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Electron Sources: Space Charge
• Space charge treatment
  – Debye length
  – Plasma frequency
  – Beam temperature
• Stationary distributions
• Emittance growth due to excess free energy
• Beam envelope equation
• Concept of equivalent beams
• Emittance compensation
• Computational aspects
In a typical bunched beam from a gun, both charge and current density are functions of transverse & longitudinal coordinates. This makes space charge dominated behavior highly nonlinear.

For beam envelope equation we will assume that \( \rho \) and \( J_z \) are independent of transverse coordinate and that the beam is not bunched (aspect ration \( << 1 \)).
“Collisional” vs. “smooth” self-fields

It would seem that the easiest approach is to calculate Lorentz force of all electrons directly (this would encompass practically all the behavior of the beam). This is not feasible because the number of evaluations for each time step is $\sim N^2$, with $N \sim 10^{10}$. Taking the fastest supercomputer with 100 TFLOPS, one estimates $\sim$ year(s) per time step (and one may need something like $\sim 10^4$ steps).

Instead, people use macroparticles in computer simulations with the same $e/m$ ratio

When $N \to \infty$ forces are smooth; when $N \to 1$ grainy collisional forces dominate. Envelope equation assumes the first scenario.

How to determine quantitatively “collisional” vs. “smooth” behavior of the space charge in the beam?
Three characteristic lengths in the bunch:

- $a$ bunch dimension; $l_p$ interparticle distance; $\lambda_D$ Debye length

Interactions due to Coulomb forces are long-range; Debye length is a measure of how ‘long’ (screen-off distance of a local perturbation in charge).

\[
\lambda_D \equiv \frac{\sigma_{v_x}}{\omega_p}
\]

for nonrelativistic case:

\[
\begin{align*}
\omega_p &= \sqrt{\frac{e^2 n}{\varepsilon_0 m}}, \\
\sigma_{v_x} &= \sqrt{\frac{k_B T}{m}} \\
\lambda_D &= \sqrt{\frac{\varepsilon_0 k_B T}{e^2 n}}
\end{align*}
\]

for relativistic case:

\[
\begin{align*}
\omega_p &= \sqrt{\frac{e^2 n}{\varepsilon_0 m\gamma^3}}, \\
\sigma_{v_x} &= \sqrt{\frac{k_B T}{m\gamma}} \\
\lambda_D &= \sqrt{\frac{\gamma^2 \varepsilon_0 k_B T}{e^2 n}}
\end{align*}
\]
Debye length: beam dynamics scenarios

\[ \lambda_D \gg a \quad \text{YES} \]

- single-particle behavior dominates; true when either energy or beam temperature is large (emittance-dominated)

\[ \lambda_D \ll a \quad \text{NO} \]

- collective forces are important

\[ \lambda_D \gg l_p \quad \text{YES} \]

- “smooth” force; Liouville’s theorem can be defined in 6-D phase space; if forces are linear, rms emittance is also conserved

\[ \epsilon_n \sim \frac{\sigma_x \sigma_{p_x}}{mc} = \frac{\gamma \sigma_x \sigma_{v_x}}{c} = \text{const} \]

\[ \gamma \sigma_x^2 k_B T = \text{const} \]

\[ \chi_{x} = \text{const} \]

\[ c = \text{const} \]

\[ \lambda_D = \text{const} \]

\[ \chi_{x} = \text{const} \]

\[ \text{iff} \]

fields of individual particles become important; one ends up having 6N-D phase space to deal with in the worst case; beam tends to develop ‘structure’
Similar to thermodynamics and plasma physics, there may exist equilibrium particle distributions (i.e. those that remain stationary). Vlasov theory allows one to find such distributions (assumes collisions are negligible, but they are the ones responsible to drive the distribution to the equilibrium!). Without the derivation, Vlasov-Maxwell equations for equilibrium distributions \( f(q_i, p_i, t) \) (i.e. no explicit time dependence, \( \partial / \partial t = 0 \)):

\[
\sum_{i=1}^{3} \left[ \frac{\partial f}{\partial q_i} \dot{q}_i + e(\vec{E} + \vec{v} \times \vec{B})_i \frac{\partial f}{\partial p_i} \right] = 0
\]

