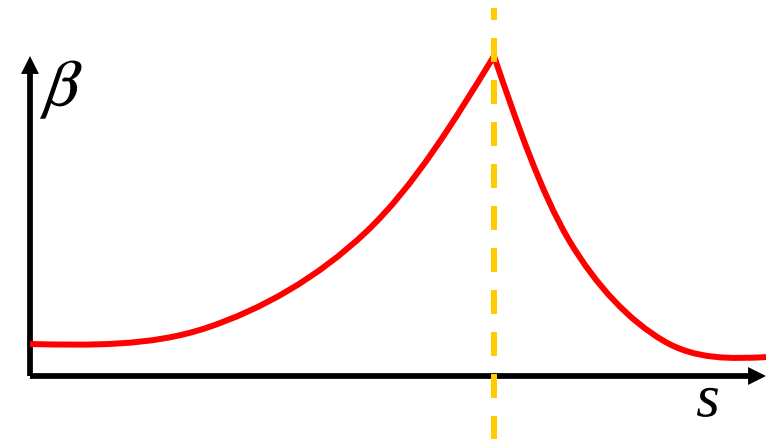


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$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix} \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}$$

$$\alpha = \alpha_0 + k\beta_0$$





From Twiss to Transport Matrix



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$$\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta_0} & 0 \\ -\frac{\alpha_0}{\sqrt{\beta_0}} & \frac{1}{\sqrt{\beta_0}} \end{pmatrix} \begin{pmatrix} \sin(\phi_0) \\ \cos(\phi_0) \end{pmatrix}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin(\psi(s) + \phi_0) \\ \cos(\psi(s) + \phi_0) \end{pmatrix}$$

$$= \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \cos \psi(s) & \sin \psi(s) \\ -\sin \psi(s) & \cos \psi(s) \end{pmatrix} \begin{pmatrix} \sin \phi_0 \\ \cos \phi_0 \end{pmatrix}$$

$$\underline{M}(s) = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \cos \psi(s) & \sin \psi(s) \\ -\sin \psi(s) & \cos \psi(s) \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta_0}} & 0 \\ \frac{\alpha_0}{\sqrt{\beta_0}} & \sqrt{\beta_0} \end{pmatrix}$$

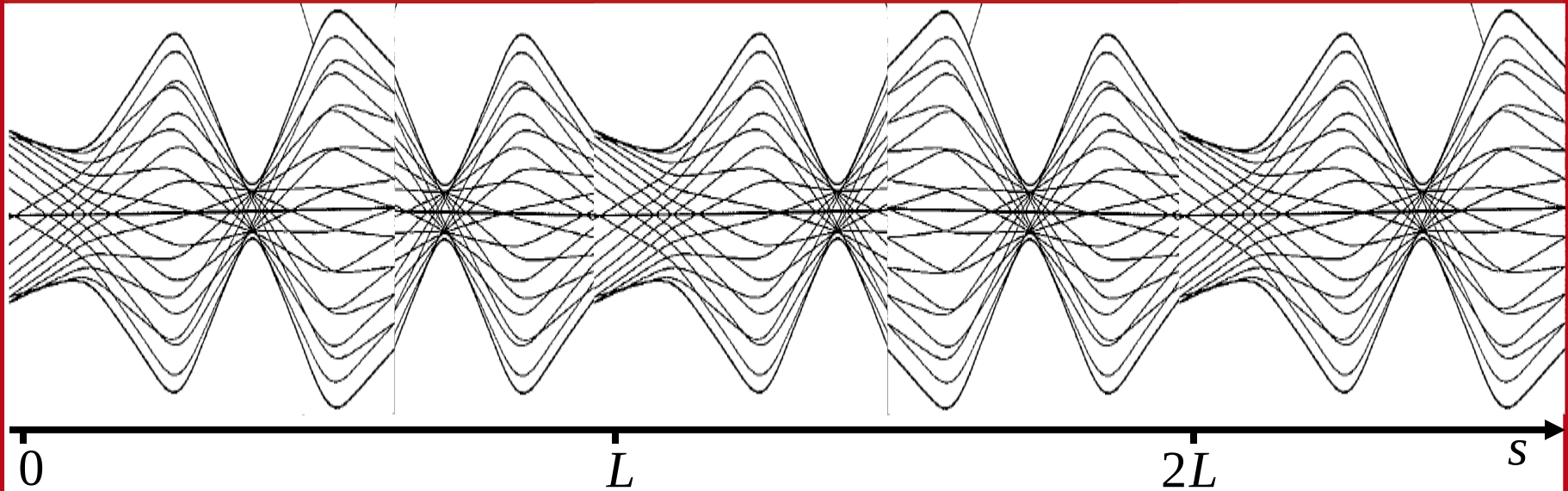
$$= \begin{pmatrix} \sqrt{\frac{\beta}{\beta_0}} [\cos \psi + \alpha_0 \sin \psi] & \sqrt{\beta_0 \beta} \sin \psi \\ \sqrt{\frac{1}{\beta_0 \beta}} [(\alpha_0 - \alpha) \cos \psi - (1 + \alpha_0 \alpha) \sin \psi] & \sqrt{\frac{\beta_0}{\beta}} [\cos \psi - \alpha \sin \psi] \end{pmatrix}$$



