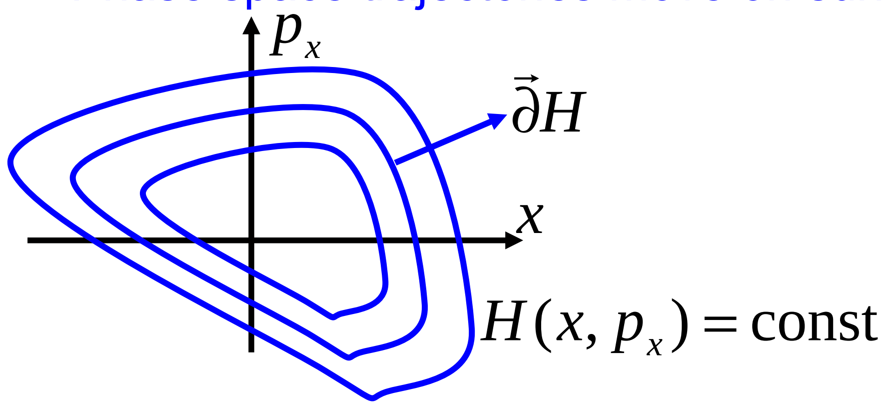
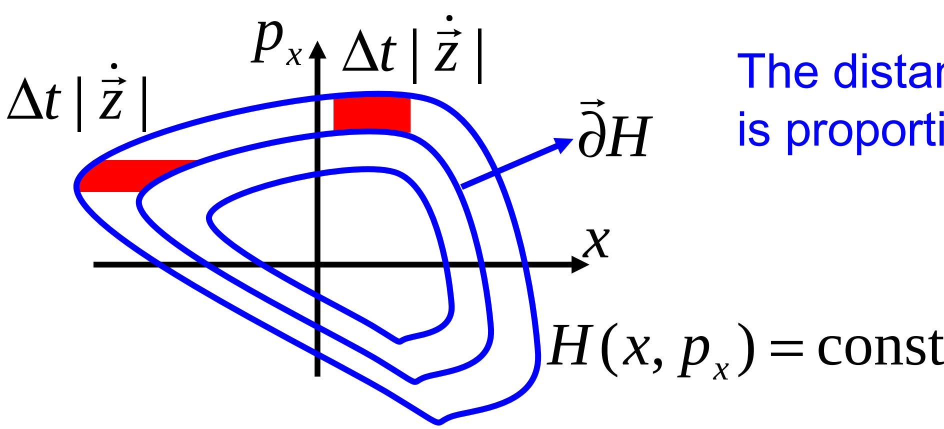
Phase space density in 2D

- Phase space trajectories move on surfaces of constant energy



$$\tfrac{d}{ds}\vec{z} = \underline{J}\,\vec{\partial} H \quad \Rightarrow \quad \tfrac{d}{ds}\vec{z} \perp \vec{\partial} H$$

- A phase space volume does not change when it is transported by Hamiltonian motion.