\[
\vec{\nabla} \times \vec{E} = 0 \quad \quad \vec{\nabla} \cdot \vec{E} = \frac{e}{\varepsilon_0} \int f \, d^3p
\]

\[
\vec{\nabla} \times \vec{B} = \mu_0 e \int \vec{v} f \, d^3p \quad \vec{\nabla} \cdot \vec{B} = 0
\]
In particular, in a constant focusing channel, equilibrium transverse density obeys a well-known Boltzmann relation

\[ n(r) = n(0) \exp \left[ -\frac{e\phi(r)}{k_B T_\perp} \right] \]

\[ \phi(r) = \phi_{\text{ext}}(r) + \frac{1}{\gamma^2} \phi_{\text{self}}(r) \]

\[ e\phi_{\text{ext}}(r) = \gamma m \omega_0^2 r^2 / 2 \]

\[ \phi_{\text{self}}(r) = -\int_0^r \int_0^r \frac{1}{\varepsilon_0 r} \hat{r} n(\hat{r}) \]
Analytically, two extreme cases:

\[ k_B T \to 0 \quad (\lambda_D / a \to 0) \quad n(r) = \begin{cases} 
  n_0 = \text{const, for } r \leq a \\ 
  0, \text{ for } r > a 
\end{cases} \quad \text{uniform} \]

\[ \varphi_{\text{self}} \to 0 \quad (\lambda_D / a \geq 1) \quad n(r) = n_0 \exp \left[ -\frac{\gamma m \omega_0^2 r^2}{2k_B T_{\perp}} \right] \quad \text{Gaussian} \]

<table>
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<th>Curve</th>
<th>( n(0)/n_0 )</th>
<th>( \lambda_D(0)/a_0 )</th>
<th>( K a^2/\epsilon^2 )</th>
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<tr>
<td>8</td>
<td>1</td>
<td>0</td>
<td>( \infty )</td>
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</table>
– **Stationary** (equilibrium) *beam distributions* (states) in a linear uniform focusing channel best described by a transverse Maxwell-Boltzmann distribution

– **Nonstationary beams** (*nonequilibrium*) such as *mismatched initial conditions* for the beam have a higher total energy per particle than the corresponding stationary beam (free energy)

– Collisional forces eventually will lead to a *new stationary state* at the higher energy per particle, result in *increased beam temperature* or emittance
Diffraction limited beam at 1 Å → $\varepsilon_x = 8$ pm at 5 GeV → $\varepsilon_{nx} = 0.08$ μm
10 MV/m gradient → $\sigma_{laser} = 0.3$ mm
Transverse temp. needed $kT = 25$ meV

Equilibrium $kT_{beam} \sim 100$ eV!!
near cathode

after the gun

relativistic energies

\[ B_\theta = \frac{\beta}{c} E_r \]

\( \perp \) force scales as \( 1 - \beta^2 = \gamma^2 \)
Let’s derive beam envelope equation (i.e. we assume that self-forces are smooth). We have almost derived the equation already (previous lecture’s paraxial ray equation). Two terms are missing – due to space charge and emittance ‘pressure’.

Uniform laminar beam in the absence of external forces:

\[
\gamma^2 \dot{r} = \frac{eI}{2\pi \varepsilon_0 a^2 \beta c} \frac{1}{\gamma^2}, \text{ using } \ddot{r} = \beta^2 \gamma^2 r'' \rightarrow r'' = \frac{eI}{2\pi \varepsilon_0 a^2 mc^3 \beta^3 \gamma^3}
\]

\[
\dot{r} = \frac{\omega_p^2}{2} r
\]

\[
r'' = \frac{K}{a^2} r
\]

\[
r_m r_m' = K \quad \text{for } r_m = a
\]

\[
\omega_p^2 = \frac{eI}{\pi \varepsilon_0 mc \beta^3 \gamma^3 a^2}
\]

\[
K = \frac{I}{I_0} \frac{2}{\beta^3 \gamma^3}
\]

