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Electron Sources: Space Charge

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- Space charge treatment
  - Debye length
  - Plasma frequency
  - Beam temperature
- Stationary distributions
- Emittance growth due to excess free energy
- Beam envelope equation
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- Emittance compensation
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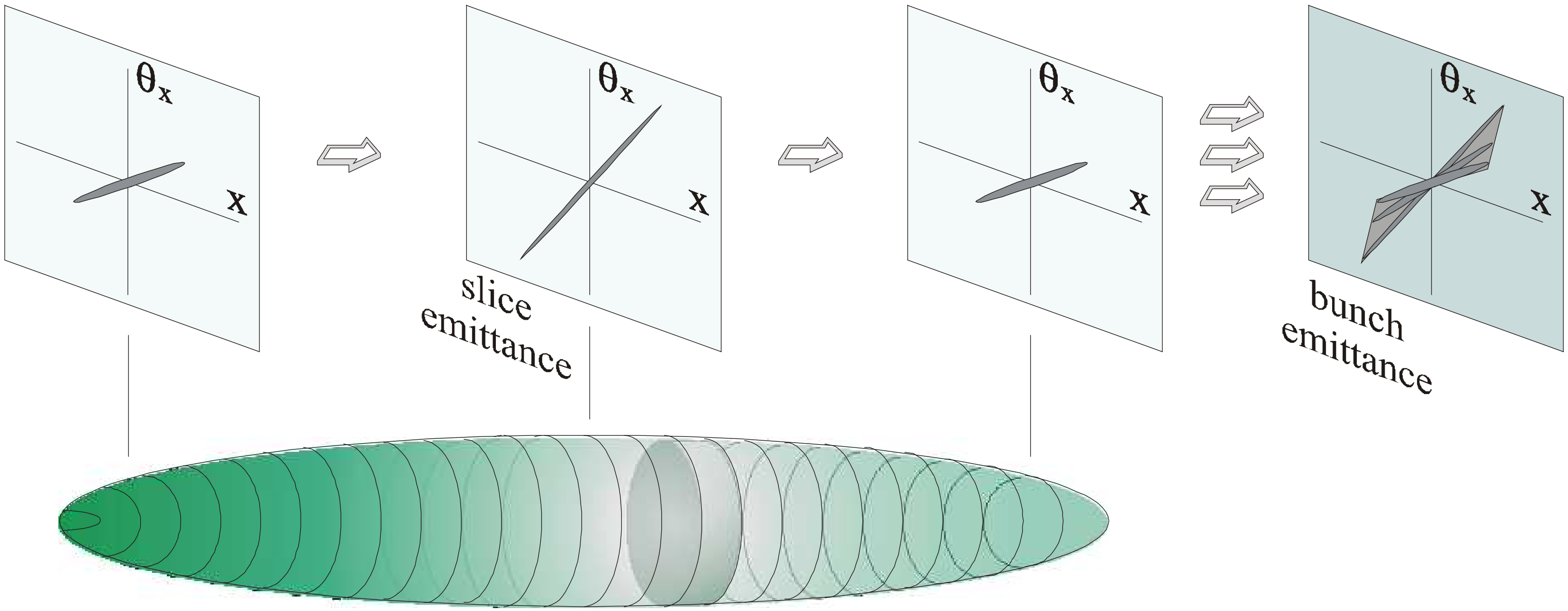
Space charge



In a typical bunched beam from a gun, both charge and current density are functions of transverse & longitudinal coordinates. This makes space charge dominated behavior highly nonlinear.

For beam envelope equation we will assume that ρ and J_z are independent of transverse coordinate and that the beam is not bunched (aspect ration << 1).

“Collisional” vs. “smooth” self-fields

It would seem that the easiest approach is to calculate Lorentz force of all electrons directly (this would encompass practically *all* the behavior of the beam). This is not feasible because the number of evaluations for each time step is ~ N^2, with N ~ 10^{10}. Taking the fastest supercomputer with 100 TFLOPS, one estimates ~ year(s) per time step (and one may need something like ~10^4 steps).

Instead, people use macroparticles in computer simulations with the same *e/m* ratio

When $N \to \infty$ forces are smooth; when $N \to 1$ grainy collisional forces dominate. Envelope equation assumes the first scenario.

How to determine quantitatively “collisional” vs. “smooth” behavior of the space charge in the beam?

Debye length and plasma frequency

Three characteristic lengths in the bunch:

a bunch dimension; l_p interparticle distance; λ_D Debye length

Interactions due to Coulomb forces are long-range; Debye length is a measure of how 'long' (screen-off distance of a local perturbation in charge).

$$\boxed{\lambda_D \equiv \frac{\sigma_{v_x}}{\omega_p}}$$

for nonrelativistic case:

$$\omega_p = \sqrt{\frac{e^2 n}{\varepsilon_0 m}}, \sigma_{v_x} = \sqrt{\frac{k_B T}{m}}$$

$$\lambda_D = \sqrt{\frac{\varepsilon_0 k_B T}{e^2 n}}$$

for relativistic case:

$$\boxed{\omega_p = \sqrt{\frac{e^2 n}{\varepsilon_0 m \gamma^3}}} \quad \sigma_{v_x} = \sqrt{\frac{k_B T}{m\gamma}}$$

$$\boxed{\lambda_D = \sqrt{\frac{\gamma^2 \varepsilon_0 k_B T}{e^2 n}}}$$

Debye length: beam dynamics scenarios

$\lambda_D \gg a$

YES

single-particle behavior dominates; true when either energy or beam temperature is large (emittance-dominated)

NO

$$\varepsilon_n \sim \frac{\sigma_x \sigma_{p_x}}{mc} = \frac{\gamma \sigma_x \sigma_{v_x}}{c} = const$$

$$\gamma \sigma_x^2 k_B T = const$$

collective forces are important

$\lambda_D \gg l_p$

YES

“smooth” force; Liouville’s theorem can be defined in 6-D phase space; **if** forces are linear rms emittance is also conserved

NO

fields of individual particles become important; one ends up having 6N-D phase space to deal with in the worst case; beam tends to develop ‘structure’

Equilibrium distribution

Similar to thermodynamics and plasma physics, there may exist equilibrium particle distributions (i.e. those that remain stationary). Vlasov theory allows one to find such distributions (assumes collisions are negligible, but they are the ones responsible to drive the distribution to the equilibrium!). Without the derivation, Vlasov-Maxwell equations for equilibrium distributions $f(q_i, p_i, t)$ (i.e. no explicit time dependence, $\partial/\partial t = 0$):

$$\sum_{i=1}^{3}\left[\frac{\partial f}{\partial q_i}\dot{q}_i + e(\vec{E}+\vec{v}\times\vec{B})_i\frac{\partial f}{\partial p_i}\right]=0$$

$$\vec{\nabla}\times\vec{E}=0 \qquad \vec{\nabla}\cdot\vec{E}=\frac{e}{\varepsilon_0}\int f\,d^3p$$

$$\vec{\nabla}\times\vec{B}=\mu_0 e\int\vec{v}f\,d^3p \qquad \vec{\nabla}\cdot\vec{B}=0$$

Equilibrium distribution (contd.)

