Eigenvalues of a Symplectic Matrix

For matrices with real coefficients:

If there is an eigenvector and eigenvalue: $\underline{M}\vec{v}_i = \lambda_i \vec{v}_i$

then the complex conjugates are also eigenvector and eigenvalue: $\underline{M}\vec{v}_i^{\,*} = \lambda_i^* \vec{v}_i^{\,*}$

For symplectic matrices:

If there are eigenvectors and eigenvalues: $\underline{M}\vec{v}_i = \lambda_i \vec{v}_i$ with $\underline{J} = \underline{M}^T \underline{J}\,\underline{M}$

then $\vec{v}_i^{\,T} \underline{J}\vec{v}_j = \vec{v}_i^{\,T} \underline{M}^T \underline{J}\,\underline{M}\vec{v}_j = \lambda_i \lambda_j \vec{v}_i^{\,T} \underline{J}\vec{v}_j \quad \Rightarrow \quad \vec{v}_i^{\,T} \underline{J}\vec{v}_j (\lambda_i \lambda_j - 1) = 0$

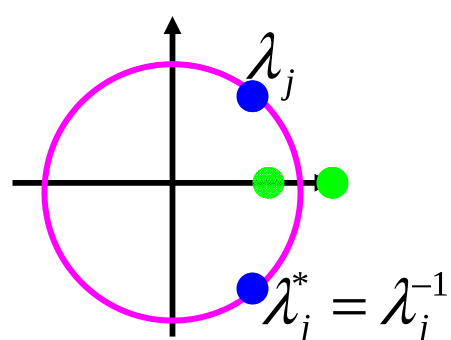
Therefore $\underline{J}\vec{v}_j$ is orthogonal to all eigenvectors with eigenvalues that are not $1/\lambda_j$. Since it cannot be orthogonal to all eigenvectors, there is at least one eigenvector with eigenvalue $1/\lambda_j$

Two dimensions: λ_j is eigenvalue

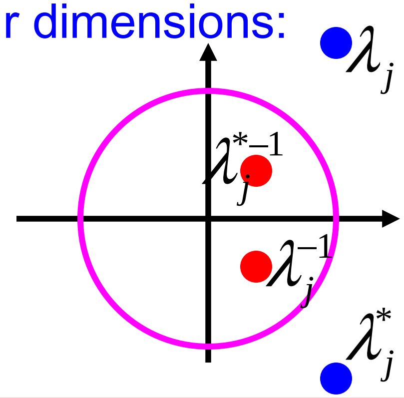
Then $1/\lambda_j$ and λ_j^* are eigenvalues

$\underline{\lambda_2 = 1/\lambda_1 = \lambda_1^*} \quad \Rightarrow \quad |\lambda_j| = 1$

$\underline{\lambda_2 = 1/\lambda_1 = \lambda_2^*}$



Four dimensions:



Variation of Constants

$$\vec{z}\,' = \vec{f}(\vec{z}, s)$$

$$\vec{z}\,' = \underline{L}(s)\vec{z} + \Delta\vec{f}(\vec{z}, s)$$

Field errors, nonlinear fields, etc can lead to $\Delta\vec{f}(\vec{z}, s)$

$$\vec{z}_H' = \underline{L}(s)\vec{z}_H \quad \Rightarrow \quad \vec{z}_H(s) = \underline{M}(s)\vec{z}_{H0} \quad \text{with} \quad \underline{M}'(s)\vec{a} = \underline{L}(s)\underline{M}(s)\vec{a}$$

$$\vec{z}(s) = \underline{M}(s)\vec{a}(s) \quad \Rightarrow \quad \vec{z}\,'(s) = \underline{M}'(s)\vec{a} + \underline{M}(s)\vec{a}\,'(s) = \underline{L}(s)\vec{z} + \Delta\vec{f}(\vec{z}, s)$$

$$\vec{a}(s) = \vec{z}_0 + \int_0^s \underline{M}^{-1}(\hat{s})\Delta\vec{f}(\vec{z}(\hat{s}), \hat{s})\,d\hat{s}$$

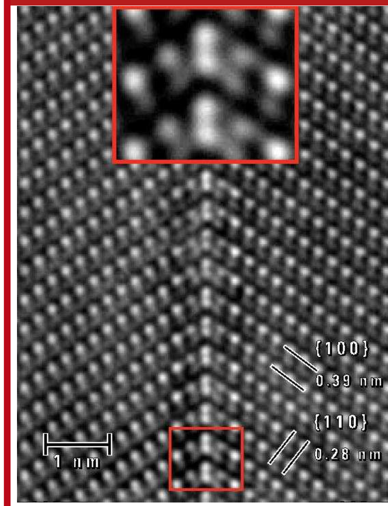
$$\vec{z}(s) = \underline{M}(s)\left\{\vec{z}_0 + \int_0^s \underline{M}^{-1}(\hat{s})\Delta\vec{f}(\vec{z}(\hat{s}), \hat{s})\,d\hat{s}\right\}$$