\[
I_0 = \frac{4\pi \varepsilon_0 mc^3}{e} \approx \frac{1}{30} \frac{mc^2}{e} = 17 \text{ kA}
\]
Nonrelativistic limit: \[ K = \frac{I}{I_0} \frac{2}{\beta^3} = \frac{eI}{2\pi \varepsilon_0 mv^3}, \text{ with } v = (2eV/m)^{1/2} \]

\[ K = \frac{I}{V^{3/2}} \left[ \frac{1}{4\pi \varepsilon_0 (2e/m)^{1/2}} \right] \]
Emittance ‘pressure’ term

\[ \Sigma = \langle \mathbf{x}^T \mathbf{x} \rangle = \begin{bmatrix} \langle xx \rangle & \langle xx' \rangle \\ \langle x'x \rangle & \langle x'x' \rangle \end{bmatrix} = \varepsilon \begin{bmatrix} B & -A \\ -A & \Gamma \end{bmatrix} \]

In a drift \(0 \rightarrow z\):

- \(x' \rightarrow x' = \text{const}\)
- \(x \rightarrow x + x'z\)

For \(\sigma_x\):

\[ \sigma_x' = -\frac{\sqrt{\varepsilon} A}{\sqrt{B}}, \quad \sigma_x'' = \frac{\sqrt{\varepsilon}}{B \sqrt{B}} \quad \text{or} \quad \sigma_x'' - \frac{\varepsilon^2}{\sigma_x^3} = 0 \]

\(B \rightarrow B - 2Az + \Gamma z^2\)

\(A \rightarrow A - \Gamma z\)

\(\Gamma \rightarrow \Gamma = \text{const}\)
From paraxial ray equation with the additional terms, one obtains

$$\sigma'' + \sigma' \frac{\gamma'}{\beta^2 \gamma} + \sigma \frac{1}{\beta^2 \gamma^2} \left[ \gamma'' \gamma + \left( \frac{eB}{2mc} \right)^2 \right] - \frac{1}{\sigma} \frac{I}{2I_0 \beta^3 \gamma^3} - \frac{1}{\sigma^3} \frac{1}{\beta^2 \gamma^2} \left[ \left( \frac{P_\theta}{mc} \right)^2 + \varepsilon_n^2 \right] = 0$$

adiabatic
RF focusing
solenoid
space charge
angular momentum
‘increases’ emittance

$$\frac{I}{2I_0 \beta \gamma} \gg \frac{\varepsilon_n^2}{\sigma^2}, \text{ or } \frac{I}{2I_0 \beta^2 \gamma^2} \gg \frac{\varepsilon_n}{\beta}$$

space charge dominated

$$\frac{I}{2I_0 \beta \gamma} \ll \frac{\varepsilon_n^2}{\sigma^2}, \text{ or } \frac{I}{2I_0 \beta^2 \gamma^2} \ll \frac{\varepsilon_n}{\beta}$$

emittance dominated
Example of beam dynamics: 80 pC charge

\( \Delta \phi_x \) (keV/c)

\( \Delta x \) (mm)

\( \Delta z \) (mm)

\( \sigma_x = 0.000 \) mm

\( \sigma_y = 0.000 \) mm

\( \sigma_z = 0.254 \) mm

\( \pi_y = 0.000 \) mm-keV

\( \pi_x = 0.077 \) mm-mrad

\( \varepsilon = 0.000 \) m
– The *actual beams have* complicated transverse profiles and phase space distribution

– How useful is beam envelope equation derived for *uniform cylindrical beams*?

– **Concept of Equivalent Beams** due to Lapostolle and Sachere (1971): “two beams having the same current and kinetic energy are *equivalent* in an approximate sense if the second moments of the distribution are the same.”

– E.g. rms sizes and emittances are *the same* for the two beams when compared at identical positions
We have seen that beam will evolve from space-charge dominated to emittance dominated regimes as it is being accelerated. At low energies, various longitudinal ‘slices’ of the bunch experience different forces due to varying current → ‘bow-tie’ phase space is common.