In particular, in a ***constant focusing channel***, equilibrium transverse density obeys a well-known Boltzmann relation

$$n(r) = n(0)\exp\left[-\frac{e\phi(r)}{k_B T_\perp}\right]$$

$$\phi(r) = \phi_{ext}(r) + \frac{1}{\gamma^2}\phi_{self}(r)$$

$$e\phi_{ext}(r) = \gamma m\omega_0^2 r^2/2$$

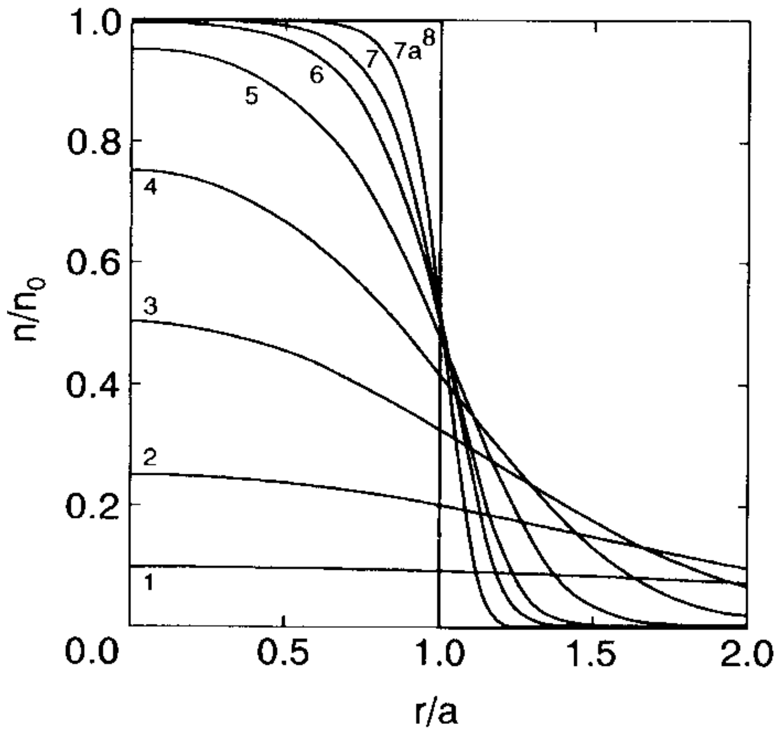
$$\varphi_{self}(r) = -\int_0^r\int_0^{\hat{r}} dr d\hat{r}\left(\frac{e}{\varepsilon_0 r}\hat{r}n(\hat{r})\right)$$

Equilibrium distribution (contd.)

Analytically, two extreme cases:

$$k_B T \to 0 \quad (\lambda_D / a \to 0) \qquad n(r) = \begin{cases} n_0 = const, \text{ for } r \le a \\ 0, \text{ for } r > a \end{cases} \qquad \textit{uniform}$$

$$\varphi_{self} \to 0 \quad (\lambda_D / a \ge 1) \qquad n(r) = n_0 \exp\left[-\frac{\gamma m \omega_0^2 r^2}{2 k_B T_\perp}\right] \qquad \textit{Gaussian}$$



Curve	$n(0)/n_0$	$\lambda_D(0)/a_0$	Ka^2/ϵ^2
1	0.1	4.82	0.054
2	0.25	1.81	0.153
3	0.5	0.795	0.396
4	0.75	0.432	0.893
5	0.95	0.229	2.51
6	0.995	0.145	6.00
7	0.9995	0.107	10.9
7a	0.999995	0.0710	24.8
8	1	0	∞

Beam heating due to free energy

– ***Stationary*** (equilibrium) ***beam distributions*** (states) in a linear uniform focusing channel best described by a transverse Maxwell-Boltzmann distribution

– ***Nonstationary beams*** (***nonequilibrium***) such as ***mismatched initial conditions*** for the beam have a higher total energy per particle than the corresponding stationary beam (free energy)

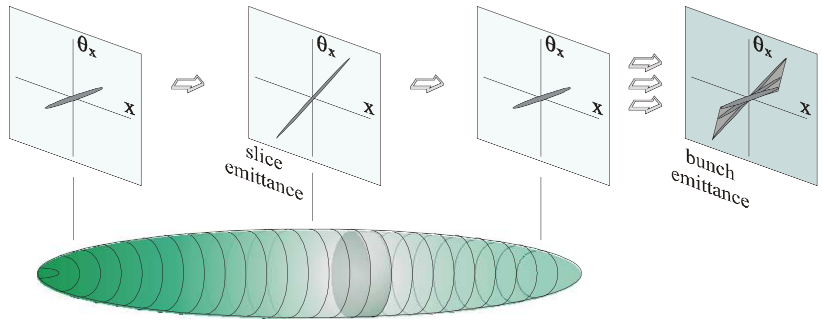
– Collisional forces eventually will lead to ***a new stationary state*** at the higher energy per particle, result in ***increased beam temperature*** or emittance

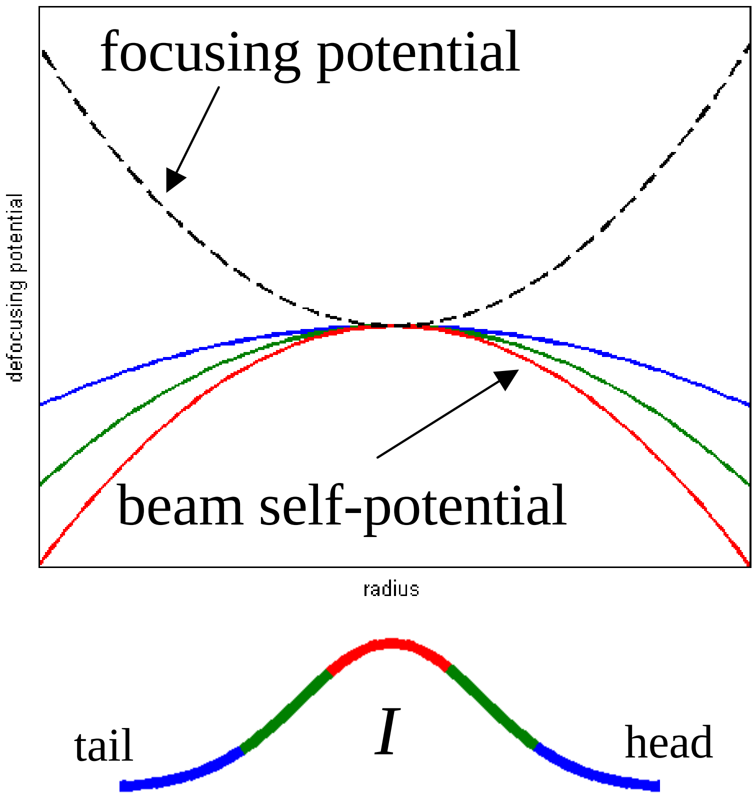
Beam temperature

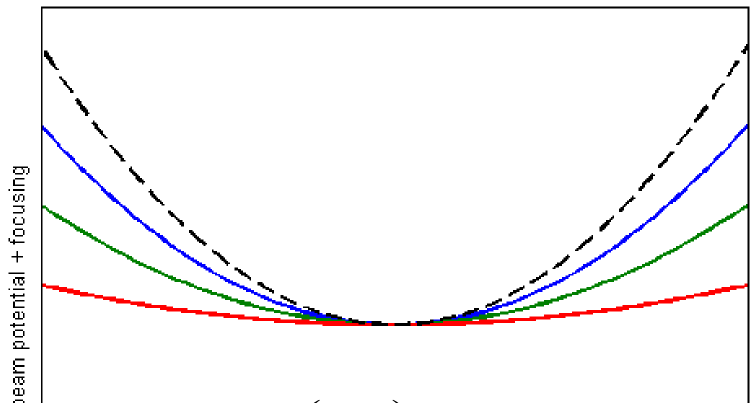
Diffraction limited beam at 1Å → ε_x = 8pm at 5GeV → ε_{nx} = 0.08 μm

