First-Order Dispersion

\[ x'' + (k_1 + \kappa^2) x = 0 \]

First order in \( x, x' \)

\[ x'' + K \cdot x = f_1 \quad \Rightarrow \quad f_1(\delta) = \kappa \delta \]

First order in \( x, x', \delta \)

\[ x'' + K \cdot x = \kappa \delta + f_2(x, x', \delta) \]

Second order in \( x, x', \delta \)

\[ x_\delta(\delta) = D \delta + D_2 \delta^2 + O^3(\delta) \]

Trajectory of a particle that starts as designed, but has relative energy deviation \( \delta \).

\[ (D'' + K \cdot D) \delta + (D_2'' + K \cdot D_2) \delta^2 = \kappa \delta + f_2(D, D', 1) \delta^2 \]

First-order dispersion:

\[ D'' + K \cdot D = \kappa \quad \Rightarrow \quad D = \int_0^s \hat{k} \sqrt{\beta \beta} \sin(\psi - \hat{\psi}) d\hat{s} \]
Second-Order Dispersion

\[ x'' + (k_1 + \kappa^2)x = 0 \]

First order in \( x, x' \)

\[ x'' + K \cdot x = f_1 \quad f_1(\delta) = \kappa \delta \]

First order in \( x, x', \delta \)

\[ x'' + K \cdot x = \kappa \delta + f_2(x, x', \delta) \]

Second order in \( x, x', \delta \)

\[ x_\delta(\delta) = D \delta + D_2 \delta^2 + O^3(\delta) \]

Trajectory of a particle that starts as designed, but has relative energy deviation \( \delta \).

\[ (D'' + K \cdot D) \delta + (D_2'' + K \cdot D_2) \delta^2 = \kappa \delta + f_2(D, D', 1) \delta^2 \]

First-order dispersion:

\[ D_2'' + K \cdot D_2 = f_2(D, D', 1) \quad \Rightarrow \quad D_2 = \int_0^s f_2(\hat{D}, \hat{D}', 1) \sqrt{\beta \beta} \sin(\psi - \hat{\psi}) d\hat{s} \]
Second-Order Achromats

\[ x'' + K \cdot x = \kappa \delta + f_2(x, x', \delta) \]

Second order in \( x, x', \delta \)

\[ f_2 = -\kappa (\delta^2 - \frac{1}{2} x^2 - 2 \kappa x \delta + \kappa^2 x^2) + k_1 x (\delta - 2 \kappa x) - \frac{1}{2} k_2 x^2 = f_2(x, x', \delta) \]

Dominant parts: \( \kappa D \leq 10^{-3}, D' \leq 10^{-1} \)

\[ f_2(D, D', 1) \approx -\kappa + k_1 D \delta - \frac{1}{2} k_2 D^2 \]

First and second-order dispersion:

\[ D = \int_0^s \hat{k} \sqrt{\hat{\beta} \hat{\beta}} \sin(\psi - \hat{\psi}) d\hat{s} \]

\[ D_2 = \int_0^s f_2(\hat{D}, \hat{D}', 1) \sqrt{\hat{\beta} \hat{\beta}} \sin(\psi - \hat{\psi}) d\hat{s} \]

In the first and last dipole of an achromat:

\[ D_2 = -D \]

In the first and last quadrupole:

\[ D_2'' + 2k_1 D_2 \approx 0 \]

In sextupoles:

\[ \Delta D_2' \approx -\frac{1}{2} k_2 L_2 D^2 \]
Example of 2nd Order Achromat Design

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A second order achromat generally needs two sextupoles, for making

\[ D_2 = 0, D_2' = 0 \]
2nd Order Isochronous Achromats

\[ x'' = -x \left( \kappa^2 + k \right) + \delta \kappa \]
\[ y'' = y k \quad , \quad \tau' = -\kappa x \]

Time of flight for a particle with energy deviation:

\[ \tau = M_{56} \delta + T_{566} \delta^2 = -\int_0^s \hat{\kappa} (\hat{D} \delta + \hat{D}_2 \delta^2) d\hat{S} \]

