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50 7μh
Z235 0.84μh

COUNTING NOTE

PULSE RESPONSE OF COAXIAL CABLES

C-11 (2)

ABSTRACT: For most cables useful in counting work, attenuation below approximately 1000 mc is due mainly to skin-effect losses and varies as the square root of frequency. For such cables the step-function response has a rise time that varies as the square of the attenuation at a given frequency. Curves are given to aid in the selection of cables for transmitting nanosecond pulses.

STEP FUNCTION RESPONSE: Mathematically ideal, lossless coaxial cables can be shown to transmit electrical pulses in the TEM mode without attenuation or distortion. However, all physically realizable cables have losses, the magnitude of which changes with frequency. Pulses transmitted through such cables suffer both attenuation and distortion. By means of the Laplace transform, the nature of the distortion can be calculated if the attenuation and phase-shift are known at all frequencies. In most of the cables presently useful in counting work, skin effect losses in the conductors are the predominate losses below about 1000 mc. Skin-effect losses produce an attenuation whose magnitude in decibels varies as the square-root of frequency. It is shown in the appendix that this results in a step function response of:

$$E_{out} = E_{in} \left(1 - \operatorname{erf} \frac{b\ell}{\sqrt{2(t-T)}}\right)$$

where

E_{out} = voltage at distance ℓ from input end of semi-infinitely long uniform cable,[1] at time t (seconds)

E_{in} = amplitude of step of voltage applied to input of cable at time $t = 0$

ℓ = distance from input end-feet

b = constant for the particular cable in question

$= 1.45 \times 10^{-8}$ A - feet^{-1} $\text{sec}^{\frac{1}{2}}$

A = attenuation of cable at 1000 mc - db/100 feet (attenuation figures for coaxial cables are commonly quoted in these units)

erf = error function[2]

T = transit time of cable defined as the value of t at which the voltage at ℓ first begins to change (considering only the step function occurring at $t = 0$, of course)

1. With negligible error in most cases E_{out} can be taken as the response at the receiving end of a cable of length ℓ, terminated in a resistor equal to its characteristic impedance.

2. As defined in Reference 1, p 256. $g(k,x) = \frac{k}{\sqrt{\pi}} e^{-k^2x^2}$

$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx = 2\int_0^x g(1,x)\,dx$

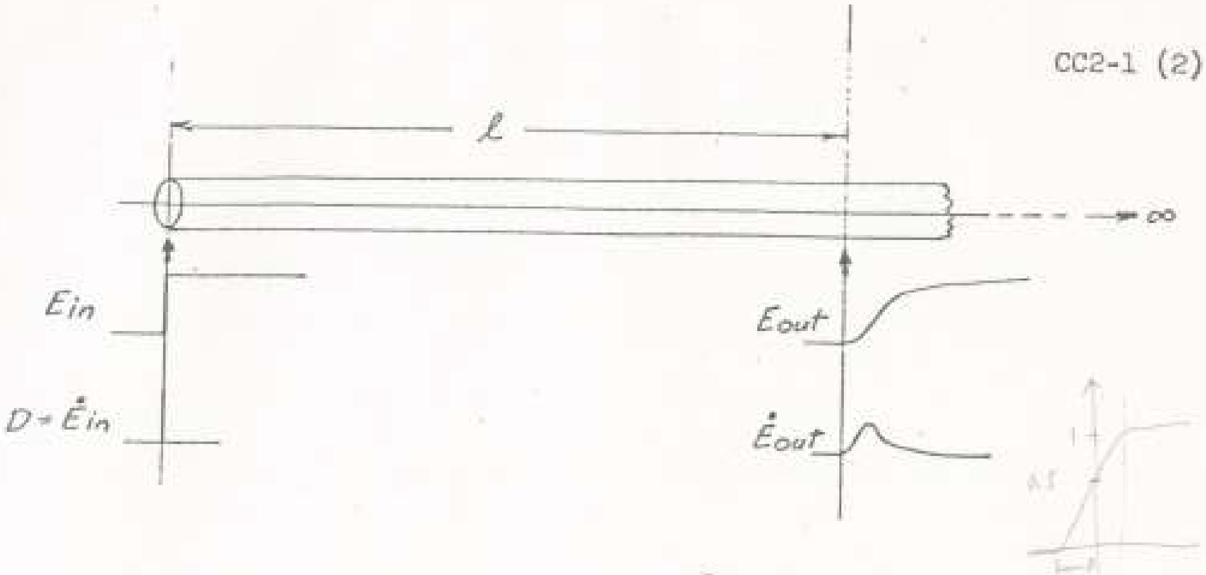


Fig. 1 - This illustrates the space relation between E_{in} and E_{out}.

Figure 1 may clarify the nomenclature involved in this relation.

A normalized curve of E_{out}/E_{in} is shown in Figure 2. The abscissa is plotted in units of T_o, the 0-50% rise time. In other words, T_o is the value of $(t-\tau)$ at which $E_{out}/E_{in} = 1/2$. For cables whose attenuation varies as the one-half power of frequency, it is convenient to calculate T_o as:

$$T_o = 4.56 \times 10^{-16} A^2 \ell^2 \text{ seconds } \left(= \left[\frac{b\ell}{0.6745}\right]^2 \right).$$

It is evident that T_o varies directly as the square of the total attenuation of the length of cable. Cables of different sizes or types may therefore be compared for rise time in terms of A, their attenuation at 1000 mc. Figures of A for most commercially available cables are given in CC2-2.

In cases where: a) the attenuation is known only at a frequency other than 1000 mc; or b) the frequency dependence of attenuation departs somewhat from the 1/2 power law (say, where $\alpha = \text{constant} \cdot f^n$, in the region $0.4 < n < 0.7$) T_o may be calculated:

$$T_o = \frac{4.56 \times 10^{-7} \alpha_f^2 \ell^2}{f}$$

where

α_f = attenuation of cable at frequency f - db/100 feet

f = frequency - cycles

In case a) the nomogram of Sec. VII CC2-2 may be useful. In case b), it has been empirically determined that reasonably accurate results are obtained where f is the frequency at which the total attenuation (i.e., $\alpha_f \ell/100$) of the cable is 6 decibels. Substituting $\alpha_f \ell/100 = 6$ db into the above gives the useful relation

$$T_o \cong 1/6f_6$$

where

f_6 = frequency at which the total attenuation of the length of cable in question is 6 db.

The times to reach other percentages of the input step amplitude are given in Table I.

TABLE I

RISE TIME CONVERSION FACTORS

X	0 to X % rise time / T_o
10	0.17
20	0.28
50	1.0
70	3.1
80	7.3
90	29.
95	110.

The 10 to 90% rise time is $(28.85-0.17)\ T_o = 28.68\ T_o$.

IMPULSE RESPONSE: The response to an impulse (delta function), of a cable having decibel attenuation proportional to the square-root of frequency, may be obtained by differentiating E_{out} above. The mathematical steps are indicated in the appendix, and the results show that, as with the step-function response, the impulse response can be represented by a universal curve, that of Figure 3. The area under this curve (coulombs) is conserved as the pulse travels along the cable. Thus the peak amplitude of the response varies as $\frac{1}{b^2\ell^2} \propto \frac{1}{A^2\ell^2}$, and the time between, for example, the half-amplitude points, varies as $b^2\ell^2$. The peak amplitude occurs at $0.152\ T_o$.