10 MV/m gradient → σ_{laser} = 0.3 mm

Transverse temp. needed ***kT = 25 meV***





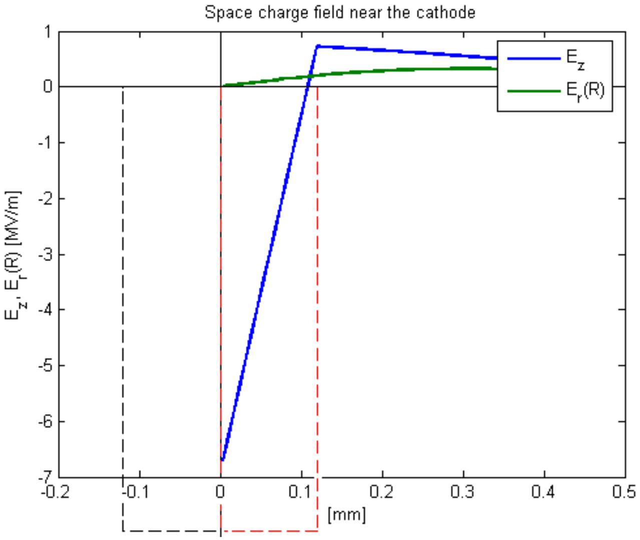


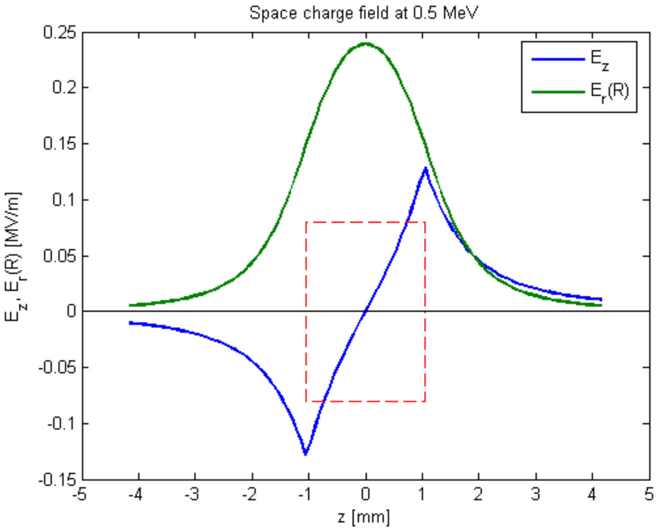
$$kT_{beam} \sim std\left(\frac{\partial F}{\partial R}\right)\sigma_x^2 \approx \frac{1}{250\varepsilon_0}\frac{eQ}{\sigma_z}$$

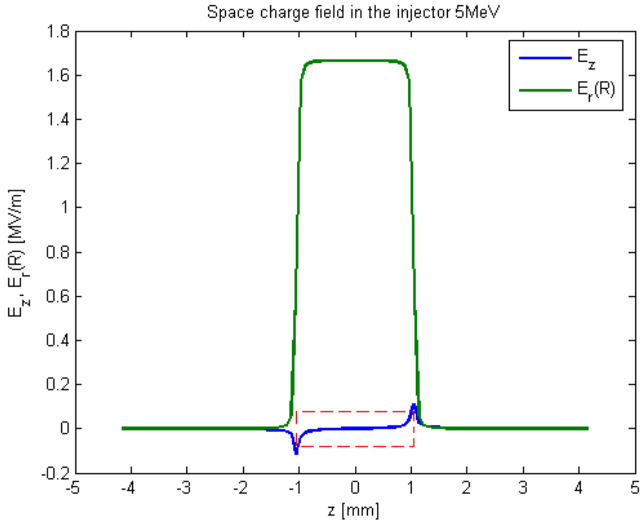
Equilibrium $\boldsymbol{kT_{beam} \sim 100\ eV!!}$

Space charge basics

near cathode	after the gun	relativistic energies







$$B_\theta = \frac{\beta}{c} E_r$$

$\perp$ force scales as 1- $\beta^2 = \gamma^2$

Perveance and characteristic current

Let's derive beam envelope equation (i.e. we assume that self-forces are smooth). We have almost derived the equation already (previous lecture's paraxial ray equation). Two terms are missing – due to space charge and emittance 'pressure'.

Uniform laminar beam in the absence of external forces:

$$\gamma m\ddot{r} = \frac{eIr}{2\pi\varepsilon_0 a^2 \beta c}\frac{1}{\gamma^2}, \text{ using } \ddot{r} = \beta^2\gamma^2 r'' \rightarrow r'' = \frac{eIr}{2\pi\varepsilon_0 a^2 mc^3\beta^3\gamma^3}$$

$$\boxed{\ddot{r} = \frac{\omega_p^2}{2} r} \qquad \boxed{r'' = \frac{K}{a^2} r} \qquad \boxed{r_m r_m'' = K} \quad \text{for } r_m = a$$

$$\omega_p^2 = \frac{eI}{\pi\varepsilon_0 mc\beta\gamma^3 a^2} \qquad \boxed{K = \frac{I}{I_0}\frac{2}{\beta^3\gamma^3}} \qquad \boxed{I_0 = \frac{4\pi\varepsilon_0 mc^3}{e} \approx \frac{1}{30}\frac{mc^2}{e} = 17\text{kA}}$$