Periodic sections in an accelerator



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$$\vec{z}(s) = \underline{M}(s,0)\vec{z}(0)$$

$$\vec{z}(L) = \underline{M}(L,0)\vec{z}(0)$$

$$\vec{z}(s+L) = \underline{M}_0(s)\vec{z}(s) \quad , \quad \underline{M}_0 = \underline{M}(s+L,s)$$

$$\vec{z}(s+nL) = \underline{M}_0^n(s)\vec{z}(s)$$



The Periodic Beta Function



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If the particle distribution in a ring or any other periodic structure is stable, it is periodic from turn to turn.

$$\rho(x, x', s + L) = \rho(x, x', s)$$

To be matched to such a beam, the Twiss parameters α , β , γ must be the same after every turn.

$$\underline{M}(s, 0) = \begin{pmatrix} \sqrt{\frac{\beta}{\beta_0}} [\cos \psi + \alpha_0 \sin \psi] & \sqrt{\beta_0 \beta} \sin \psi \\ \sqrt{\frac{1}{\beta_0 \beta}} [(\alpha_0 - \alpha) \cos \psi - (1 + \alpha_0 \alpha) \sin \psi] & \sqrt{\frac{\beta_0}{\beta}} [\cos \psi - \alpha \sin \psi] \end{pmatrix}$$

$$\underline{M}_p(s) = \begin{pmatrix} \cos \mu + \alpha \sin \mu & \beta \sin \mu \\ -\gamma \sin \mu & \cos \mu - \alpha \sin \mu \end{pmatrix} = \underline{1} \cdot \cos \mu + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \sin \mu$$

$$\mu = \psi(s + L) - \psi(s)$$



One Turn Matrix to Periodic Twiss



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The periodic Twiss parameters are the solution of a nonlinear differential equation with periodic boundary conditions:

$$\begin{aligned} \beta' &= -2\alpha & \text{with } \beta(L) &= \beta(0) \\ \alpha' &= k\beta - \frac{1+\alpha^2}{\beta} & \text{with } \alpha(L) &= \alpha(0) \end{aligned}$$

$$\mu = \int_0^L \frac{1}{\beta(\hat{s})} d\hat{s}$$

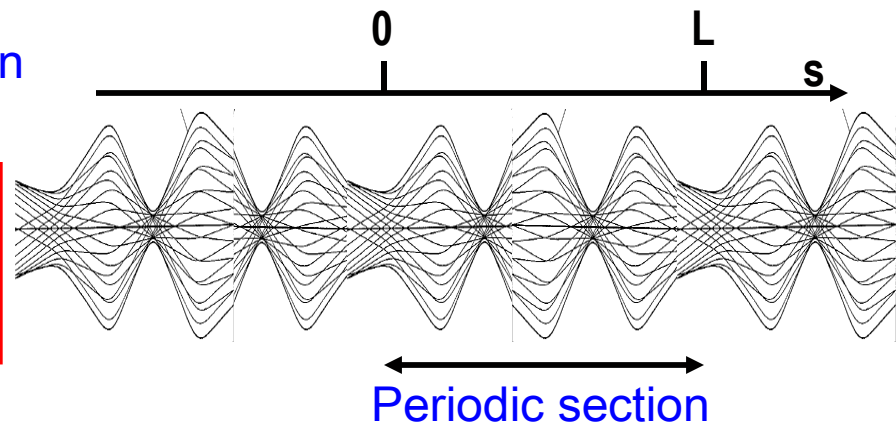
Note: $\beta(s) > 0$

They can be computed easily from matrix $\underline{M}$:

$$\underline{M}(s, s+L) = \underline{M}_0(s)$$

$$= \underline{1} \cdot \cos \mu + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \sin \mu$$

Stable beam motion and thus a periodic beta function can only exist when $\text{Tr}[\underline{M}] < 2$.



$$\cos \mu = \frac{1}{2} \text{Tr}[\underline{M}_0(s)]$$

$$\beta = \underline{M}_{0,12} \frac{1}{\sin \mu}$$

$$\alpha = (\underline{M}_{0,11} - \underline{M}_{0,22}) \frac{1}{2 \sin \mu}$$

$$\gamma = \frac{1+\alpha^2}{\beta}$$



The Periodic-Cell Phase Advance



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$$\mu = 2\pi\nu = \psi(s+L) - \psi(s)$$

It is a property of the cell structure and does not depend on where the cell starts.

$$\underline{M}_0(s) = \underline{1} \cdot \cos \mu + \begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix} \sin \mu$$

$$\begin{aligned} 2 \cos \mu(s) &= \text{Tr}[\underline{M}_0(s)] = \text{Tr}[\underline{M}(s,0)\underline{M}_0(0)\underline{M}^{-1}(s+L,L)] \\ &= \text{Tr}[\underline{M}_0(0)] = 2 \cos \mu(0) \end{aligned}$$

$$\underline{M}_0^n = \underline{1} \cdot \cos n\mu + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \sin n\mu$$