RESPONSE TO OTHER PULSE SHAPES: It will be noted that, since the rise time T_o is proportional to ℓ^2, if two equal lengths of a given type of cable are cascaded, the rise time of the combination is four times the rise time of either length alone. This is in contrast to the well-known case of amplifiers of "Gaussian" frequency response[3], in which the rise time varies as the square root of the number of identical sections. For this reason, and also because the characteristic step-or impulse-function responses of cables and of "Gaussian" amplifiers are so different, the rule-of-thumb that the over-all rise time= $\sqrt{\text{sum of squares of individual rise times}}$ is not applicable either with cables alone, or where cables are combined with Gaussian elements. Instead, the overall response of a system with cables and other elements may be obtained graphically or with the standard convolution integrals[4] using either the step-or impulse-function response of the cables.

3. Appendix.

4. Reference 1, pp 112-120.

RECTANGULAR PULSE RESPONSE, CLIPPING LINES: The response of a cable to a rectangular pulse of a duration T can be found by a simple application of superposition. The rectangular pulse is considered to consist of a positive step-function at $t = 0$, followed by a negative step-function at $t = T$. The amplitude reduction of such a pulse as a function of the distance it has traveled along the selected coaxial cables is shown in Figures 5, 6 and 7. Figure 5 includes a curve showing the time-stretching of the output pulse with respect to the input pulse. By suitably changing the length scale in the way indicated on the figure, the two curves of Figure 5 can be applied to any pulse duration and any cable for which attenuation varies as the square root of frequency. The amount of time-stretching of any output pulse can therefore be determined from Figure 5 by knowing the value E_{out}/E_{in} for the pulse, where E_{out} is the peak amplitude of the output pulse, and E_{in} is the amplitude of the input pulse.

The relative merits of various coaxial cables as conductors of pulses from multiplier phototubes or other current generators can be estimated from the curves of Figure 6 which are replotted from Figure 5. Use figure 6a for pulses of $T = 10^{-8}$ second; 6b for $T = 10^{-9}$; 6c for $T = 10^{-10}$. Note that the input is a rectangular current pulse of 1 ampere amplitude. At the input end of the cable, therefore, the voltage amplitude of the rectangular pulse is Z_o volts, where Z_o is the characteristic impedance of the line. The curves show, for example, that for an input current pulse of $T = 10^{-9}$ second, the peak voltage of the output pulse at the end of a 75 foot run of RG 114 would be the same as that at the output end of a 75 foot run of RG 63, even though the voltage developed at the input end of the RG 114 would be 185/125 times the voltage at the input of the RG 63.

In Figure 7 are shown some specific output pulse shapes together with the lengths of commonly used cables that give the corresponding output pulse shape for an input pulse of $T = 10^{-9}$ second. These pulse shapes were determined from Figure 2 in the way mentioned above.

The curves of Figures 5, 6 and 7 also apply[5] to clipping lines if the input is a step-function and $T = 2$ times the electrical length of the clipping line. This is true whether the clipping line is located at the input or output end of the transmission line. The minimum 0-100% rise-time of a clipped pulse is $0.15\ T_o$. Clipping lines of electrical length less than $0.075\ T_o$ will not decrease the rise-time, but will only decrease the amplitude of the output pulse.

The curves and data are intended to present the properties of the coaxial cables, and therefore do not include the effect of quantities that depend on the way in which the cables are used. Examples of such quantities are the rise-time of multiplier phototube output pulses and imperfect cable terminations. The curves and data also do not take into account the inevitable small variations of characteristic impedance along the line. These impedance variations will generally degrade the rise-time of the output pulse by reflecting portions of the faster rising parts of the pulse being transmitted.

5. Provided the clipping line is short enough that its attenuation may be neglected.

EXPERIMENTAL VERIFICATION: Photographs of the responses of several cable types to step-function inputs are shown in Figure 8. These photographs were all taken from displays on a DuMont K1056 cathode ray tube connected as shown in the block diagram of Figure 8. Figure 8a shows the step from the pulse generator delayed only by 25 nanoseconds of cable inserted at A-A in Figure 9. The rise time of the pulse generator-oscilloscope combination is about 0.45 nanosecond, and therefore obscures the shape of the leading edge of the waveform of some of the better cables. The typical 1 - erf shape is plainly seen in Figure 8f, for RG 63.

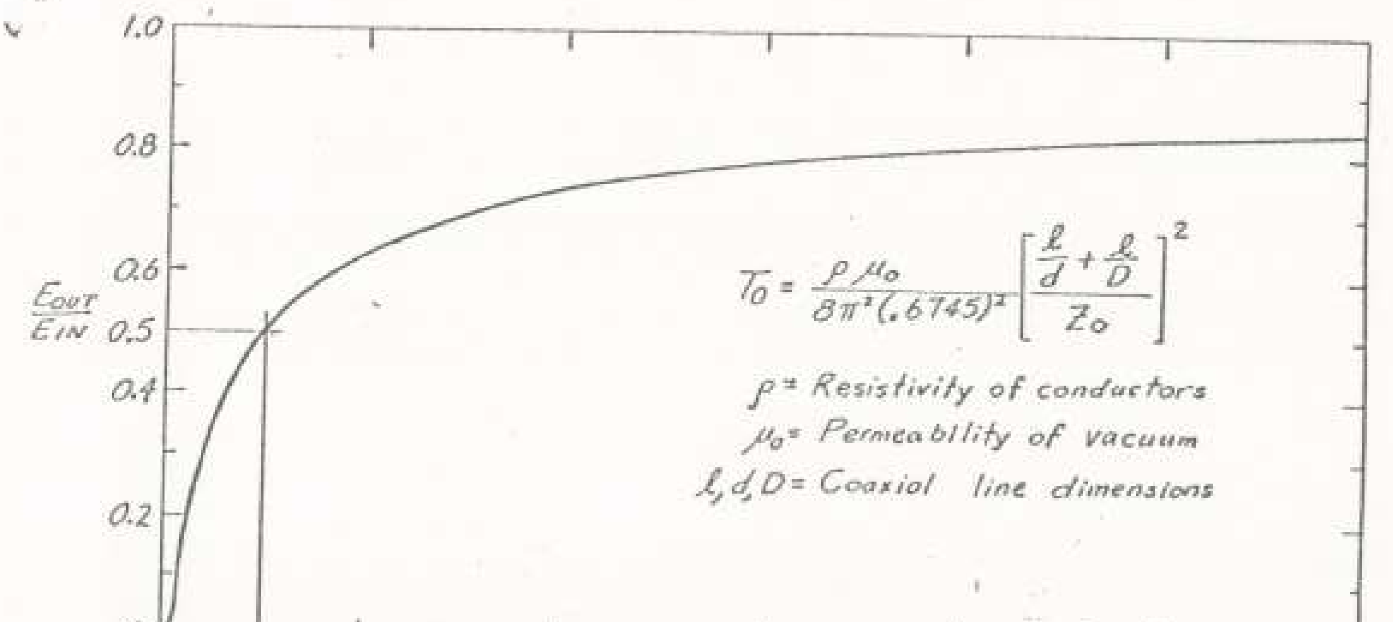


Fig. 2. Step-function response of transmission lines for which decibel attenuation varies as the square root of frequency. The time T_0 is defined as the interval measured from the start of the output pulse to the point at which $E_{out} = 0.5\ E_{in}$. To depends on the transmission line parameters; the relation for coaxial structures with negligible dielectric loss is given in the figure. In Fig. 4, T_0 is plotted as a function of cable type and length.