Perveance and generalized perveance

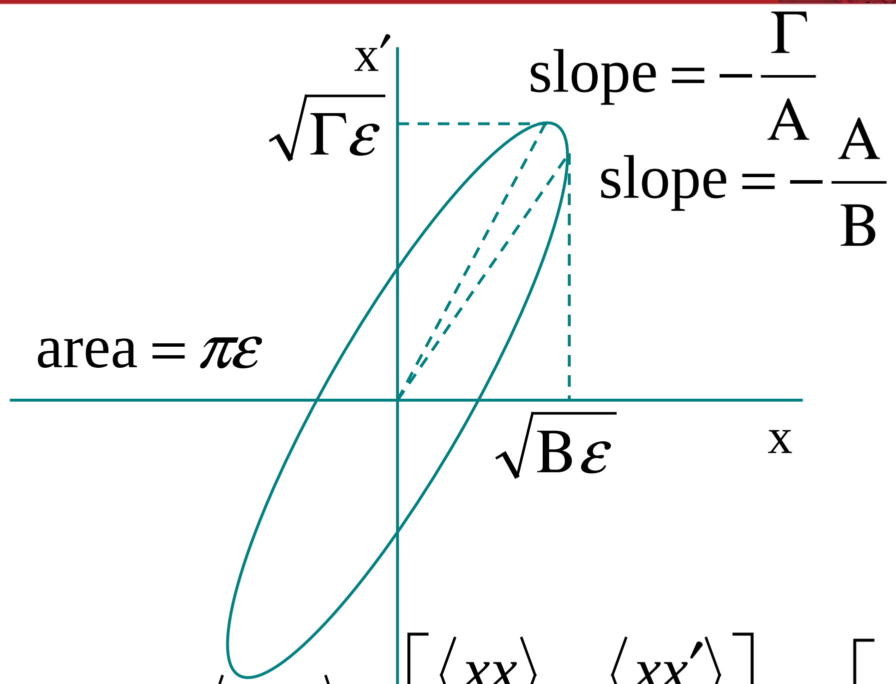
Nonrelativistic limit: $K = \frac{I}{I_0}\frac{2}{\beta^3} = \frac{eI}{2\pi\varepsilon_0 m v^3}$, with $v = (2eV/m)^{1/2}$

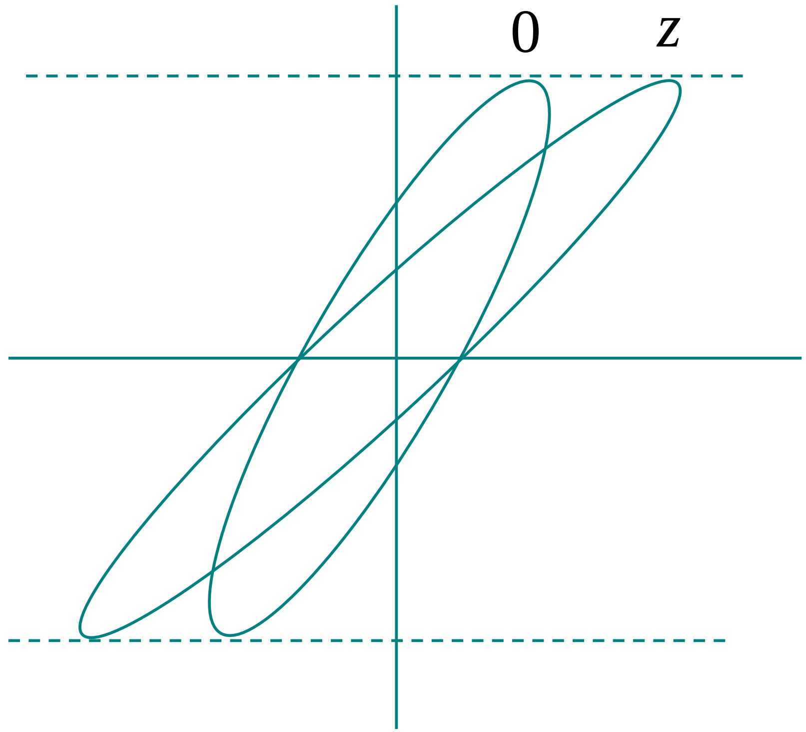
$$K = \frac{I}{V^{3/2}}\left[\frac{1}{4\pi\varepsilon_0 (2e/m)^{1/2}}\right]$$

generalized perveance

perveance

Emittance ‘pressure’ term





$\det[\ldots] = 1$

$$\Sigma = \langle \vec{x}^T \vec{x} \rangle = \begin{bmatrix} \langle xx \rangle & \langle xx' \rangle \\ \langle x'x \rangle & \langle x'x' \rangle \end{bmatrix} = \varepsilon \begin{bmatrix} B & -A \\ -A & \Gamma \end{bmatrix}$$

In a drift $0 \to z$: $\begin{array}{l} x' \to x' = const \\ x \to x + x'z \end{array}$ and $\begin{array}{l} B \to B - 2Az + \Gamma z^2 \\ A \to A - \Gamma z \\ \Gamma \to \Gamma = const \end{array}$

For σ_x: $\sigma_x' = -\frac{\sqrt{\varepsilon}A}{\sqrt{B}}, \quad \sigma_x'' = \frac{\sqrt{\varepsilon}}{B\sqrt{B}}$ or $\boxed{\sigma_x'' - \frac{\varepsilon^2}{\sigma_x^3} = 0}$

Beam envelope equation

From paraxial ray equation with the additional terms, one obtains

$$\sigma'' + \sigma' \frac{\gamma'}{\beta^2\gamma} + \sigma \frac{1}{\beta^2\gamma^2}\left[\frac{\gamma''\gamma}{2} + \left(\frac{eB}{2mc}\right)^2\right] - \frac{1}{\sigma}\frac{I}{2I_0\beta^3\gamma^3} - \frac{1}{\sigma^3}\frac{1}{\beta^2\gamma^2}\left[\left(\frac{P_\theta}{mc}\right)^2 + \varepsilon_n^2\right] = 0$$

adiabatic

RF focusing of cavity edge

solenoid

space charge

angular momentum ‘increases’ emittance

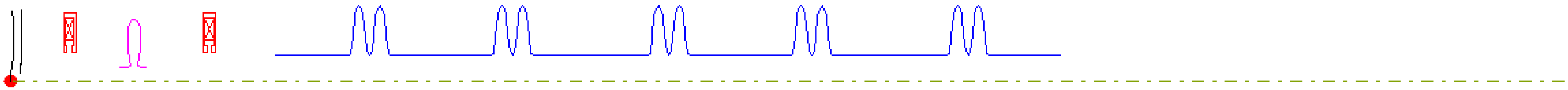
$$\frac{I}{2I_0\beta\gamma} >> \frac{\varepsilon_n^2}{\sigma^2}, \text{ or } \frac{I}{2I_0\beta^2\gamma^2} >> \frac{\varepsilon_n}{\mathrm{B}}$$