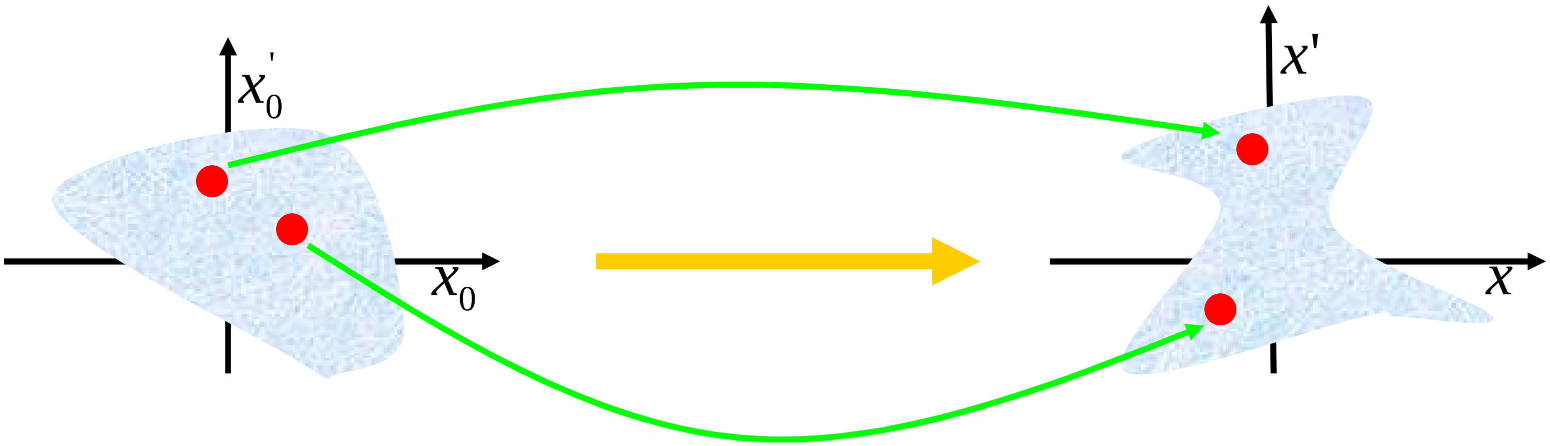


The distance d of lines with equal energy is proportional to $1/|\vec{\partial} H| \propto |\dot{\vec{z}}\,|^{-1}$

$$d * \Delta t\,|\dot{\vec{z}}\,| = \text{const}$$

Lioville's Theorem

- A phase space volume does not change when it is transported by Hamiltonian motion. $\vec{z}(s) = \underline{M}(s) \cdot \vec{z}_0 \quad \text{with} \quad \det[\underline{M}(s)] = +1$



Volume = $V = \iint_V d^n\vec{z} = \iint_{V_0} \left|\frac{\partial \vec{z}}{\partial \vec{z}_0}\right| d^n\vec{z}_0 = \iint_{V_0} \left|\underline{M}\right| d^n\vec{z}_0 = \iint_{V_0} d^n\vec{z}_0 = V_0$

Hamiltonian Motion $\rightarrow$ $V = V_0$

But Hamiltonian requires symplecticity, which is much more than just $\det[\underline{M}(s)] = +1$

Significance of Hamiltonian

The equations of motion can be determined by one function:

$$\tfrac{d}{ds} x = \partial_{p_x} H(\vec{z},s)\,, \quad \tfrac{d}{ds} p_x = -\partial_x H(\vec{z},s)\,, \quad \ldots$$

$$\tfrac{d}{ds}\vec{z} = \underline{J}\,\vec{\partial} H(\vec{z},s) = \vec{F}(\vec{z},s) \quad \text{with} \quad \underline{J} = \mathrm{diag}(\underline{J}_2)\,, \quad \underline{J}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The force has a Hamiltonian Jacobi Matrix:

A linear force: $\vec{F}(\vec{z},s) = \underline{F}(s)\cdot\vec{z}$

The Jacobi Matrix of a linear force: $\underline{F}(s)$

The general Jacobi Matrix : $F_{ij} = \partial_{z_j} F_i$ or $\underline{F} = \left(\vec{\partial}\,\vec{F}^T\right)^T$

Hamiltonian Matrices: $\underline{F}\,\underline{J} + \underline{J}\,\underline{F}^T = 0$

Prove : $F_{ij} = \partial_{z_j} F_i = \partial_{z_j} J_{ik} \partial_{z_k} H = J_{ik}\,\partial_k \partial_j H \quad \Rightarrow \quad \underline{F} = \underline{J}\,\underline{DH}$

$$\underline{F}\,\underline{J} + \underline{J}\,\underline{F}^T = \underline{J}\,\underline{D}\,H\,\underline{J} + \underline{J}\,\underline{D}^T H\,\underline{J}^T = 0$$