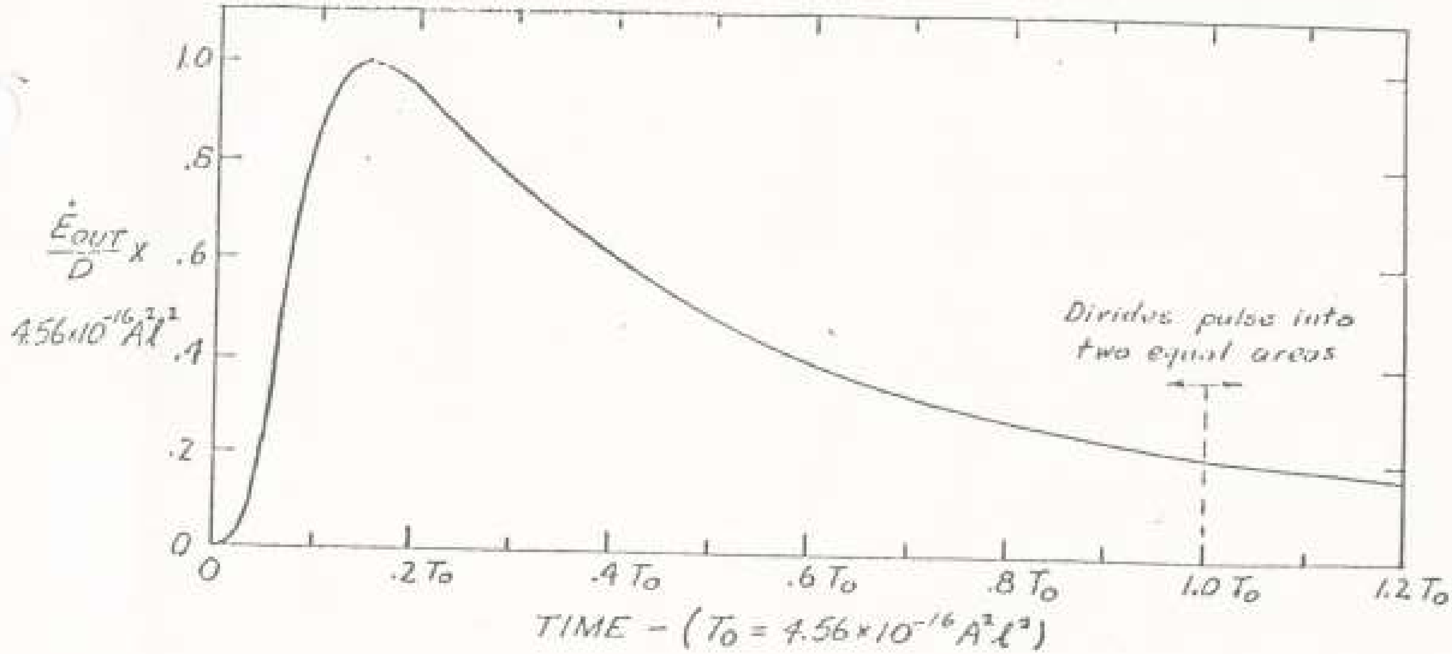


Fig. 3. Delta-function response of transmission lines for which attenuation varies as the square-root of frequency. As given in the text, A is the attenuation in db/100 feet at 1000 mc, ℓ is the cable length in feet, and D is the volt-second product of the input delta function. This curve is the time derivative of the curve of Fig. 2.

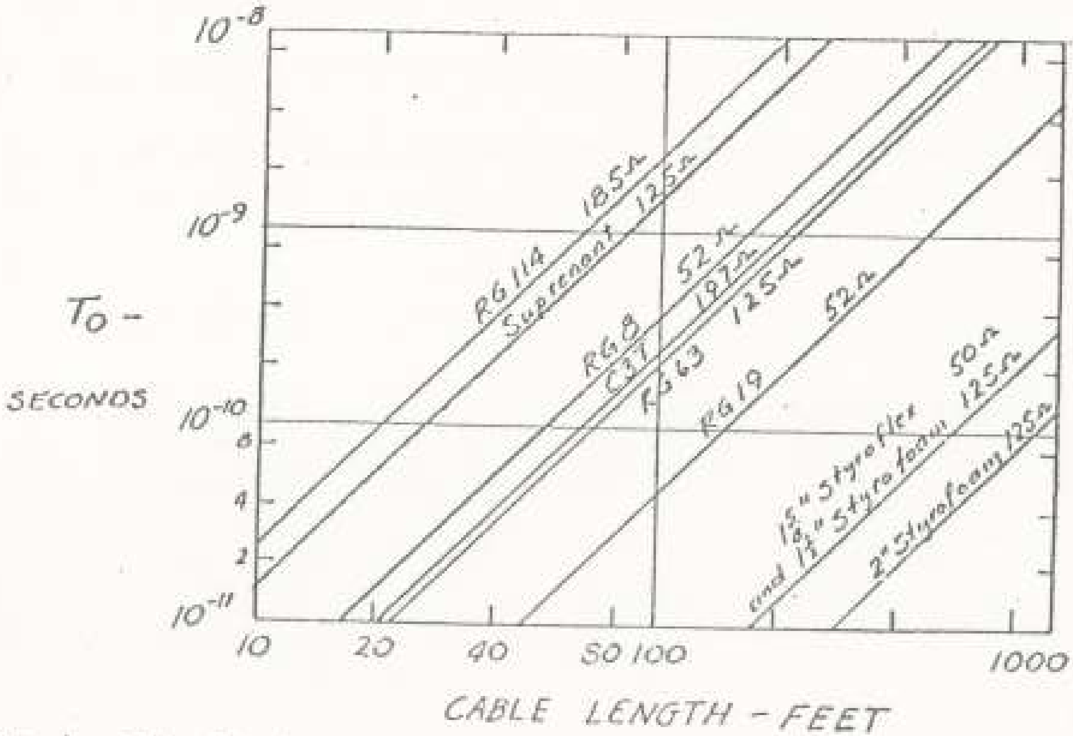


Fig. 4. Calculated variation of T_O with cable length for typical coaxial cables. To obtain the values of T_O for other cable types see CC2-2.