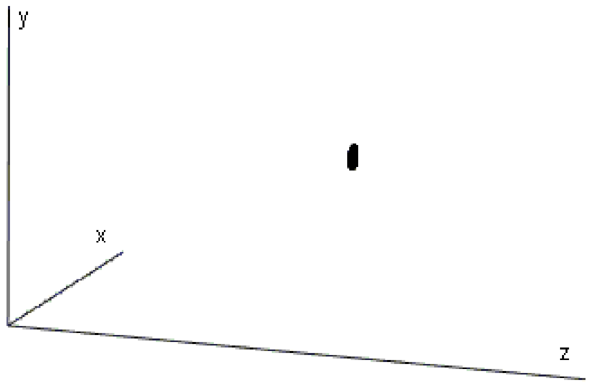
space charge dominated

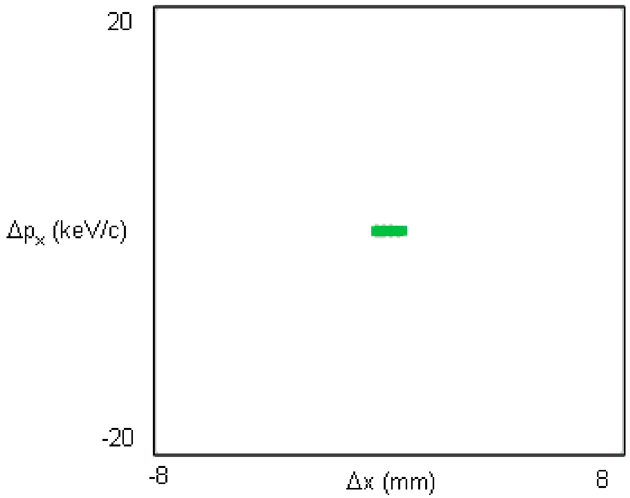
$$\frac{I}{2I_0\beta\gamma} << \frac{\varepsilon_n^2}{\sigma^2}, \text{ or } \frac{I}{2I_0\beta^2\gamma^2} << \frac{\varepsilon_n}{\mathrm{B}}$$

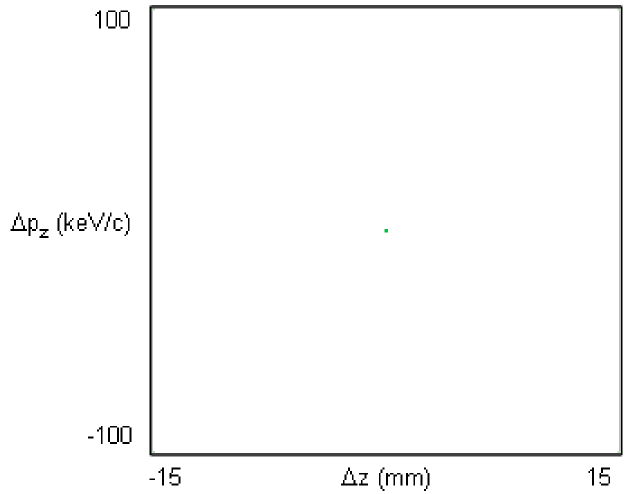
emittance dominated

Example of beam dynamics: 80 pC charge







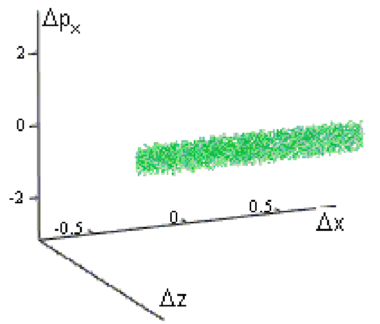


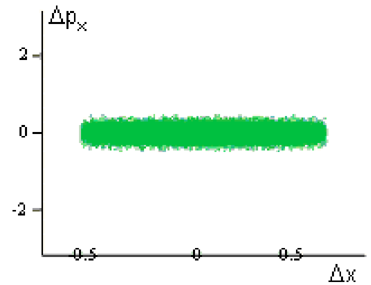
$z = 0.000$ m

$p_z = 0.000$ MeV/c

$\sigma_x = 0.294$ mm $\quad \varepsilon_x = 0.077$ mm-mrad

$\sigma_z = 0.000$ mm $\quad \varepsilon_z = 0.000$ mm-keV

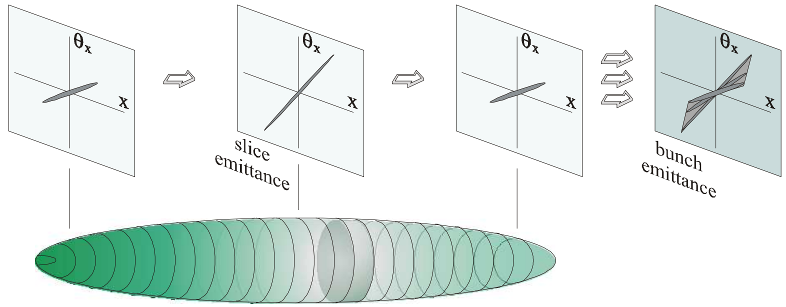




Equivalent Beams Concept

- The ***actual beams have*** complicated transverse profiles and phase space distribution
- How useful is beam envelope equation derived for ***uniform cylindrical beams***?
- ***Concept of Equivalent Beams*** due to Lapostolle and Sachere (1971): "two beams having the same current and kinetic energy are *equivalent* in an approximate sense if the second moments of the distribution are the same."
- E.g. rms sizes and emittances are ***the same*** for the two beams when compared at identical positions

Emittance evolution due to space charge



We have seen that beam will evolve from space-charge dominated to emittance dominated regimes as it is being accelerated. At low energies, various longitudinal ‘slices’ of the bunch experience different forces due to varying current → ‘bow-tie’ phase space is common.

Important realization is that much of these space charge emittance growth may be reversible through appropriate focusing (and drifts), a so-called emittance compensation. Obviously, this emittance compensation should take place before (or rather as) the beam becomes ultrarelativistic (and emittance-dominated).

Emittance growth in RF guns (without comp.)

Kim analyzed emittance growth due to space charge and RF (NIM A **275** (1989) 201-218). His analysis applies to beam in RF guns. He has found:

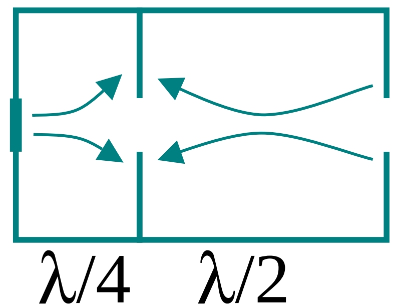
$$\varepsilon_x^{rf} = \frac{\alpha k_{RF}^3 \sigma_x^2 \sigma_z^2}{\sqrt{2}} \qquad \boxed{\alpha \downarrow} \qquad\qquad \alpha = \frac{eE_0}{2mc^2 k_{RF}}$$

$$\varepsilon_x^{sc} = \frac{\pi}{4}\frac{1}{\alpha k_{RF}}\frac{1}{\sin\phi_0}\frac{I_{peak}}{I_0}\mu_x(A) \qquad \boxed{\alpha \uparrow} \qquad\qquad \mu_{x,gauss} \approx \frac{1}{3A+5}$$

e.g. 100 MV/m RF gun (λ = 10.5 cm): α = 1.64, phase (to minimize rf emittance) ϕ_0 = 71° ($\phi \rightarrow$ 90°), laser width and length σ_x = 3.5 mm, σ_z = 0.6 mm

$$\varepsilon_x^{rf} = 1.1\,\text{mm-mrad}$$

$$\varepsilon_x^{sc} = 4.0\,\text{mm-mrad}\ (1.3\,\text{mm-mrad for uniform})$$



Emittance Compensation

Carlsten noted in simulations that emittance can be brought down and gave a simple explanation for the effect (NIM A **285** (1989) 313-319).