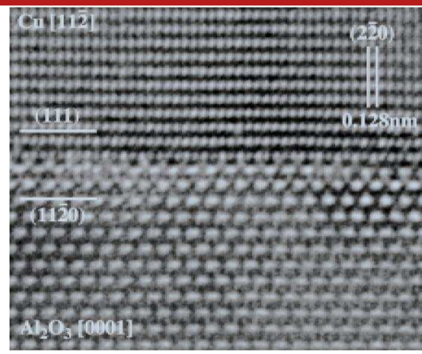
$$= \vec{z}_H(s) + \int_0^s \underline{M}(s-\hat{s})\Delta\vec{f}(\vec{z}(\hat{s}), \hat{s})\,d\hat{s}$$

Perturbations are propagated from s to s'

Aberration Correction

CHESS & LEPP





$$w_2(s) = w_H(s) + C(s) w_0^2 \overline{w}_0 + \ldots$$

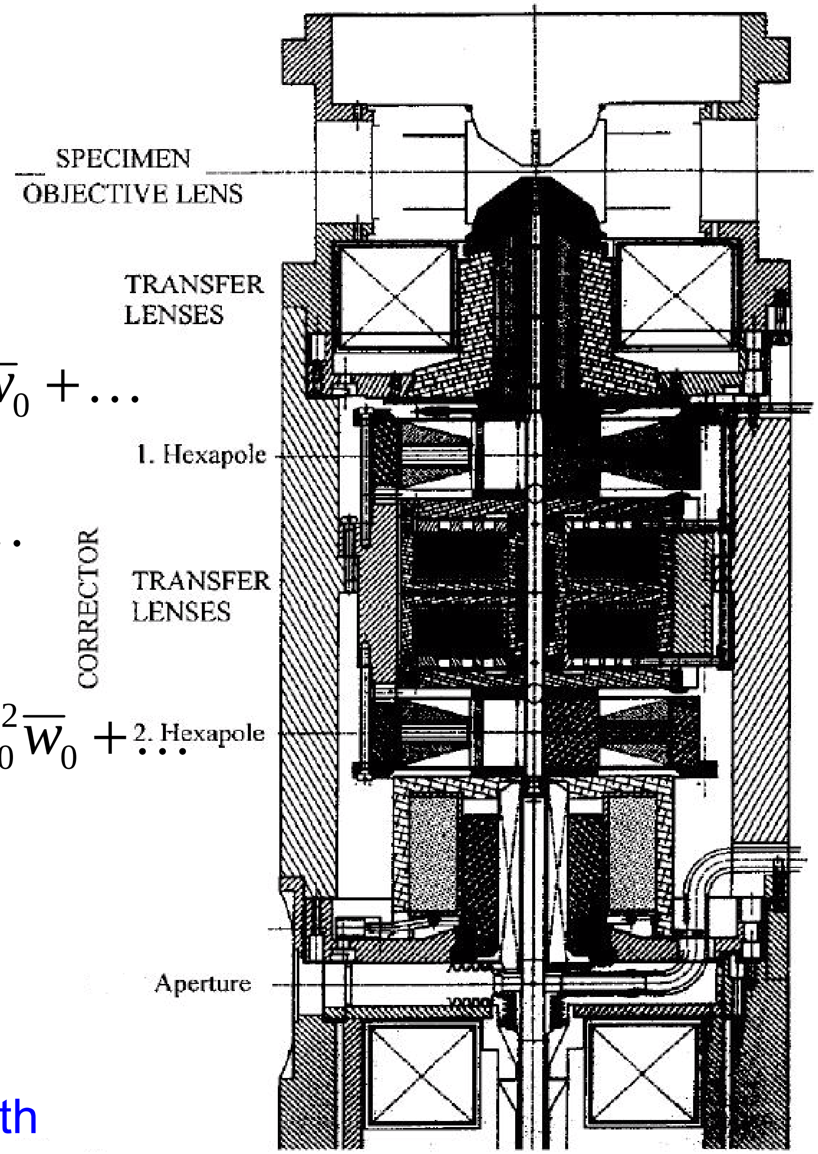
$$w_2(s) = w_H(s) + A(s) \overline{w}_0^2 + B(s) w_0^2 \overline{w}_0 + \ldots$$

$$w_2(s) = w_H(s) + \cancel{A(s) \overline{w}_0^2} + 2B(s) w_0^2 \overline{w}_0 + \ldots$$

2B cancels C !