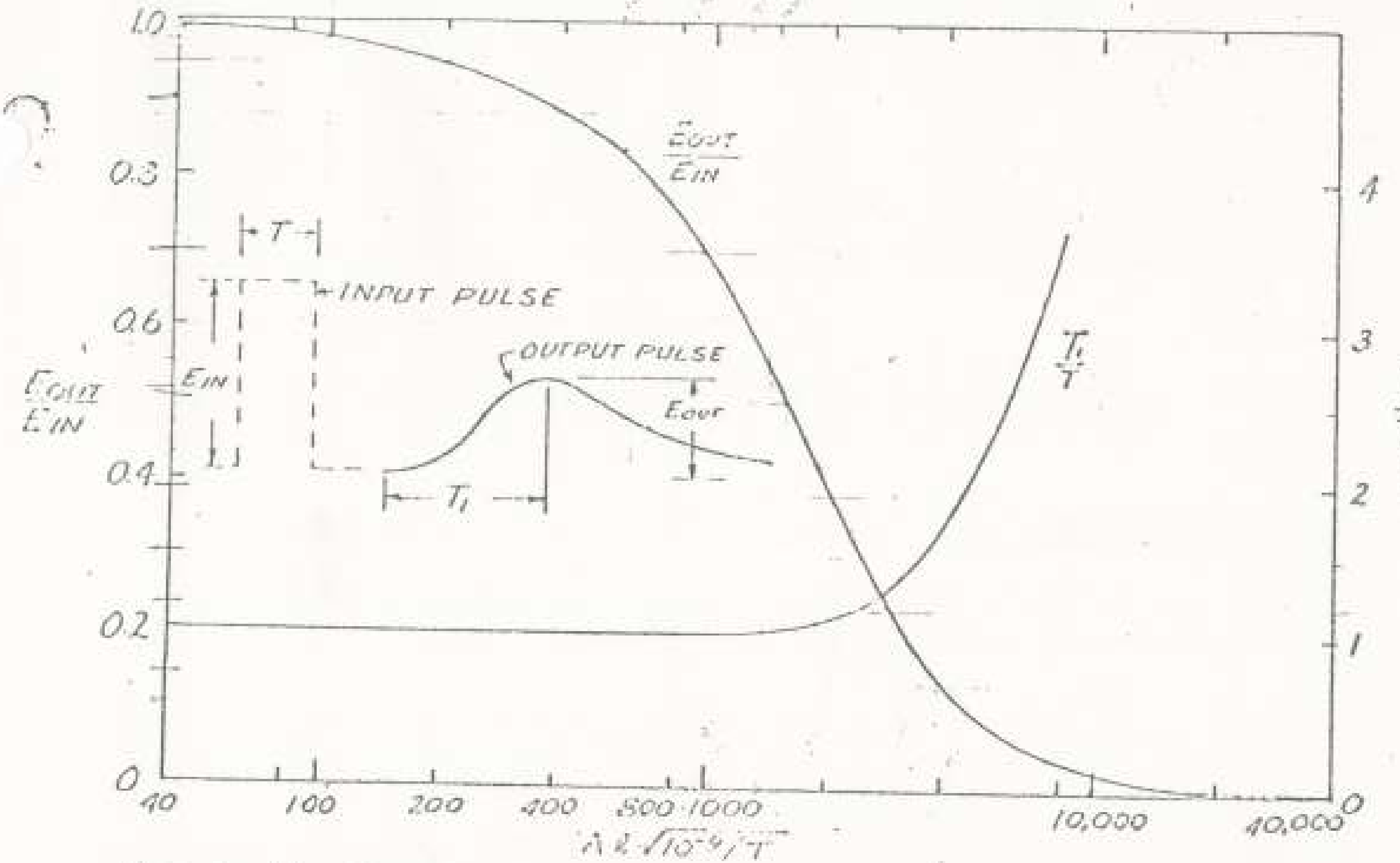
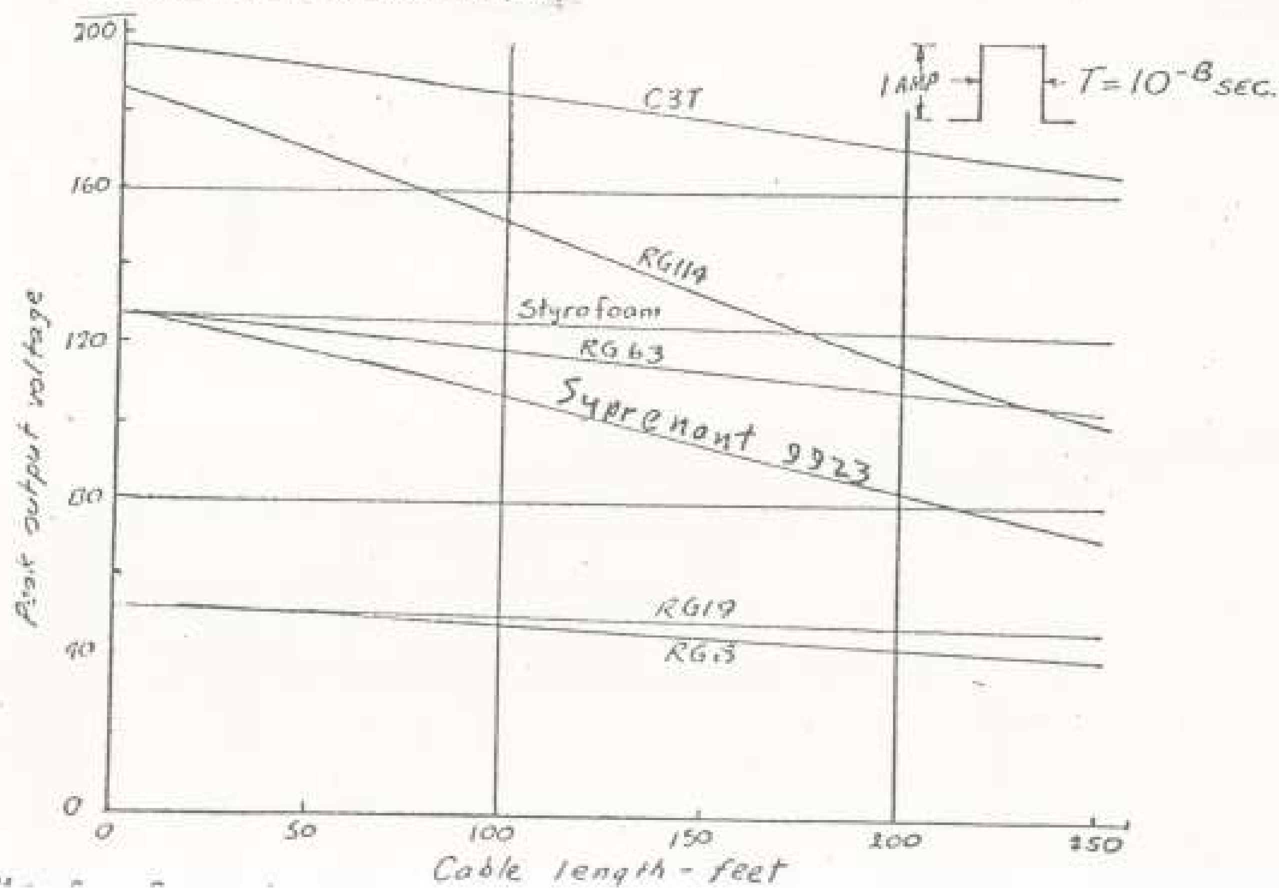


Fig. 5. The time-stretching and amplitude-reduction of an originally rectangular pulse plotted as a function of A, the attenuation of the cable at 1000 mc in db/100 ft.; ℓ, the length in feet; and T, the duration of the input pulse in seconds. Attenuation figures may be obtained from CC2-2. As an example, for RG63, A is 7 db/100 ft. Thus if T were 10^{-9} sec, and ℓ were 100 feet, the chart should be entered at an abcissa of 700.