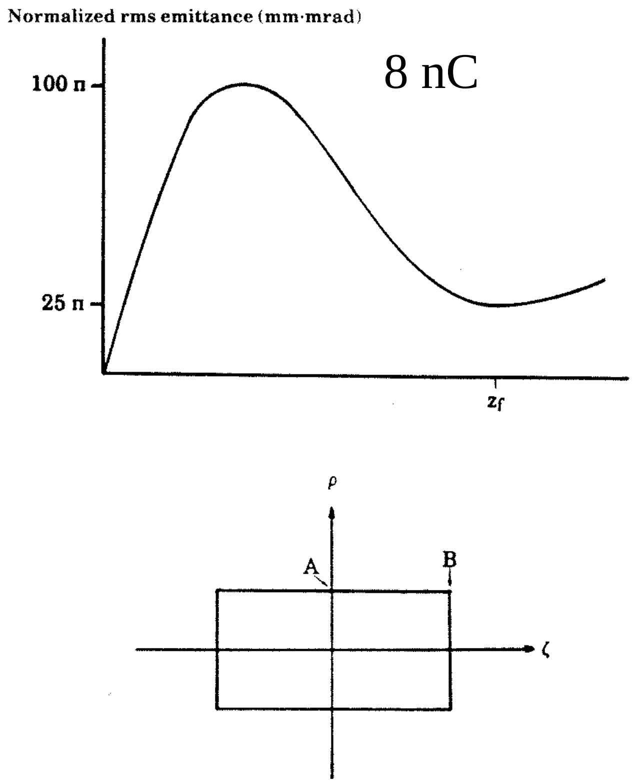
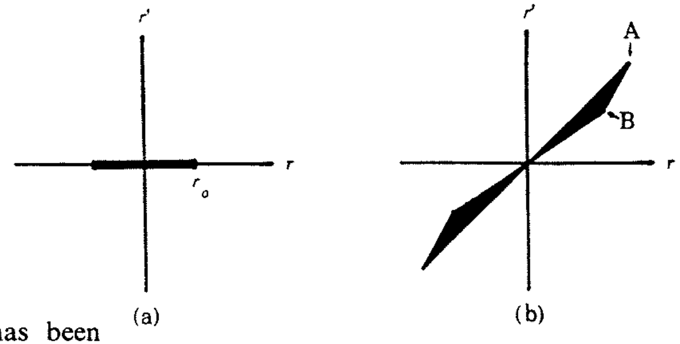


Fig. 2. Typical transverse emittance versus beamline position plot for a photoelectric injector, showing quick initial growth and subsequent reduction for a slug beam and physical description of a slug beam, with internal coordinates ρ and ζ.

5. Conclusion

A photoelectric injector design analysis has been presented. The emittance growth from the dominant mechanism has been shown to be eliminated with a simple lens configuration, leaving only a small residual emittance resulting from the other mechanisms.



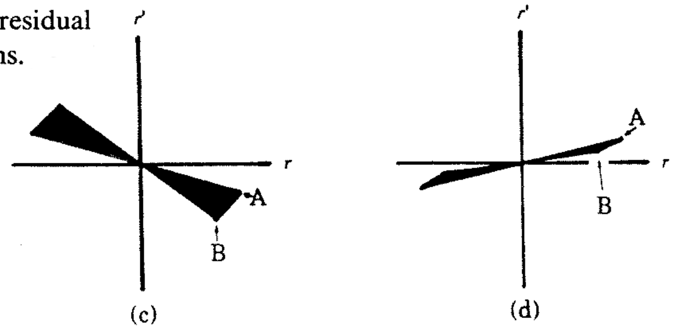


Fig. 3. Transverse phase-space plots showing emittance growth and reduction. (a) Initial phase-space plot with very small emittance. (b) Phase space plot after drift z_1 to lens, showing the emittance growth due to the different expansion rates of points A and B. (c) Phase-space plot immediately after lens, showing rotation due to the lens. The emittance is unchanged because we assume the lens is linear. (d) Phase space plot after drift z behind lens, showing the emittance reduction due to the different expansion rates of points A and B.

Emittance compensation from envelope eqn.

Serafini and Rosenzweig used envelope equation to explain emittance compensation (Phys. Rev. E **55** (1997) 7565).

$$\sigma'' + \sigma' \frac{\gamma'}{\beta^2\gamma} + \sigma K_r - \frac{I(\zeta)}{\sigma 2 I_0 \beta^3 \gamma^3} - \frac{\varepsilon_n^2}{\sigma^3 \beta^2 \gamma^2} = 0$$

includes solenoid and RF

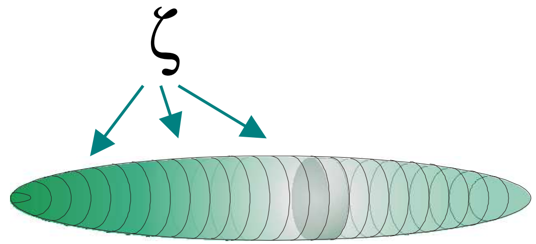
ζ tags long. slice in the bunch

For space charge dominated case in absence of acceleration

$$\sigma'' + \cancel{\sigma' \frac{\gamma'}{\beta^2\gamma}} + \sigma K_r - \frac{I(\zeta)}{2\sigma I_0 \beta^3 \gamma^3} - \cancel{\frac{\varepsilon_n^2}{\sigma^3 \beta^2 \gamma^2}} = 0$$

Brillouin flow $\sigma'' \to 0$:

$$\sigma_{eq} = \sqrt{\frac{I(\zeta)}{2 K_r I_0 \beta^3 \gamma^3}}$$



Emittance compensation

Small oscillations near equilibrium: $\delta\sigma'' + \delta\sigma\left[2K_r \cancelto{0}{-\frac{\delta\sigma}{\sigma_{eq}} + \left(\frac{\delta\sigma}{\sigma_{eq}}\right)^2} - \ldots\right] = 0$

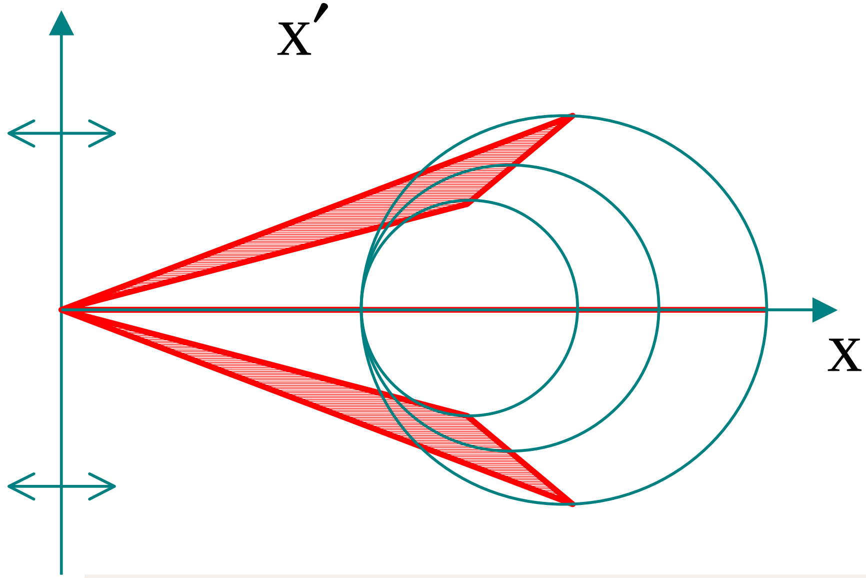
Important: frequency of small oscillations around equilibrium does not depend on ζ. E.g. for beam with $\sigma'(0,\zeta) = 0$ and $\sigma(0,\zeta) = \sigma_{eq}(\zeta) + \delta\sigma(\zeta) = \sigma_0$:

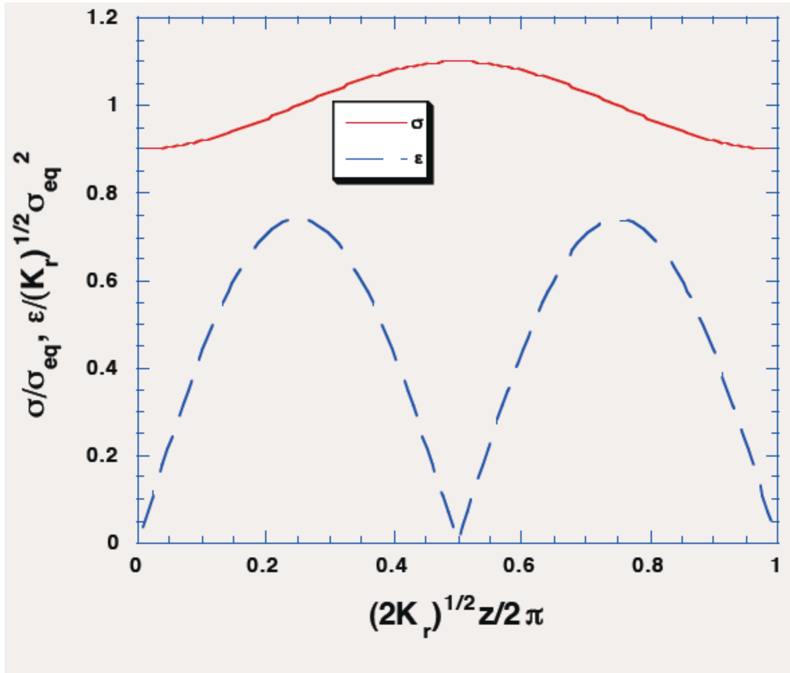
$$\begin{cases} \sigma(z,\zeta) = \sigma_{eq}(\zeta) + \delta\sigma(\zeta)\cos\left(\sqrt{2K_r}\,z\right) \\ \sigma'(z,\zeta) = -\sqrt{2K_r}\,\delta\sigma(\zeta)\sin\left(\sqrt{2K_r}\,z\right) \end{cases}$$

$$\varepsilon = \frac{1}{2}\sqrt{\langle r^2\rangle\langle r'^2\rangle - \langle r'r\rangle^2} \approx \sqrt{2K_r}\,\langle\sigma_{eq}\rangle\sigma_0\left|\sin(\sqrt{2K_r}\,z)\right|$$

Emittance compensation

- slices oscillate in phase space around different equilibria but with the same frequency
- ‘projected’ emittance reversible oscillations when $\delta\sigma/\sigma_{eq} << 1$, unharmonicity shows up when $\delta\sigma$ is not small
- ignores the fact that beam aspect ratio can be >> 1 (e.g. at the cathode)





Small amplitude oscillation model

Emittance compensation

Including acceleration term and transforming from (σ,z) → (τ,y) in the limit $\gamma >> 1$

$$\frac{d^2\tau}{dy^2}+\Omega^2\tau=\frac{e^{-y}}{\tau}$$

$y \equiv \ln\frac{\gamma}{\gamma_0}, \tau \equiv \sigma\gamma'\sqrt{\gamma_0/(I(\zeta)/2I_0)}$, Ω represents solenoid & RF focusing

Particular solution that represents generalized Brillouin flow or 'invariant envelope':

$$\tau_{eq}=\frac{2e^{-y/2}}{\sqrt{1+4\Omega^2}}, \sigma_{eq}=\frac{2}{\gamma'}\sqrt{\frac{I(\zeta)}{\gamma 2I_0}\frac{1}{1+4\Omega^2}}$$

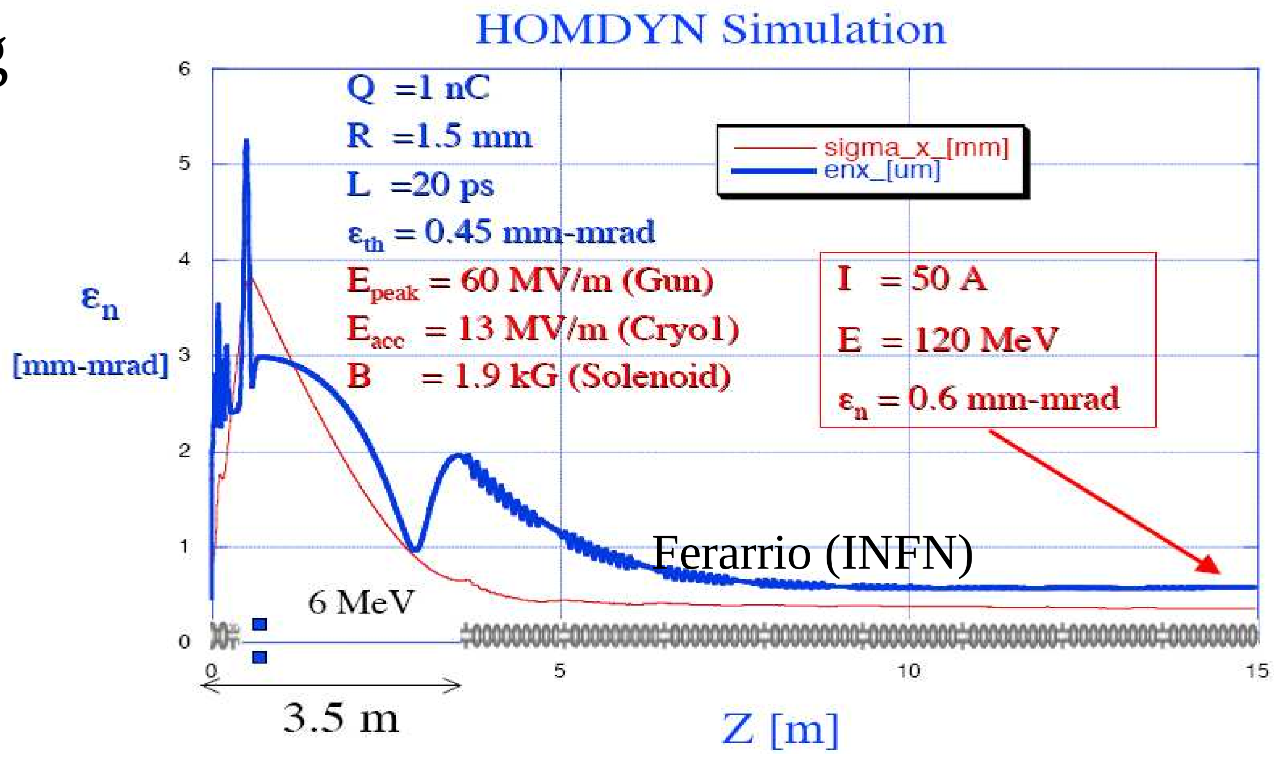
$\frac{\gamma\sigma'_{eq}}{\sigma_{eq}}=-\frac{\gamma'}{2}$ phase space angle is independent of slice ζ

Matching beam to 'invariant envelope' can lead to 'damping' of projected rms emittance.