Important realization is that much of these space charge emittance growth may be reversible through appropriate focusing (and drifts), a so-called emittance compensation. Obviously, this emittance compensation should take place before (or rather as) the beam becomes ultrarelativistic (and emittance-dominated).
Kim analyzed emittance growth due to space charge and RF (NIM A \textbf{275} (1989) 201-218). His analysis applies to beam in RF guns. He has found:

\[
\varepsilon_x^{rf} = \frac{\alpha k_{RF}^3 \sigma_x^2 \sigma_z^2}{\sqrt{2}}
\]

\[
\varepsilon_x^{sc} = \frac{\pi}{4} \frac{1}{\alpha k_{RF}} \frac{1}{\sin \phi_0} \frac{I_{peak}}{I_0} \mu_x(A)
\]

\[
\alpha = \frac{eE_0}{2mc^2k_{RF}}
\]

\[
\mu_{x,\text{gauss}} \approx \frac{1}{3A + 5}
\]

e.g. 100 MV/m RF gun ($\lambda = 10.5$ cm): $\alpha = 1.64$, phase (to minimize rf emittance) $\phi_0 = 71^\circ$ ($\phi \rightarrow 90^\circ$), laser width and length $\sigma_x = 3.5$ mm, $\sigma_z = 0.6$ mm

\[
\varepsilon_x^{rf} = 1.1$ mm - mrad
\]

\[
\varepsilon_x^{sc} = 4.0$ mm - mrad (1.3 mm - mrad for uniform)
Carlsten noted in simulations that emittance can be brought down and gave a simple explanation for the effect (NIM A 285 (1989) 313-319).

Fig. 2. Typical transverse emittance versus beamline position plot for a photoelectric injector, showing quick initial growth and subsequent reduction for a slug beam and physical description of a slug beam, with internal coordinates $\rho$ and $\xi$.

5. Conclusion

A photoelectric injector design analysis has been presented. The emittance growth from the dominant mechanism has been shown to be eliminated with a simple lens configuration, leaving only a small residual emittance resulting from the other mechanisms.

Fig. 3. Transverse phase-space plots showing emittance growth and reduction. (a) Initial phase-space plot with very small emittance. (b) Phase space plot after drift $z_1$ to lens, showing the emittance growth due to the different expansion rates of points A and B. (c) Phase-space plot immediately after lens, showing rotation due to the lens. The emittance is unchanged because we assume the lens is linear. (d) Phase space plot after drift $z$ behind lens, showing the emittance reduction due to the different expansion rates of points A and B.

\[
\sigma'' + \sigma' \frac{\gamma'}{\beta^2 \gamma} + \sigma K_r - \frac{I(\zeta)}{\sigma_0 I_0 \beta^3 \gamma^3} - \frac{\varepsilon_n^2}{\sigma^3 \beta^2 \gamma^2} = 0
\]

includes solenoid and RF \(\zeta\) tags long. slice in the bunch

For space charge dominated case in absence of acceleration

\[
\sigma'' + \sigma' \frac{\gamma'}{\beta^2 \gamma} + \sigma K_r - \frac{I(\zeta)}{2 \sigma_0 I_0 \beta^3 \gamma^3} - \frac{\varepsilon_n^2}{\sigma^3 \beta^2 \gamma^2} = 0
\]

Brillouin flow \(\sigma'' \to 0\): \(\sigma_{eq} = \sqrt{\frac{I(\zeta)}{2 K_r I_0 \beta^3 \gamma^3}}\)
Small oscillations near equilibrium: 
\[ \delta \sigma'' + \delta \sigma \left[ 2K_r - \frac{\delta \sigma}{\sigma_{eq}} + \left( \frac{\delta \sigma}{\sigma_{eq}} \right)^2 - \ldots \right] = 0 \]

Important: frequency of small oscillations around equilibrium does not depend on \( \zeta \). E.g. for beam with \( \sigma'(0,\zeta) = 0 \) and \( \sigma(0,\zeta) = \sigma_{eq}(\zeta) + \delta \sigma(\zeta) = \sigma_0 \): 

\[
\begin{align*}
\sigma(z, \zeta) &= \sigma_{eq}(\zeta) + \delta \sigma(\zeta) \cos \left( \sqrt{2K_r} z \right) \\
\sigma'(z, \zeta) &= -\sqrt{2K_r} \delta \sigma(\zeta) \sin \left( \sqrt{2K_r} z \right)
\end{align*}
\]