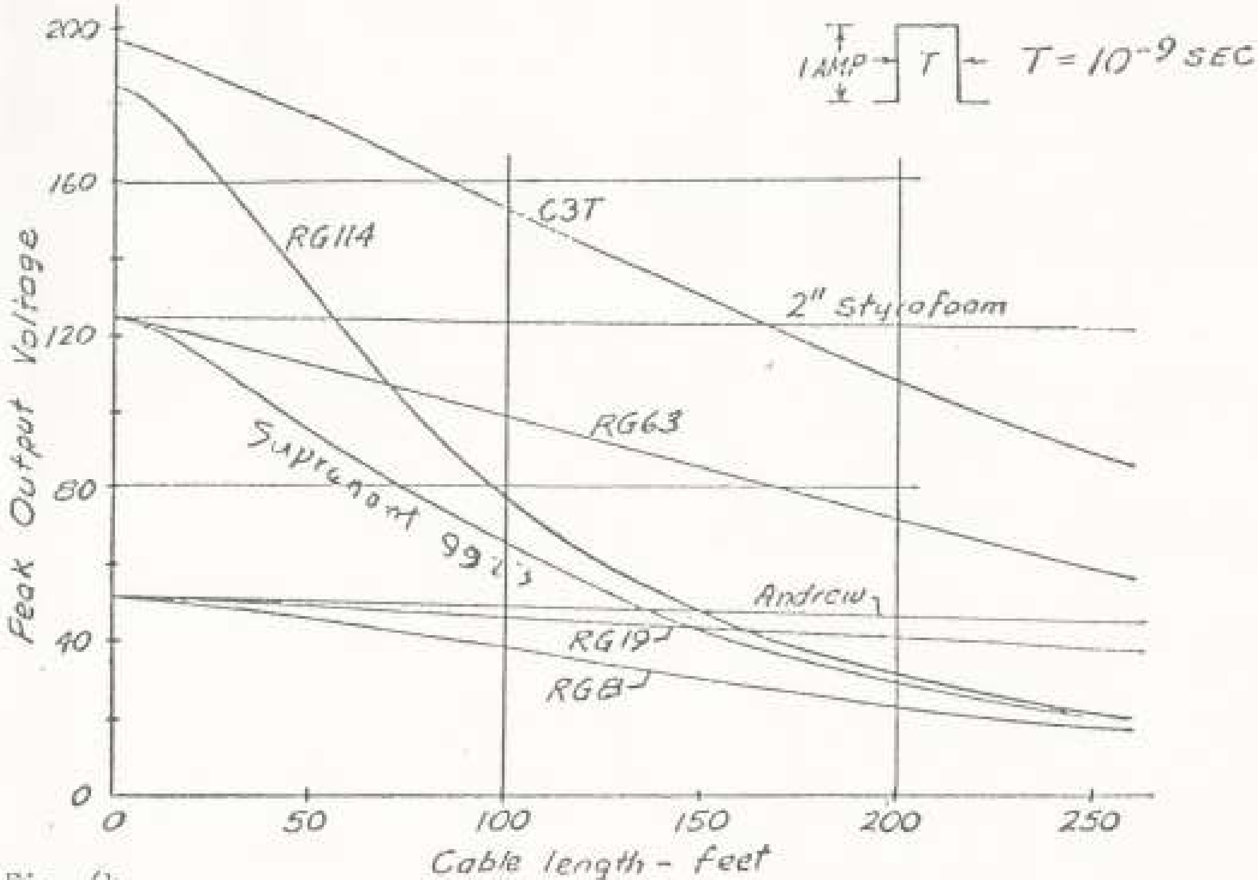


Fig. 6b

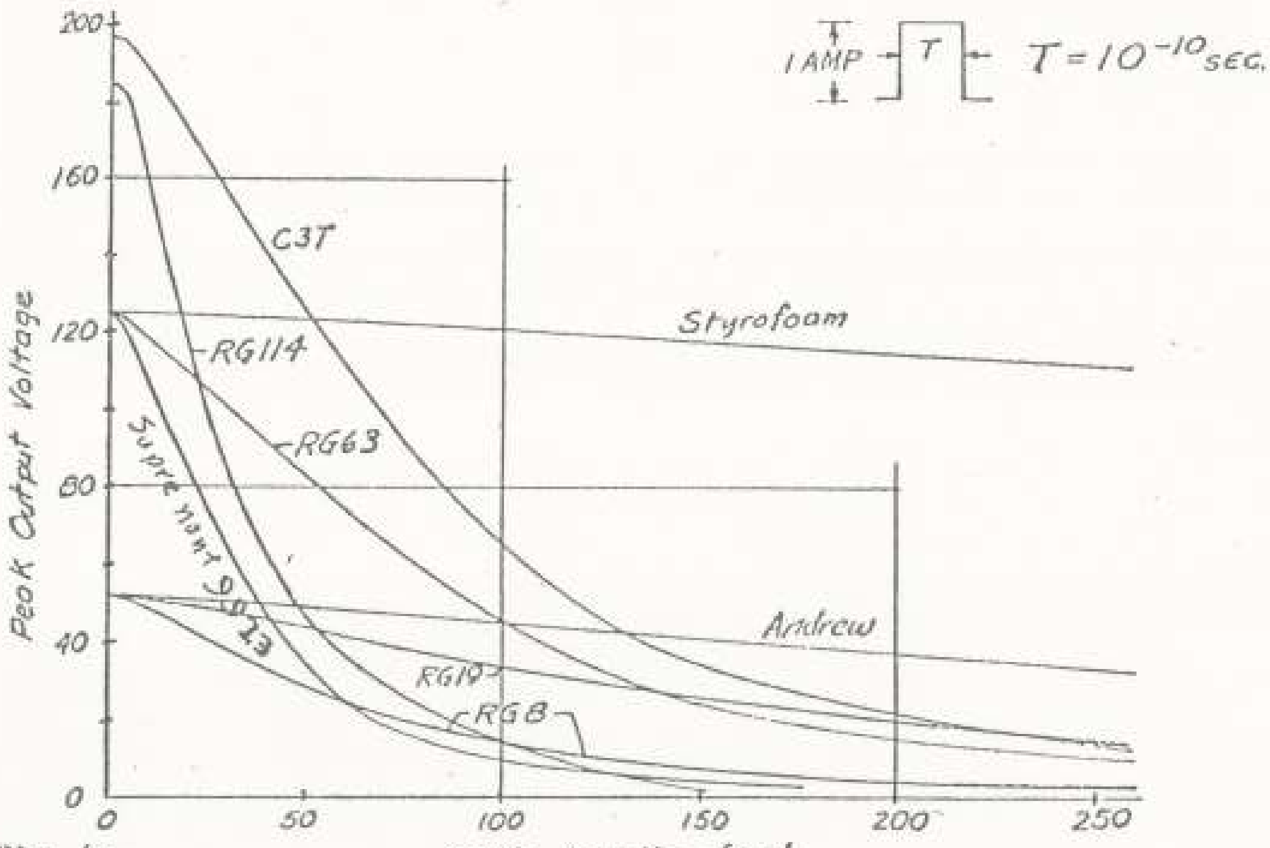
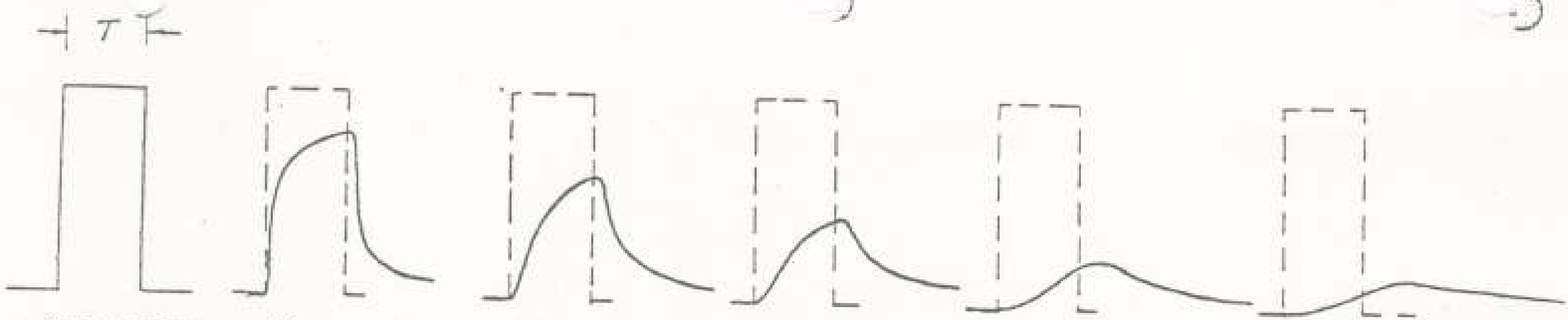


Fig. 6c.

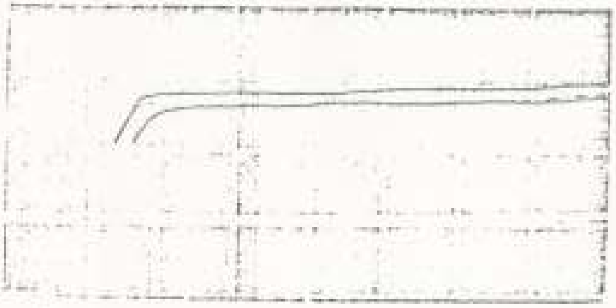
Fig. 6. Peak amplitude of the output voltage pulse from some typical coaxial cables as a function of cable length. The assumed inputs are rectangular current pulses of 1 ampere amplitude and durations of 10^{-8}, 10^{-9} and 10^{-10} seconds. These curves are all replotted from Fig. 5 with suitable scale changes.