Numerical approach

Using the beam envelope equation for individual slices leads to a recipe for emittance compensation, which works for simple cases (e.g. matching beam into long focusing channel / linac in the injector). For other more complicated scenarios one should solve the equations numerically (example: code HOMDYN).

Finally, particle tracking is indispensable for analysis and design of the injector where the assumptions made are invalid or theory is too complicated to be useful.



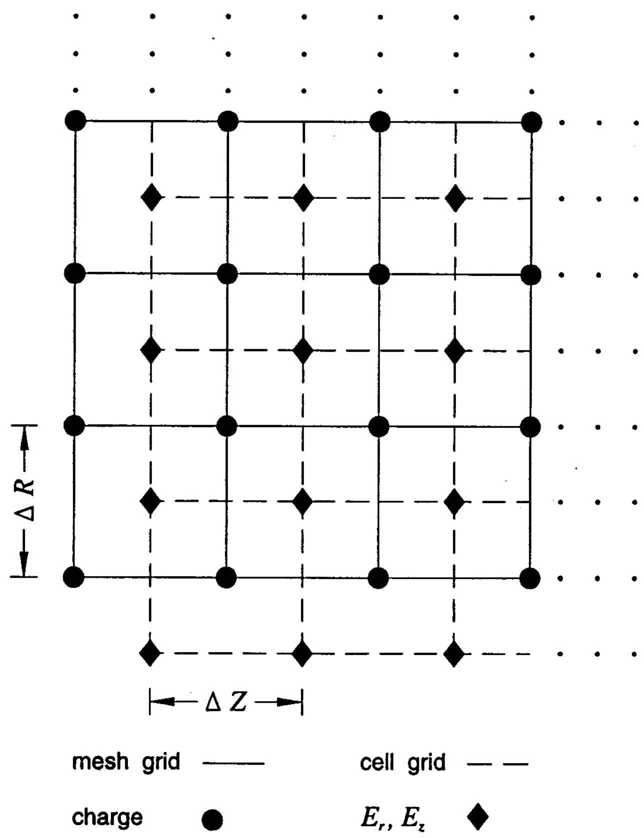
Common space charge code strategy

- Majority of injector design codes use the strategy:
  - Assume nearly monochromatic beam
  - Solve Poisson equation in the rest frame
    - various meshing strategies (simple uniform mesh for fast FFT methods, nonequidistant adaptive mesh for distributions with varying density)
    - Essentially removes granularity of the force
  - Lorentz back-transform and apply forces including 3D field maps of external elements (cavities, magnets, etc.)

Space charge codes

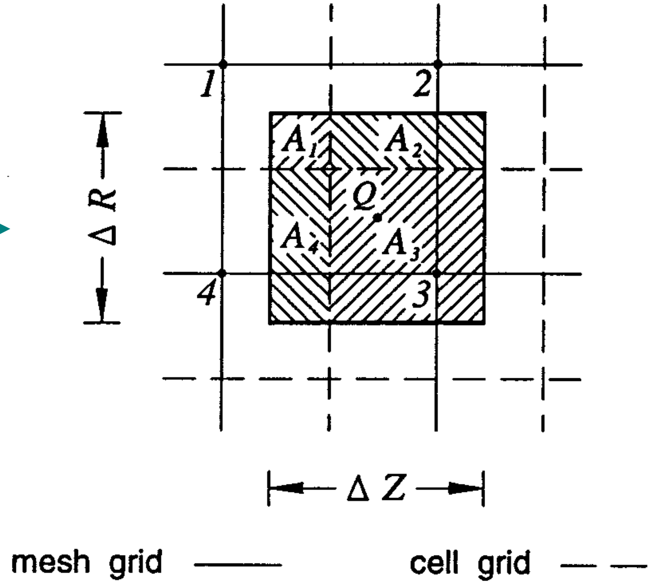
Different approaches are used (e.g. envelope equation integration, macroparticle tracking, various meshing scenarios, etc.).
Mesh method works as following:

1) transform to rest frame of the reference particle
2) create mesh (charge) and cell grid (electrostatic fields)
3) create table containing values of electrostatic field at any cell due to a unit charge at any mesh vertex (does not need to be recalculated each time step)



Space charge codes

4) assign macroparticle charges to mesh nodes, e.g. 1,2,3, and 4 vertices get QA_1/A, QA_2/A, QA_3/A, and QA_4/A respectively, where $A_1+A_2+A_3+A_4 = A = \Delta Z \Delta R$

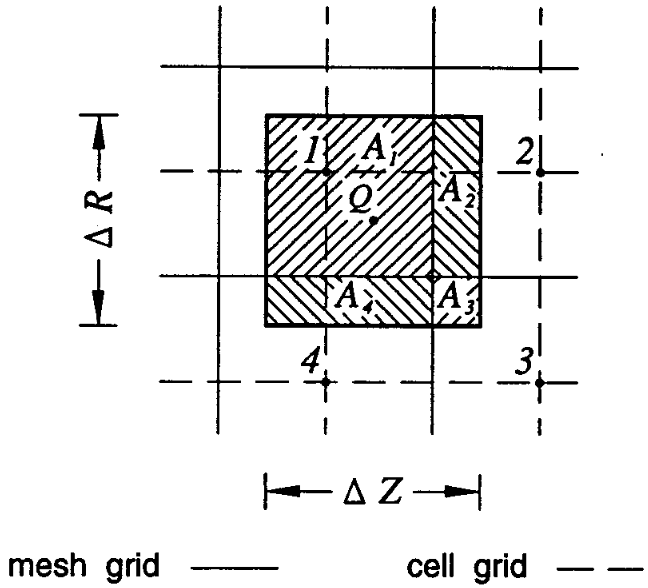


5) calculate field at each cell by using mesh charges and table, e.g.

$$\vec{E}(1), \vec{E}(2), \vec{E}(3), \vec{E}(4)$$

6) find fields at macroparticle position by weighting

$$(A_1\vec{E}(1) + A_2\vec{E}(2) + A_3\vec{E}(3) + A_4\vec{E})/A$$



7) Apply force to each macroparticle

8) Lorentz back-transform to the lab frame

Limitations

- Limitations of this method include:
  - Ignores interaction of charged beam with conducting walls of the chamber (wake fields; can be added ad hoc)
  - Fails if the beam has large energy spread
  - Most of the time removing excessive granularity from space charge is justified because the actual beam has many more particles than ‘virtual’ beam
  - However, the mesh method may lead to artificial ‘over-smoothing’ of the forces, underestimate intrabeam scat.
- There are powerful self-consistent Particle-In-Cell (PIC) codes. These require use of supercomputers.