A second order isochronous achromat generally needs two sextupoles, for making \( D_2 = 0, D_2' = 0 \)
Phase space transport without achromat

\[ D_2 = -D \]

\[ D = \frac{x'}{x} \]
Phase space transport with achromat

\[ D_2 = -D \]

\[ D = 2D \]
Energy Dependent Phase Advance

\[ \Delta \psi = \Delta kl(\hat{s}) \frac{\hat{\beta}}{\hat{s}} \sin^2(\psi - \hat{\psi}) \]

\[ \xi = \text{energy dependence of betatron phase} \]
\[ \text{often called chromaticity} \]

\[ \nu(\delta) = \nu + \frac{\partial \nu}{\partial \delta} \delta + \ldots \]
\[ \xi = \frac{\partial \nu}{\partial \delta} \text{ with } \nu = \frac{\psi}{2\pi} \]

Natural chromaticity \( \xi_0 \) = energy dependence of phase advance due to \( k_1 \) only

\[ f_2 = -\kappa (\delta^2 - \frac{1}{2} x^2 - 2kx\delta + k^2 x^2) + k_1 x (\delta - 2k\kappa) - \frac{1}{2} k_2 x^2 \approx k_1 x \delta - k_2 D x \delta \]

\[ \xi_{x0} = -\frac{1}{2\pi} \int \beta_x(\hat{s}) k_1(\hat{s}) \sin^2(\psi - \hat{\psi}) d\hat{s} \]

Particles with energy difference oscillate around the periodic dispersion leading to a quadrupole effect in sextupoles that also shifts the tune:

\[ \xi_x = \frac{1}{2\pi} \int \beta_x(-k_1 + D x k_2) \sin^2(\psi - \hat{\psi}) d\hat{s} \]
The energy dependent part of the beta function is often called the chromatic beta beat.

\[
\Delta \beta \frac{\beta_x}{\beta} = -\Delta k l(\hat{s}) \hat{\beta} \sin 2(\psi - \hat{\psi})
\]

\[
\beta_x(\delta) = \beta_x + \frac{\partial \beta_x}{\partial \delta} \delta + \ldots
\]

\[
f_2 = -\kappa(\delta^2 - \frac{1}{2} x'^2 - 2 \kappa x \delta + \kappa^2 x^2) + k_1 x (\delta - 2 \kappa x) - \frac{1}{2} k_2 x^2 \approx k_1 x \delta - k_2 D_x \delta
\]

Chromatic beta beat:

\[
\frac{1}{\beta_x} \frac{\partial \beta_x}{\partial \delta} = \int \hat{\beta}_x (k_1 - D_x k_2) \sin 2(\psi - \hat{\psi}) d\hat{s}
\]
… leading to an x-Ray ERL

Key: 1) injector, 2) north linac, 3) turn-around arc, 4) south linac, 5) south x-ray beamlines, 6) CESR turn-around, 7) north x-ray beamlines, 8) 1st beam dump, 9) 2nd beam dump and 10) distributed cryoplant. Tunnel cross-section of 12’ ID shown on lower right.
HIGH BRIGHTNESS, HIGH CURRENT INJECTOR DESIGN FOR THE
CORNELL ERL PROTOTYPE

2003 Particle Accelerator Conference
I.V. Bazarov† and C.K. Sinclair.

Space charge emittance growth
\( \Delta \varepsilon = 0.1 \mu \text{m} \)

CSR emittance growth
\( \Delta \varepsilon = 0.1 \mu \text{m} \)
Cornell Injector prototype: Verification of beam production

- dump
- diagnostics
- SC injector
- gun
- Power sup.
- Cathode
- Photo emitter
- gun
Assembly of the injector accelerator

Done at Cornell – no company could do that today!
The Klystron as Power Source

Power < 1.5MW

Power < 40MW pulsed

\[ P = \eta U_0 I_{\text{beam}}, \quad \eta \leq 65\% \]

- DC acceleration to several 10kV, 100kV pulsed
- Energy modulation with a cavity
- Time of flight density modulation
- Excitation of a cavity with output coupler

Time of flight bunching

Only works for non-relativistic electrons

I up to > 10A

\[ I = \mathcal{P} \eta \]

\[ \mathcal{P} \geq 1.5 \text{MW} \]
Good Luck to you and to the ERL Project

**Key:** 1) injector, 2) north linac, 3) turn-around arc, 4) south linac, 5) south x-ray beamlines, 6) CESR turn-around, 7) north x-ray beamlines, 8) 1st beam dump, 9) 2nd beam dump and 10) distributed cryoplant. Tunnel cross-section of 12’ ID shown on lower right.