Quadratic in
sextupole strength

Linear in
solenoid strength



Orbit distortions

$$x' = a$$

$$a' = -(\kappa^2 + k)x + \Delta f$$

The extra force can for example come from an erroneous dipole field or from a correction coil: $\Delta f = \frac{q}{p}\Delta B_y = \Delta\kappa$

Variation of constants: $\vec{z} = \underline{M}\vec{z}_0 + \Delta\vec{z} \text{ with } \Delta\vec{z} = \int_0^s \underline{M}(s-\hat{s})\begin{pmatrix} 0 \\ \Delta\kappa(\hat{s}) \end{pmatrix} d\hat{s}$

$$\Delta\vec{z} = \int_0^L \begin{pmatrix} -\sqrt{\beta\hat{\beta}}\sin(\psi-\hat{\psi}) \\ \sqrt{\frac{\hat{\beta}}{\beta}}[\cos(\psi-\hat{\psi}) + \alpha\sin(\psi-\hat{\psi})] \end{pmatrix} \Delta\kappa(\hat{s})d\hat{s}$$

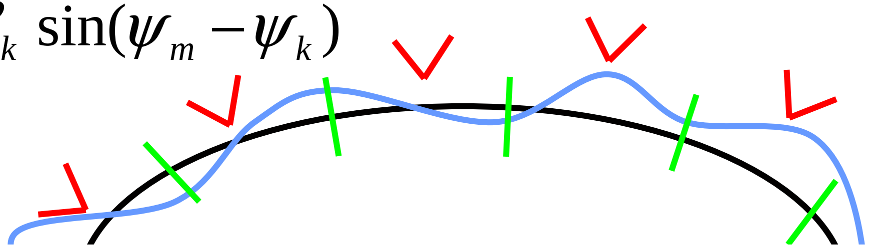
$$\boxed{\Delta x(s) = \sum_k \Delta\vartheta_k \sqrt{\beta(s)\beta_k}\sin(\psi(s)-\psi_k)}$$

Orbit Correction

When the closed orbit $x_{\text{co}}^{\text{old}}(s_m)$ is measured at beam position monitors (BPMs, index m) and is influenced by corrector magnets (index k), then the monitor readings before and after changing the kick angles created in the correctors by $\Delta\vartheta_k$ are related by

$$x_{\text{co}}^{\text{new}}(s_m) = x_{\text{co}}^{\text{old}}(s_m) + \sum_k \Delta\vartheta_k \sqrt{\beta_m \beta_k}\,\sin(\psi_m - \psi_k)$$

$$= x_{\text{co}}^{\text{old}}(s_m) + \sum_k O_{mk}\Delta\vartheta_k$$



$$\vec{x}_{\text{co}}^{\text{new}} = \vec{x}_{\text{co}}^{\text{old}} + \underline{O}\Delta\vec{\vartheta}$$

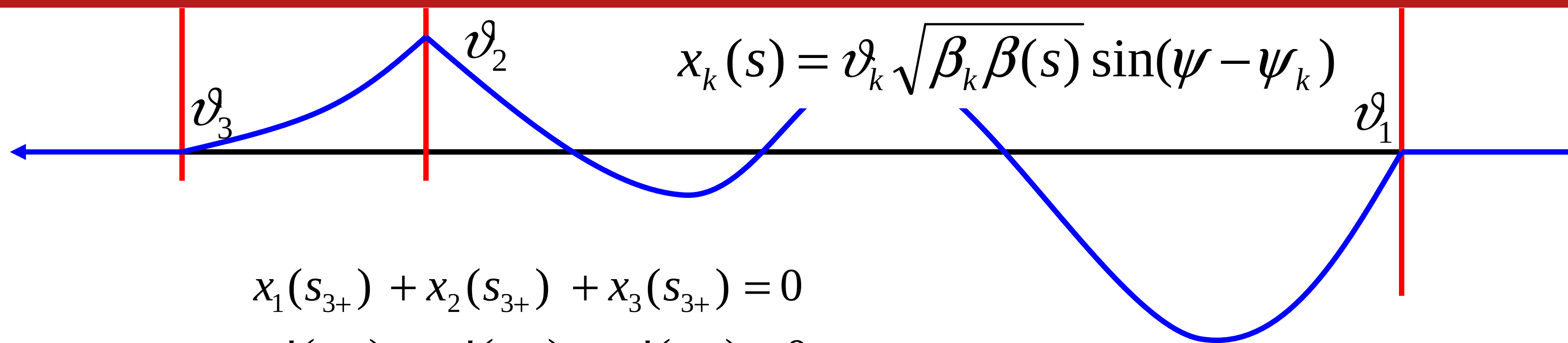
$$\boxed{\Delta\vec{\vartheta} = -\underline{O}^{-1}\vec{x}_{\text{co}}^{\text{old}} \quad \Rightarrow \quad \vec{x}_{\text{co}}^{\text{new}} = 0}$$