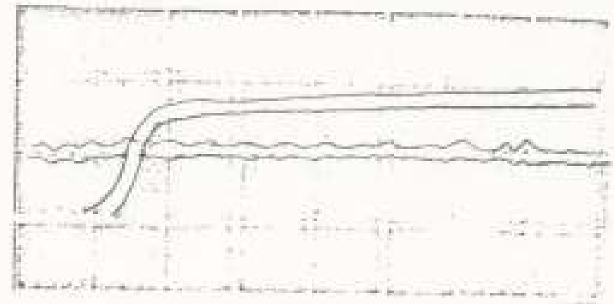
For T = 10^{-9} sec., the output pulse will have the shape and amplitude shown for the following cable lengths.

Cable type	(a)	(b)	(c)	(d)	(e)
RG 8	77	145	240	350	400
RG 19	200	380	600	900	1100
RG 63	95	180	290	430	500
2" Styrofoam	1200	2300	3700	5500	6400
C3T	90	170	280	400	470
RG 114	37	70	110	170	200

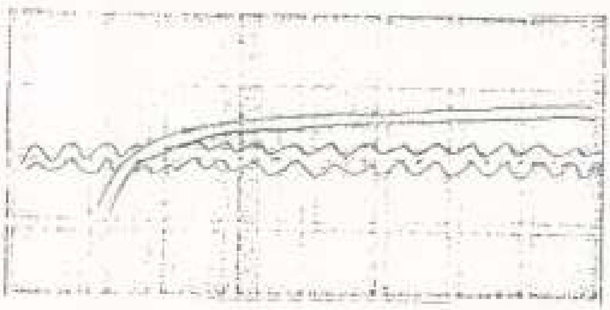
Fig. 7. The above waveforms show the deterioration of an originally rectangular pulse as it travels along a transmission line for which the decibel attenuation varies as the square root of frequency. For comparison purposes, the input pulse is also shown with each output waveform. The figures listed above give the cable lengths that will cause the distortion shown when T = 10^{-9} second. To find the cable lengths for which the output pulse will have the same form relative to the input pulse for other input pulse durations, multiply the above lengths by $\sqrt{T}$, where T is the input pulse duration in millimicroseconds.



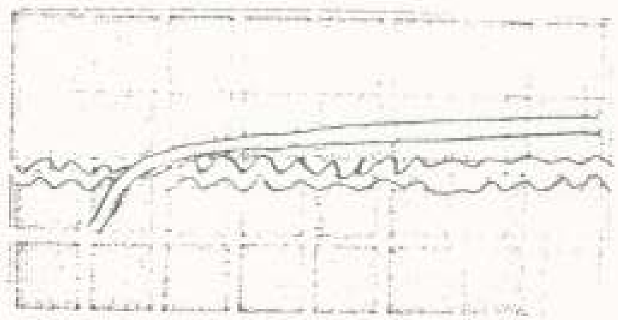
a. Input pulse.



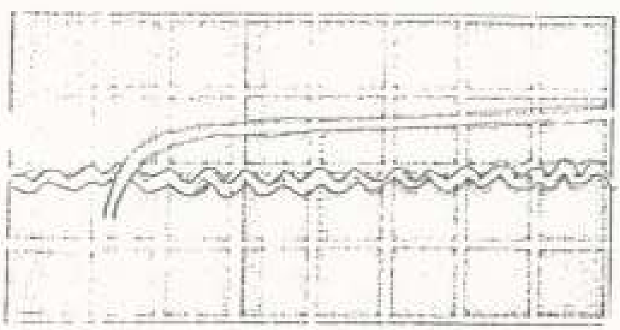
b. Styroflex: $Z_o = 50$; d = 317 mus.



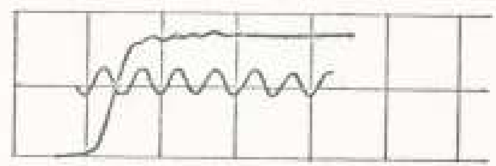
c. RG 9: $Z_o = 51$; d = 300 mus.
(RG 8 gives same response.)



f. RG 63: $Z_o = 125$; d = 300 mus.
(21-342 and 21-406 give same response.)

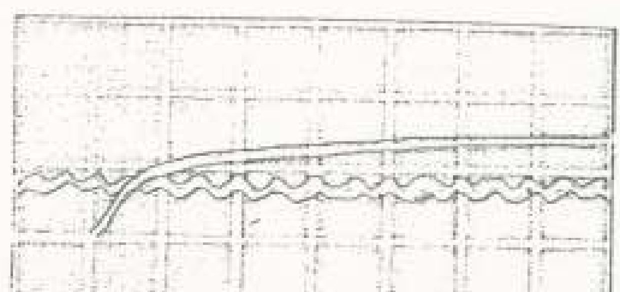


c. RG 59: $Z_o = 52$; d = 300 mus.



g. Styrofoam: $Z_o = 125$; d = 258 mus

d. Andrew: $Z_o = 52$; d = 326 mus.



h. C3T: $Z_o = 197$; d = 300 mus.

Fig. 8. Photographs of the leading parts of pulses before and after transmission through some coaxial transmission lines used in counting work. All photographs taken with Dumont K1056 cathode-ray tube. a) Pulse applied to input end of transmission lines. b-h) Pulse appearing at output end of transmission line; d = elecical length in millimicroseconds. Frequency of timing wave is 1000 mc. Trace g) replotted from Fig. 4b to about same time scale as other traces of this series. Part of the upward tilt is caused by cathode-ray tube distortion. An estimate of the amount of this distortion can be made by referring to the three traces in a), where the lower trace is a zero reference.