H → Symplectic Flows

The flow of a Hamiltonian equation of motion has a symplectic Jacobi Matrix

The flow or transport map: $\vec{z}(s) = \vec{M}(s, \vec{z}_0)$

A linear flow: $\vec{z}(s) = \underline{M}(s) \cdot \vec{z}_0$

The Jacobi Matrix of a linear flow: $\underline{M}(s)$

The general Jacobi Matrix : $M_{ij} = \partial_{z_{0j}} M_i$ or $\underline{M} = \left(\vec{\partial}_0 \vec{M}^T\right)^T$

The Symplectic Group SP(2N) : $\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J}$

$$\tfrac{d}{ds}\vec{z} = \tfrac{d}{ds}\vec{M}(s, \vec{z}_0) = \underline{J}\,\vec{\nabla}H = \vec{F}$$

$$\tfrac{d}{ds} M_{ij} = \partial_{z_{0j}} F_i(\vec{z}, s) = \partial_{z_{0j}} M_k \partial_{z_k} F_i(\vec{z}, s)$$

$$\tfrac{d}{ds}\underline{M}(s, \vec{z}_0) = \underline{F}(\vec{z}, s)\underline{M}(s, \vec{z}_0)$$

$$K = \underline{M}\,\underline{J}\,\underline{M}^T$$

$$\tfrac{d}{ds}\underline{K} = \tfrac{d}{ds}\underline{M}\,\underline{J}\,\underline{M}^T + \underline{M}\,\underline{J}\,\tfrac{d}{ds}\underline{M}^T = \underline{F}\,\underline{M}\,\underline{J}\,\underline{M}^T + \underline{M}\,\underline{J}\,\underline{M}^T\underline{F}^T = \underline{F}\,\underline{K} + \underline{K}\,\underline{F}^T$$

$\underline{K} = \underline{J}$ is a solution. Since this is a linear ODE , $\underline{K} = \underline{J}$ is the unique solution.

Symplectic Flows → H

For every symplectic transport map there is a Hamilton function

The flow or transport map: $\vec{z}(s) = \vec{M}(s, \vec{z}_0)$

Force vector: $\vec{h}(\vec{z}, s) = -\underline{J}\left[\frac{d}{ds}\vec{M}(s, \vec{z}_0)\right]_{\vec{z}_0 = \vec{M}^{-1}(\vec{z}, s)}$

Since then: $\frac{d}{ds}\vec{z} = \underline{J}\vec{h}(\vec{z}, s)$

There is a Hamilton function H with: $\vec{h} = \vec{\partial} H$

If and only if: $\partial_{z_j} h_i = \partial_{z_i} h_j \quad \Rightarrow \quad \underline{h} = \underline{h}^T$

$$\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J} \quad \Rightarrow \quad \begin{cases} \frac{d}{ds}\underline{M}\,\underline{J}\,\underline{M}^T = -\underline{M}\,\underline{J}\,\frac{d}{ds}\underline{M}^T \\ \underline{M}^{-1} = -\underline{J}\,\underline{M}^T\,\underline{J} \end{cases}$$

$$\vec{h} \circ \vec{M} = -\underline{J}\,\frac{d}{ds}\vec{M}$$

$$\underline{h}(\vec{M})\underline{M} = -\underline{J}\,\frac{d}{ds}\underline{M}$$

$$\underline{h}(\vec{M}) = -\underline{J}\,\frac{d}{ds}\underline{M}\,\underline{M}^{-1} = \underline{J}\,\frac{d}{ds}\underline{M}\,\underline{J}\,\underline{M}^T\,\underline{J} = -\underline{J}\,\underline{M}\,\underline{J}\,\frac{d}{ds}\underline{M}^T\,\underline{J} = \underline{M}^{-T}\,\frac{d}{ds}\underline{M}^T\,\underline{J} = \underline{h}^T$$

Symplectic Representations

Hamiltonian

$$\vec{z}' = \underline{J}\,\vec{\partial} H(\vec{z}, s)$$

ODE

$$\vec{z}' = \vec{F}\,, \quad \underline{F}\,\underline{J} + \underline{J}\,\underline{F}^T = 0$$

Symplectic transport map

$$\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J}$$