It is often better not to try to correct the

closed orbit at the the BPMs to zero in this way since

1. computation of the inverse can be numerically unstable, so that small errors in the old closed orbit measurement lead to a large error in the corrector coil settings.
2. A zero orbit at all BPMs can be a bad orbit inbetween BPMs

Closed Orbit Bumps



$$x_k(s) = \vartheta_k \sqrt{\beta_k \beta(s)} \sin(\psi - \psi_k)$$

$$x_1(s_{3+}) + x_2(s_{3+}) + x_3(s_{3+}) = 0$$

$$x_1'(s_{3+}) + x_2'(s_{3+}) + x_3'(s_{3+}) = 0$$

$$\vartheta_1 \sqrt{\beta_1} \sin(\psi_3 - \psi_1) + \vartheta_2 \sqrt{\beta_2} \sin(\psi_3 - \psi_2) = 0$$

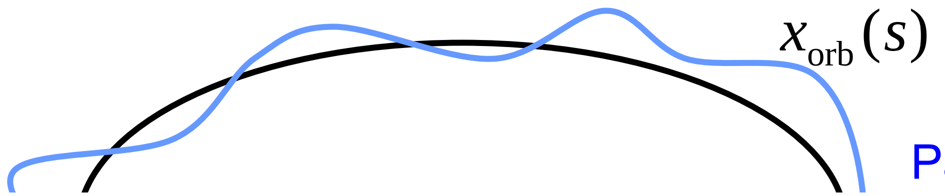
$$\vartheta_1 \sqrt{\beta_1} \cos(\psi_3 - \psi_1) + \vartheta_2 \sqrt{\beta_2} \cos(\psi_3 - \psi_2) + \vartheta_3 \sqrt{\beta_3} = 0$$

$$\vartheta_1 \sqrt{\beta_1} \sin(\psi_2 - \psi_1) = \vartheta_3 \sqrt{\beta_3} \sin(\psi_3 - \psi_2)$$

$$-\vartheta_2 \sqrt{\beta_2} \sin(\psi_2 - \psi_1) = \vartheta_3 \sqrt{\beta_3} \sin(\psi_3 - \psi_1)$$

$$\boxed{\vartheta_1 : \vartheta_2 : \vartheta_3 = \beta_1^{-\frac{1}{2}} \sin\psi_{32} : -\beta_2^{-\frac{1}{2}} \sin\psi_{31} : \beta_3^{-\frac{1}{2}} \sin\psi_{21}}$$

Oscillations around a distorted Orbit



Particles oscillate around this periodic orbit, not around the design orbit.

$$\vec{z} = \vec{z}_\beta + \vec{z}_{\text{orb}}$$

$$\vec{z}_{\text{orb}}(s) = \underline{M}\,\vec{z}_{\text{orb}}(0) + \Delta\vec{z}(s)$$

$$\vec{z}_\beta(s) + \vec{z}_{\text{orb}}(s) = \vec{z}(s) = \underline{M}\,\vec{z}(0) + \Delta\vec{z}(s) = \underline{M}[\vec{z}_\beta(0) + \vec{z}_{\text{orb}}(0)] + \Delta\vec{z}(s)$$

$$= \underline{M}\,\vec{z}_\beta(0) + \vec{z}_{\text{orb}}(s)$$

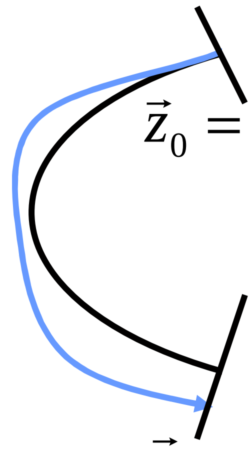
$$\boxed{\vec{z}_\beta(L) = \underline{M}_0 \vec{z}_\beta(0)}$$

The distorted orbit does not change the linear transport matrix.

Dispersion Integral

$$x' = a$$

$$a' = -(\kappa^2 + k)x + \kappa\delta$$



$$\vec{z} = \underline{M}\,\vec{z}_0 + \int_0^s \underline{M}(s-\hat{s})\begin{pmatrix} 0 \\ \delta\kappa(\hat{s}) \end{pmatrix} d\hat{s}$$

$$\Rightarrow \quad \vec{D}(s) = \int_0^s \underline{M}(s-\hat{s})\begin{pmatrix} 0 \\ \kappa(\hat{s}) \end{pmatrix} ds'$$

$$\Delta\kappa = \delta\kappa$$

$$\boxed{D(s) = \sqrt{\beta(s)}\int_0^s \kappa(\hat{s})\sqrt{\beta(\hat{s})}\,\sin(\psi(s) - \psi(\hat{s}))\,d\hat{s}}$$