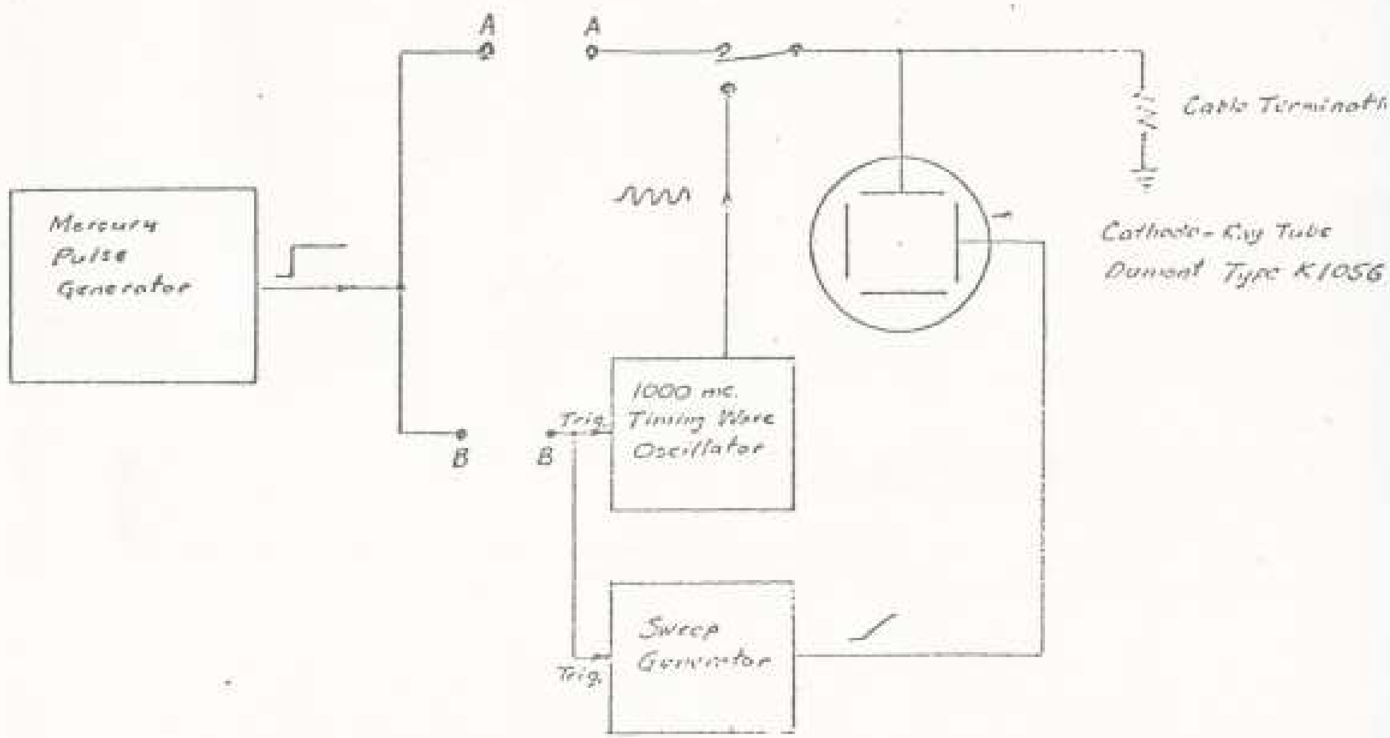


Fig. 9. Block diagram of equipment used to take the cable response photographs of Figure 8. The cable to be tested is placed between points A-A. A time dela of about 25 mus. less than that of A-A is placed between B-B.

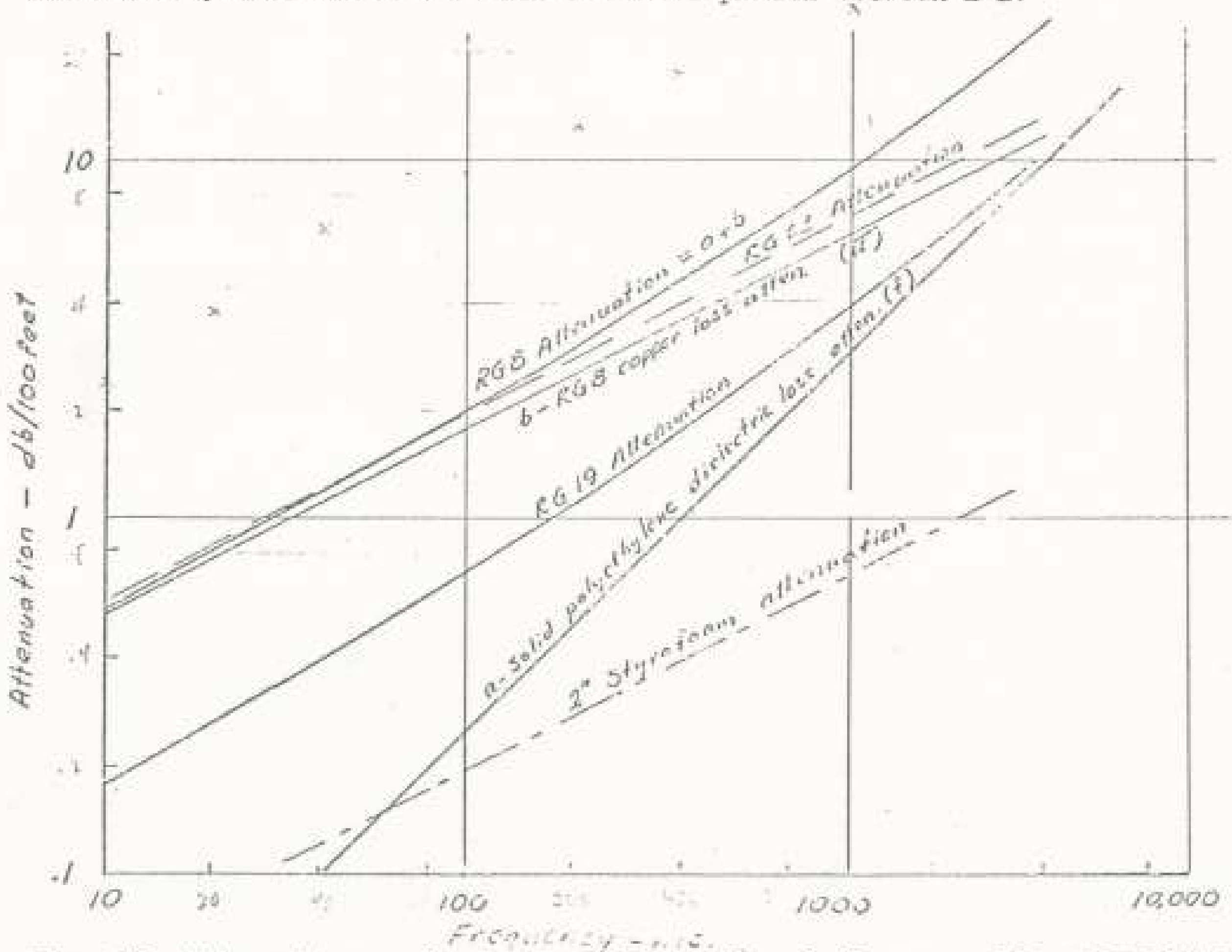


Fig. 10 Attenuation vs frequency. Both RG8 and 19 have solid polyethylene dielectrics, and their attenuation curves are asymptotic to curve a at higher frequencies. RG63 has a semi-solid polyethylene dielectric, and Styrofoam uses

APPENDIX: A brief derivation of the cable responses described in the previous section is presented. The derivation follows that given in Reference 2. Another derivation is given in Reference 3.

The two important sources of attenuation in coaxial cables are conductor losses and dielectric losses. Examination of the equations for calculating these two losses, as given in CC2-2 Sec. VIII, shows that conductor losses (skin-effect losses) vary as the square root of the frequency, and are functions of the cable Z_0 and diameter, while dielectric losses of cables having solid dielectric vary directly with frequency, and are independent of cable diameter and impedance. This is shown in Fig. 10. Note that below $\sim$ 1000 Mc for RG 8, and below $\sim$ 4000 Mc for Styrofoam, conductor losses predominate.

Because of the skin effect, the penetration of current into the conductors of coaxial cables decreases as the frequency increases. This results in increased conductor resistance because of the smaller useful cross section of the conductor. The distributed inductance of the cable thereby also changes. The total distributed inductance, L, may be thought of as existing in two parts: (a) that represented by magnetic flux in the dielectric, L_1; (b) that represented by flux in the conductors, L_2. As the current penetration decreases, L_2 likewise decreases, and it is shown* that L_2 is related to the distributed conductor resistance, R, by $2\pi f L_2 = R$ (and $R = \text{constant} \times \sqrt{f}$). L_1, of course, does not vary with frequency

The transfer function of a coaxial cable may be represented by

$$\frac{E_{out}}{E_{in}} (\omega) = e^{-[\alpha(\omega)\ell + j\beta(\omega)\ell]},$$

where

E_{out} = voltage at receiving end of cable of length ℓ,

E_{in} = voltage at sending end of cable of length ℓ

α = attenuation of cable in nepers/unit length,

β = phase shift of cable in radians/unit length,

ω = frequency in radians/second = $2\pi f$.

The quantities α and β may be expressed in terms of the distributed R, L, and C of the transmission line (the shunt conductance loss of the dielectric is assumed negligible) as

$$\beta = \omega\sqrt{LC}$$

and $$\alpha = R/2Z_0.$$

*Ref. 4, p. 239.