\[
\varepsilon = \frac{1}{2} \sqrt{\left\langle r^2 \right\rangle \left\langle r'^2 \right\rangle - \left\langle r' r \right\rangle^2} \approx \sqrt{2K_r \left\langle \sigma_{eq} \right\rangle \sigma_0} \left| \sin\left( \sqrt{2K_r} z \right) \right|
\]
• slices oscillate in phase space around different equilibria but with the same frequency
• ‘projected’ emittance reversible oscillations when $\delta\sigma/\sigma_{eq} \ll 1$, unharmonicity shows up when $\delta\sigma$ is not small
• ignores the fact that beam aspect ratio can be $\gg 1$ (e.g. at the cathode)
Including acceleration term and transforming from \((\sigma, z) \rightarrow (\tau, y)\) in the limit \(\gamma >> 1\)

\[
\frac{d^2 \tau}{dy^2} + \Omega^2 \tau = \frac{e^{-y}}{\tau}
\]

\(y \equiv \ln \frac{\gamma}{\gamma_0}, \quad \tau \equiv \sigma \gamma' \sqrt{\gamma_0 / (I(\zeta) / 2I_0)}\), \(\Omega\) represents solenoid & RF focusing

Particular solution that represents generalized Brillouin flow or ‘invariant envelope’:

\[
\tau_{eq} = \frac{2e^{-y/2}}{\sqrt{1 + 4\Omega^2}}, \quad \sigma_{eq} = \frac{2}{\gamma'} \sqrt{\frac{I(\zeta)}{\gamma 2I_0}} \frac{1}{1 + 4\Omega^2}
\]

\[
\frac{\gamma \sigma'_{eq}}{\sigma_{eq}} = -\frac{\gamma'}{2}
\]

phase space angle is independent of slice \(\zeta\)

Matching beam to ‘invariant envelope’ can lead to ‘damping’ of projected rms emittance.
Using the beam envelope equation for individual slices leads to a recipe for emittance compensation, which works for simple cases (e.g. matching beam into long focusing channel / linac in the injector). For other more complicated scenarios one should solve the equations numerically (example: code HOMDYN).

Finally, particle tracking is indispensable for analysis and design of the injector where the assumptions made are invalid or theory is too complicated to be useful.
• Majority of injector design codes use the strategy:
  – Assume nearly monochromatic beam
  – Solve Poisson equation in the rest frame
    • various meshing strategies (simple uniform mesh for fast FFT methods, nonequidistant adaptive mesh for distributions with varying density)
    • Essentially removes granularity of the force
  – Lorentz back-transform and apply forces including 3D field maps of external elements (cavities, magnets, etc.)
Different approaches are used (e.g. envelope equation integration, macroparticle tracking, various meshing scenarios, etc.). Mesh method works as following:

1) transform to rest frame of the reference particle
2) create mesh (charge) and cell grid (electrostatic fields)
3) create table containing values of electrostatic field at any cell due to a unit charge at any mesh vertex (does not need to be recalculated each time step)
4) assign macroparticle charges to mesh nodes, e.g. 1,2,3, and 4 vertices get QA$_1$/A, QA$_2$/A, QA$_3$/A, and QA$_4$/A respectively, where A$_1$+A$_2$+A$_3$+A$_4$ = A = $\Delta Z \Delta R$

5) calculate field at each cell by using mesh charges and table, e.g. $\vec{E}(1)$, $\vec{E}(2)$, $\vec{E}(3)$, $\vec{E}(4)$

6) find fields at macroparticle position by weighting $(A_1 \vec{E}(1) + A_2 \vec{E}(2) + A_3 \vec{E}(3) + A_4 \vec{E}) / A$

7) Apply force to each macroparticle

8) Lorentz back-transform to the lab frame
• Limitations of this method include:
  – Ignores interaction of charged beam with conducting walls of the chamber (wake fields; can be added ad hoc)
  – Fails if the beam has large energy spread
  – Most of the time removing excessive granularity from space charge is justified because the actual beam has many more particles than ‘virtual’ beam
  – However, the mesh method may lead to artificial ‘over-smoothing’ of the forces, underestimate intrabeam scat.

• There are powerful self-consistent Particle-In-Cell (PIC) codes. These require use of supercomputers.