But $R = \sqrt{\frac{f\mu_c}{\pi\sigma}} \left(\frac{1}{D} + \frac{1}{d}\right)$

Therefore $\alpha = b\sqrt{\omega}$,

where

$$b = \sqrt{\frac{\mu_c}{2\sigma}} \times \left(\frac{1}{D} + \frac{1}{d}\right) / 2\pi Z_0 ,$$

a constant for the cable,

$= 1.45 \times 10^{-8}$ A (see p. 1),

μ_c = permeability of the conductor material,

σ = conductivity of the conductor material,

D = inside diameter of outer conductor,

d = outside diameter of inner conductor.

The quantity β may be separated into two parts by substituting $L_1 + L_2 = L$:

$$\beta = \omega\sqrt{(L_1 + L_2)C}$$

$$\cong \omega\sqrt{L_1C}\left(1 + \tfrac{1}{2}\ \frac{L_2}{L_1}\right), \text{ since usually } L_2 << L_1 ;$$

$$\beta = \beta_1 + \beta_2 ,$$

where

$$\beta_1 = \omega\sqrt{L_1C},$$

and

$$\beta_2 = \omega/2\sqrt{C/L_1} \cdot L_2.$$

Substituting $Z_0 \cong \sqrt{L_1/C}$, $L_2 = R/\omega$,

One obtains $\beta_2 = R/2Z_0 = \alpha = b\sqrt{\omega}$.

The term β_1 is independent of frequency and is the phase constant that the cable would have if the skin depth were zero. The second term, β_2, is the frequency-sensitive part of β, which arises because of the skin effect. Substituting into the transfer function equation,

$$E_{out} = E_{in}\, e^{-(\alpha + j\beta)\,\ell} ,$$

one has $E_{out} = E_{in}\, e^{-(b\sqrt{\omega} + jb\sqrt{\omega})\,\ell} \cdot e^{-j\omega\sqrt{L_1C}\cdot\,\ell}$.

Since the velocity of propagation along a cable is the inverse of the product of distributed capacity and inductance, the term $\ell\sqrt{L_1C}$ is the transit time, τ, through the cable of length ℓ at frequencies at which the skin depth is essentially zero,

$$E_{out} = E_{in}\, e^{-b\sqrt{2j\omega}\ \ell}\ \ e^{-j\omega\tau}.$$

The Laplace transform may be obtained formally by substituting the Laplace variable, s, for $j\omega$;

$$E_{out}(s) = E_{in}(s)\, e^{-b\sqrt{2s}\,\ell} \cdot e^{-s\tau}.$$

Let $E_{in}(s) = 1/s$, a step function of amplitude E_{in}. The inverse transform for $E_{out}(t)$ is

$$E_{out}(t) = E_{in}\left(1 - \text{erf}\ \frac{b\ell}{\sqrt{2(t-\tau)}}\right).$$

This function is plotted in Fig. 2 for $E_{in} = 1$.

The function 1 - erf ( ) is also obtained as the solution to certain problems in diffusion and heat transfer, and is also the step-function response of a distributed RC or LR transmission line. It might also be mentioned that practically it is difficult to determine τ, the time at which the response begins, because its slope changes slowly at first.

The response to a delta function of volt-second product, D, may be obtained by letting $E_{in}(s) = D$, or by differentiating $E_{out}(t)$. The result is

$$\dot{E}_{out} = D\ \frac{1}{\sqrt{\pi}\,(t-\tau)}\ \ \frac{b\ell}{\sqrt{2(t-\tau)}}\ \ e^{-\left(\frac{b^2\ell^2}{2(t-\tau)}\right)}.$$

Normalizing and multiplying both sides by $b^2\ell^2$, one gets

$$\frac{\dot{E}_{out}}{D}\ b^2\ell^2 = \frac{2}{\sqrt{\pi}}\left(\frac{b\ell}{\sqrt{2(t-\tau)}}\right)^3 e^{-\left(\frac{b^2\ell^2}{2(t-\tau)}\right)}.$$

This shows that the impulse response of a cable having decibel attenuation varying as the square root of frequency is a universal function. The same curve may be used to represent the response at all points along the cable provided the abscissa (amplitude) scale is divided by $b^2\ell^2$ and the ordinate (time) scale is multiplied by $b^2\ell^2$. The function is plotted in Fig. 3.

It is instructive to inquire into the role played by the (assumed to be lossless) dielectric in affecting the conductor losses and thereby the rise time of the cable. With the assumption $D > d$ (cf nomograph of Sec. VII, CC2-2), the equation for a may be written

$$a \cong \sqrt{\frac{f\mu_c}{\pi\sigma}} \quad \frac{1}{d}$$

or

$$\ln a \cong \ln \sqrt{\frac{f\mu_c}{\pi\sigma}} - \ln d.$$

The quantity $\ln d$ can be found from

$$Z_0 = \sqrt{\frac{\mu_d}{\epsilon_d}} \quad \frac{1}{2\pi} \quad \ln D/d ,$$

where μ_d = permeability of dielectric,

ϵ_d = permittivity of dielectric.

Rearranging,

$$\ln d = 2\pi Z_0 \sqrt{\frac{\epsilon_d}{\mu_d}} - \ln D.$$

Thus,

$$\ln a = \ln \sqrt{\frac{f\mu_c}{\pi\sigma}} - 2\pi Z_0 \sqrt{\frac{\epsilon_d}{\mu_d}} - \ln D.$$

Therefore, in the situation in which ϵ_d is the only variable, one has

$$\frac{da}{a} = \frac{\pi Z_0}{\sqrt{\mu\epsilon_d}} \quad d\,\epsilon_d$$

Remembering that a is the attenuation per length of cable, one sees that to transmit a pulse over a given distance where Z_0, the outer diameter D, and the conductor material are fixed, dielectrics of the minimum ϵ_d should be used to obtain the shortest rise time.

Note that this result may not apply to other situations such as when a given delay of minimum rise time is to be built into a certain volume.

The result that the rise time of a cable varies as the length squared may also be obtained by using the principle of time-bandwidth invariance, which is that compressing a time function by a certain factor expands its spectrum in frequency and reduces it in amplitude by the same factor.* Or, in symbols,

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*Many other useful relations are contained in the Hewlett-Packard Journal, Vol. 7 No. 3, Nov. 1955, and also in a "Table of Important Transforms," issued by Hewlett-Packard.

for $\quad f(t) \rightarrow f(kt)$

then $\quad F(\omega) \rightarrow \frac{1}{|K|} F\left(\frac{\omega}{K}\right)$

Suppose that one has a device with a transfer function that can be expressed as

$$\frac{E_{out}}{E_{in}}(\omega)_1 = \exp(-\omega^n) = F(\omega) .$$

If one cascades m such devices, the over-all transfer function is

$$\frac{E_{out}}{E_{in}}(\omega)_m = \exp(-m\omega^n) = \exp[-(m^{1/n}\omega)^n] = F(m^{1/n}\omega) .$$

In this case, it is seen that $K = 1/m^{1/n}$, and thus the time-response function is stretched in time by the factor $m^{1/n}$. Some common values of n are: (a) n = 0.5 for the cable case discussed above; (b) n = 1 for cables in which attenuation is owing mainly to the dielectric; (c) n = 2 for "Gaussian frequency response" amplifiers. Case (c) includes many distributed amplifiers (e.g., H. P. 460A, 460B) and is approximated by transfer functions of a single RC or L/R time constant. The relation between rise time and n, where m is the number of identical cascaded units, is summarized as follows

Value of n	Rise time varies as
0.5	m^2
1	m
2	$m^{\frac{1}{2}}$

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